Vanishing theorems for quaternionic Kähler manifolds

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Abstract

In this article we discuss a peculiar interplay between the representation theory of the holonomy group of a Riemannian manifold, the Weitzenböck formula for the Hodge–Laplace operator on forms and the Lichnerowicz formula for twisted Dirac operators. For quaternionic Kähler manifolds this leads to simple proofs of eigenvalue estimates for Dirac and Laplace operators. We determine which representations may contribute to harmonic forms and prove the vanishing of certain odd Betti numbers on compact quaternionic Kähler manifolds of negative scalar curvature. We simplify the proofs of several related results in the positive case.

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1 Introduction

Since decades the Weitzenböck formulas for the Dirac operator on Clifford bundles have inspired intensive and important research. The full Weitzenböck machinery is now beginning to take its definite place in differential geometry incorporating recent ideas about Kato inequalities (cf. [CGH99]) and more and more representation theory. It is inevitable to get the impression that geometrically interesting operators like the Hodge–Laplace or the Dirac operator can be defined abstractly apart from their original setting. In particular it is thus possible to compare geometric differential operators defined on completely different vector bundles. In this article we will describe the impact of this idea and discuss potential applications for quaternionic Kähler manifolds in detail.

Studying manifolds of special holonomy may lead to new insights into underlying structures and concepts of differential geometry. In fact the primary feature of a manifold of special holonomy is its richness in geometric vector bundles $\pi(M)$ corresponding to the representations π of the holonomy group. In this article we will use Meyer's interpretation

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(cf. [Me71]) of the Weitzenböck formula for the Hodge–Laplacian Δ to define an elliptic selfadjoint second order differential operator

$$\Delta_\pi : \quad \Gamma\,\pi(M) \;\longrightarrow\; \Gamma\,\pi(M)$$

for every geometric vector bundle $\pi(M)$. For a homogeneous vector bundle on a symmetric space G/K the operator Δ_π becomes the Casimir of G. Moreover Δ_π agrees with the Hodge–Laplacian Δ for all parallel subbundles $\pi(M)$ of the differential forms. This immediately implies the generalized Lefschetz decomposition of the de Rham cohomology

$$H^\bullet_{dR}(\,M,\,\mathbb{C}\,) \;=\; \bigoplus_\pi \operatorname{Hom}_{\,\mathrm{Hol}}(\,\pi,\,\Lambda^\bullet\mathbb{C}^{n*}\,) \otimes \ker\,\Delta_\pi$$

where the sum is over all irreducible representations π of the holonomy group Hol. Considering Δ_π as a generalization of the Casimir of a symmetric space to arbitrary Riemannian manifolds it is only natural to derive formulas linking this operator to other second order differential operators. In particular we will generalize Parthasarathy's formula which expresses the twisted Dirac operator on symmetric spaces in terms of the Casimir.

The general result for twisted spinor bundles can be applied to a very prominent family of twisted spinor bundles on quaternionic Kähler manifolds. The indices of the twisted Dirac operators in this family are of fundamental importance in studying quaternionic Kähler manifolds in general. Our main technical result is a general eigenvalue estimate for the Dirac operators in this family leading to an interpretation of their kernels in terms of eigenspaces of operators Δ_π corresponding to the minimal eigenvalue. This enables us to give an explicit description of all representations contributing to harmonic forms. In the case of positive scalar curvature $\kappa > 0$ we obtain new proofs for results of S. Salamon on the Betti numbers (c.f. [Sal82]) and the strong Lefschetz Theorem 6.3. The same techniques can be applied to obtain completely new results on the cohomology of quaternionic Kähler manifolds with negative scalar curvature. Our main result here is the vanishing of all odd Betti numbers up to degree n for a quaternionic Kähler manifold of dimension $4n$.

Theorem 1.1. *(Weak Lefschetz theorem for quaternionic Kähler manifolds with $\kappa < 0$)* *Let $(M^{4n},\,g)$ be a quaternionic Kähler manifold of negative scalar curvature $\kappa < 0$. Then the odd Betti numbers $b_{2k+1}(M)$ vanish for $2k+1 < n$. In general the Betti numbers of M satisfy for all $k < n$ the inequalities*

$$b_{2k}(\,M\,) \;\leq\; b_{2k+4}(\,M\,) \qquad \textit{and} \qquad b_{2k+1}(\,M\,) \;\leq\; b_{2k+3}(\,M\,)$$

For quaternionic Kähler manifolds of negative scalar curvature the wedge product with the parallel Kraines form Ω still descends to an injective map on the level of cohomology in all degrees one could possibly hope for. Philosophically however this is not really the strong Lefschetz theorem, because the space of primitive forms decomposes non–trivially into different isotypical components with respect to the holonomy group.

2 Holonomy groups and Weitzenböck formulas

In this section we will discuss the classical Weitzenböck formula for the Hodge–Laplacian or more general for the Dirac operator on a Clifford bundle (cf. [Be], [Po81] or [Sal89]) and introduce the operator Δ_π. Our approach is in some sense similar to [Me71] and [Ch57] although our formulation of the cohomology decomposition as well as our definition of the operator Δ_π as an abstract object without reference to differential forms seems to be new. The basic example of a Clifford bundle is the bundle of exterior forms $\Lambda^\bullet T^*M$ endowed with the scalar product induced by the metric on M and Clifford multiplication with tangent vectors

$$\star : \; T_pM \times \Lambda^\bullet T_p^*M \longrightarrow \Lambda^\bullet T_p^*M, \qquad (X,\omega) \longmapsto X \star \omega$$

defined by $X \star \omega := X^\sharp \wedge \omega - X \lrcorner \omega$. The Levi–Civita–connection induces a connection ∇ on $\Lambda^\bullet T^*M$ and an associated second order elliptic differential operator $\nabla^*\nabla := -\sum_i \nabla^2_{E_i,E_i}$ where $\nabla^2_{X,Y} := \nabla_X \nabla_Y - \nabla_{\nabla_X Y}$ and the sum is over a local orthonormal base $\{E_i\}$. On the other hand we have the exterior differential d and its formal adjoint d^* as natural first order differential operators on $\Lambda^\bullet T^*M$ linked to $\nabla^*\nabla$ by the classical Weitzenböck formula

$$\Delta \;:=\; (d+d^*)^2 \;=\; \nabla^*\nabla + \frac{1}{2}\sum_{ij} E_i \star E_j \star R_{E_i,E_j} \tag{1}$$

where $R_{X,Y}$ is the curvature endomorphism of $\Lambda^\bullet T_p^*M$. However the connection on $\Lambda^\bullet T^*M$ is induced by a connection on TM and consequently the curvature endomorphism $R_{X,Y}$ is just the curvature endomorphism of T_pM in a different representation, namely the representation

$$\bullet : \; \mathfrak{so}(T_pM) \times \Lambda^\bullet T_p^*M \longrightarrow \Lambda^\bullet T_p^*M, \qquad (X,\omega) \longmapsto X \bullet \omega$$

of the Lie algebra $\mathfrak{so}(T_pM)$ of $\mathbf{SO}\,(T_pM)$ on the exterior algebra induced by its representation on T_pM. The canonical identification of $\mathfrak{so}(T_pM)$ with the bivectors $\Lambda^2 T_pM$ characterized by $\langle (X\wedge Y)\bullet A, B\rangle := \langle X\wedge Y, A\wedge B\rangle$ reads $(X\wedge Y)\bullet A := \langle X, A\rangle Y - \langle Y, A\rangle X$ and defines a unique bivector $R(X\wedge Y)$ via:

$$\langle R(X\wedge Y)\bullet Z, W\rangle \;:=\; \langle R_{X,Y}Z, W\rangle \qquad\qquad R(X\wedge Y) \;=\; \frac{1}{2}\sum_i E_i \wedge R_{X,Y}E_i$$

In the spirit of this identification the representation of $\mathfrak{so}(T_pM)$ on $\Lambda^\bullet T_p^*M$ is given by $(X\wedge Y)\bullet \;=\; Y^\sharp \wedge X \lrcorner - X^\sharp \wedge Y \lrcorner$. In particular, the classical Weitzenböck formula becomes

$$\Delta \;=\; \nabla^*\nabla + \frac{1}{2}\sum_{ij} (E_i\wedge E_j)\bullet R(E_i\wedge E_j)\bullet$$

because both potentially troublesome inhomogeneous terms cancel by the first Bianchi identity leaving us with a curvature term depending linearly on the curvature tensor:

$$R \;:=\; \frac{1}{4}\sum_{ij} (E_i\wedge E_j)\cdot R(E_i\wedge E_j) \qquad \in \quad \mathrm{Sym}^2(\Lambda^2 T_pM)\;.$$

It will be convenient to compose the identification $\Lambda^2 T_pM \xrightarrow{\cong} \mathfrak{so}(T_pM)$ with the quantization map $q : \mathrm{Sym}^2\mathfrak{so}(T_pM) \longrightarrow \mathcal{U}\,\mathfrak{so}(T_pM)$, $X^2 \longmapsto X^2$, into the universal enveloping algebra of $\mathfrak{so}(T_pM)$ to get an element $q(R) \in \mathcal{U}\,\mathfrak{so}(T_pM)$ with:

$$\Delta \quad = \quad \nabla^*\nabla \;+\; 2\,q(R) \tag{2}$$

Writing the well known classical Weitzenböck formula (1) this way we can bring the holonomy group of the underlying manifold into play. Recall that the holonomy group $\mathrm{Hol}_pM \subset \mathbf{O}\,(T_pM)$ is the closure of the group of all parallel transports along piecewise smooth loops in $p \in M$. We will assume that M is connected so that the holonomy groups in different points p and $\tilde{p}$ are conjugated by parallel transport $T_pM \longrightarrow T_{\tilde{p}}M$. Choosing a suitable representative $\mathrm{Hol} \subset \mathbf{O}_n\mathbb{R}$ with $n := \dim M$ of their common conjugacy class acting on the abstract vector space $\mathbb{R}^n$ we can define the holonomy bundle of M:

$$\mathrm{Hol}\,(M) \quad := \quad \{\, f : \mathbb{R}^n \longrightarrow T_pM \mid\ p \in M \text{ and } f \text{ isometry with } f(\mathrm{Hol}\,) \;=\; \mathrm{Hol}_pM \,\}\ .$$

The holonomy bundle is a reduction of the orthonormal frame bundle $\mathbf{O}\,(M)$ to a principal bundle with structure group Hol, which is stable under parallel transport. Consequently the Levi–Civita connection is tangent to $\mathrm{Hol}\,(M)$ and descends to a connection on $\mathrm{Hol}\,(M)$.

With the Levi–Civita connection being tangent to the holonomy bundle $\mathrm{Hol}\,(M)$ its curvature tensor R takes values in the holonomy algebra $\mathfrak{hol}_pM$ at every point $p \in M$, so that $R \in \mathrm{Sym}^2\mathfrak{hol}_pM \subset \mathrm{Sym}^2\Lambda^2T_pM$ and $q(R) \in \mathcal{U}\,\mathfrak{hol}_pM$. However by definition every point $f \in \mathrm{Hol}\,(M)$ identifies $\mathfrak{hol}_pM$ with $\mathfrak{hol}$ making $q(R)$ a $\mathcal{U}\,\mathfrak{hol}$–valued function on $\mathrm{Hol}\,(M)$. For an arbitrary irreducible complex representation π of Hol the associated vector bundle $\pi(M) := \mathrm{Hol}\,(M) \times_{\mathrm{Hol}} \pi$ over M is endowed with the connection induced from the Levi–Civita connection. Moreover there is a canonical second order differential operator defined on sections of $\pi(M)$:

$$\Delta_\pi \quad := \quad \nabla^*\nabla \;+\; 2\,q(R) \tag{3}$$

It is evident from the Weitzenböck formula (1) written as in (2) that the diagram

$$\begin{array}{ccc} \pi(M) & \xrightarrow{\;\Delta_\pi\;} & \pi(M) \\ F\big\downarrow & & \big\downarrow F \\ \Lambda^\bullet T^*M \otimes_{\mathbb{R}} \mathbb{C} & \xrightarrow{\;\Delta\;} & \Lambda^\bullet T^*M \otimes_{\mathbb{R}} \mathbb{C} \end{array}$$

commutes for any $F \in \mathrm{Hom}_{\mathrm{Hol}}(\pi, \Lambda^\bullet\mathbb{C}^{n*})$ or equivalently for any globally parallel embedding $F : \pi(M) \longrightarrow \Lambda^\bullet T^*M \otimes_{\mathbb{R}} \mathbb{C}$. Hence the pointwise decomposition of $\Lambda^\bullet T^*_pM \otimes_{\mathbb{R}} \mathbb{C}$ into irreducible complex representations of Hol_pM becomes a global decomposition of any eigenspace of Δ, e. g. we have for its kernel:

$$H^\bullet_{dR}(M,\,\mathbb{C}) \;=\; \bigoplus_\pi \mathrm{Hom}_{\mathrm{Hol}}(\,\pi,\,\Lambda^\bullet\mathbb{C}^{n*}\,) \otimes \ker\,\Delta_\pi$$

The same kind of reasoning is possible for the Dirac operator on spinors, assuming the manifold M to be spin and taking $\mathrm{Hol}_p M$ to be its spin holonomy group. Ignoring for the moment the Lichnerowicz result that the curvature term reduces to multiplication by the scalar curvature and employing the formula $(X \wedge Y) \bullet := \frac{1}{2}(X \star Y \star + \langle X, Y \rangle)$ for the representation of $\mathfrak{so}(T_p M)$ on the spinor bundle $\mathbf{S}(M)$ we can proceed from (1) directly to:

$$D^2 \quad = \quad \nabla^* \nabla \; + \; 4\, q(R)\,. \tag{4}$$

In particular, all eigenspaces of D^2 decompose globally according to the pointwise decomposition of the spinor bundle under the spin holonomy group $\mathrm{Hol}_p M$. From Lichnerowicz's result we already know that $q(R)$ acts by scalar multiplication with $\frac{\kappa}{16}$ on $\mathbf{S}(M)$, where κ is the scalar curvature of M. Hence we can read equation (4) as

$$D^2 \Big|_\pi \quad = \quad \Delta_\pi \; + \; \frac{\kappa}{8}$$

where the restriction to π is a short hand notation for any globally parallel embedding $F : \; \pi(M) \longrightarrow \mathbf{S}(M)$ induced by some non–trivial $F \in \mathrm{Hom}_{\,\mathrm{Hol}}(\pi, \mathbf{S})$. Written in this way formula (4) is seen to be a generalization of the Parthasarathy formula for the Dirac square D^2 on a symmetric space G/K, because in this case the operators Δ_π defined above on sections of $\pi(M)$ all become the Casimir of G.

Counterexamples to the idea that eigenspaces of intrinsically defined differential operators always decompose globally according to the pointwise decomposition under the holonomy group are easily found among twisted Dirac operators. Consider therefore a geometric vector bundle $\mathcal{R}(M) := \mathrm{Hol}\,(M) \times_{\mathrm{Hol}} \mathcal{R}$ associated to the holonomy bundle via some not necessarily irreducible representation $\mathcal{R}$ of the holonomy group. The Levi–Civita connection on $\mathrm{Hol}\,(M)$ defines a connection on this vector bundle, whose curvature endomorphism is given through the representation $\bullet : \; \mathfrak{hol}_p M \times \mathcal{R}_p(M) \longrightarrow \mathcal{R}_p(M)$ of the Lie algebra $\mathfrak{hol}_p M$ on $\mathcal{R}_p(M)$. The twisted Dirac operator $D_\mathcal{R}$ is a first order differential operator acting on sections of the tensor product $(\mathbf{S} \otimes \mathcal{R})(M)$. It satisfies a twisted Weitzenböck formula derived from (1):

$$D^2_\mathcal{R} \; = \; \nabla^*\nabla \; + \; \frac{1}{2} \sum_{ij} \Big(E_i \star E_j \star R(E_i \wedge E_j) \bullet \otimes \mathrm{id}_\mathcal{R} \; + \; E_i \star E_j \star \otimes R(E_i \wedge E_j) \bullet \Big) \tag{5}$$

Trying to balance the apparent asymmetry in (5) between the spinor bundle and the twist we may rewrite the action of $q(R)$ on the tensor product $\mathbf{S} \otimes \mathcal{R}$ in the following asymmetric way in order to cast equation (5) into a form similar to (4):

$$\begin{aligned} q(R) \; = \; & \frac{1}{2} \sum_{ij} \Big(\, (E_i \wedge E_j) \bullet R(E_i \wedge E_j) \bullet \otimes \mathrm{id}_\mathcal{R} \; + \; (E_i \wedge E_j) \bullet \otimes R(E_i \wedge E_j) \bullet \, \Big) \\ & \qquad - \; q(R) \otimes \mathrm{id}_\mathcal{R} \; + \; \mathrm{id}_\mathbf{S} \otimes q(R)\,. \end{aligned}$$

With Lichnerowicz's result $q(R) = \frac{\kappa}{16}$ for the spinor representation $\mathbf{S}$ equation (5) becomes:

$$D^2_\mathcal{R} \quad = \quad \Delta_{\mathbf{S} \otimes \mathcal{R}} + \frac{\kappa}{8} \otimes \mathrm{id}_\mathcal{R} \; - \; \mathrm{id}_\mathbf{S} \otimes 2\, q(R)\,. \tag{6}$$

In conclusion, the squares $D^2_{\mathcal{R}}$ of twisted Dirac operators will in general not respect the decomposition of $(\mathbf{S} \otimes \mathcal{R})(M)$ into parallel subbundles because of the critical summand $\mathrm{id}_{\mathbf{S}} \otimes 2\,q(R)$. Nevertheless, if $q(R)$ acts by scalar multiplication not only on $\mathbf{S}$ but on $\mathcal{R}$, too, the global decomposition of the eigenspaces of $D^2_{\mathcal{R}}$ according to the pointwise decomposition of $\mathbf{S} \otimes \mathcal{R}$ is restored.

Equation (6) is the key relation of this article and forms the cornerstone and motivation of all statements and calculations to come. In fact, we can take advantage of equation (6) even if the manifold in question is not spin, because the twisted Dirac operator may be well defined on the vector bundle $(\mathbf{S} \otimes \mathcal{R})(M)$ although M is neither spin nor $\mathbf{S}(M)$ or $\mathcal{R}(M)$ are well defined vector bundles.

3 Quaternionic Kähler holonomy

In this section we introduce the main notions of quaternionic Kähler holonomy based on the group $\mathrm{Hol} = \mathbf{Sp}(1) \cdot \mathbf{Sp}(n)$ with $n \geq 2$. Very few examples of compact manifolds with this particular holonomy group are known, and it is a deep result that in every quaternionic dimension n there are up to isometry only finitely many of these manifolds with positive scalar curvature $\kappa > 0$ ([LeBSa94]). In fact, the only known examples with $\kappa > 0$ are symmetric spaces, the so-called Wolf spaces.

Consider an abstract complex vector space $E \cong \mathbb{C}^{2n}$, $n \geq 2$ endowed with a symplectic form $\sigma \in \Lambda^2 E^*$ and an adapted, positive quaternionic structure J, i. e. a conjugate linear map $J : E \longrightarrow E$ satisfying $J^2 = -1$, $\sigma(Je_1, Je_2) = \overline{\sigma(e_1, e_2)}$ and $\sigma(e, Je) > 0$ for all $e_1, e_2 \in E$ and $e \neq 0$. The symplectic form σ induces mutually inverse isomorphisms $\sharp : E \longrightarrow E^*$, $e \longmapsto \sigma(e, \cdot)$ and $\flat : E^* \longrightarrow E$. Similar to the representation of $\Lambda^2 T_pM$ on T_pM considered in the first section there is an action

$$\bullet : \ \mathrm{Sym}^2 E \times E \longrightarrow E, \qquad (e_1e_2, e) \longmapsto (e_1e_2) \bullet e := \sigma(e_1, e)e_2 + \sigma(e_2, e)e_1$$

of the second symmetric power $\mathrm{Sym}^2 E$ on E. This action is skew symplectic and commutes with J for all real elements of $\mathrm{Sym}^2 E$ thus identifying the real subspace with the Lie algebra $\mathfrak{sp}(n)$ of $\mathbf{Sp}(n)$.

Let $H \cong \mathbb{C}^2$ be another abstract vector space with the same structures: a symplectic form $\sigma \in \Lambda^2 H^*$ and an adapted, positive quaternionic structure J. The tensor product $H \otimes E$ of these two vector spaces carries a real structure $J \otimes J$ and a complex bilinear symmetric form $\langle \, , \, \rangle := \sigma \otimes \sigma$, which is positive definite on the real subspace. The subgroup of $O(H \otimes E)$ preserving the real subspace and the tensor product decomposition is isomorphic to the group $\mathbf{Sp}(1) \cdot \mathbf{Sp}(n) := \mathbf{Sp}(1) \times \mathbf{Sp}(n)/\mathbb{Z}_2$ and the conjugacy class of this subgroup defines quaternionic Kähler geometry.

In particular a quaternionic Kähler manifold is a Riemannian manifold of dimension $4n$, $n \geq 2$ with a parallel isomorphism of the complexified tangent bundle $TM \otimes_{\mathbb{R}} \mathbb{C}$ with the tensor product $H \otimes E$ of two locally defined symplectic bundles H and E. These bundles are only locally defined because the two representations H and E do not factor through the projection $\mathbf{Sp}(1) \times \mathbf{Sp}(n) \longrightarrow \mathbf{Sp}(1) \cdot \mathbf{Sp}(n)$. In passing from representation theory to geometry we always have to check, whether the representations factor through the projection $\mathbf{Sp}(1)\times\mathbf{Sp}(n) \longrightarrow \mathbf{Sp}(1)\cdot\mathbf{Sp}(n)$. Things get actually simpler in some respect, as the spinor representation $\mathbf{S}$ of $\mathbf{Sp}(1)\times\mathbf{Sp}(n)$ factors through to a representation of $\mathbf{Sp}(1)\cdot\mathbf{Sp}(n)$ whenever n is even. Thus all quaternionic Kähler manifolds of even quaternionic dimension n are spin:

Proposition 3.1. *(The signed spinor representation ([BaS83], [Wan89]))*
The spinor representation $\mathbf{S}$ *of* $\mathbf{Sp}(1)\times\mathbf{Sp}(n)$ *decomposes into the direct sum*

$$\mathbf{S} \quad=\quad \bigoplus_{r=0}^{n}\mathbf{S}_r \quad:=\quad \bigoplus_{r=0}^{n}\mathrm{Sym}^{r}H\otimes\Lambda_{\circ}^{n-r}E \tag{7}$$

where $\Lambda_{\circ}^{n-r}E$ *is the kernel of the contraction* $\sigma:\ \Lambda^{n-r}E \longrightarrow \Lambda^{n-r-2}E$ *with the symplectic form. For the canonical quaternionic orientation of* $H\otimes E$*, induced by the Kraines form, the half spin representations are given by:*

$$\mathbf{S}^{+} \quad:=\quad \bigoplus_{r\equiv n\,(2)}\mathbf{S}_r \qquad\qquad \mathbf{S}^{-} \quad:=\quad \bigoplus_{r\not\equiv n\,(2)}\mathbf{S}_r\,.$$

The delicate point in a constructive proof of this proposition is the choice of Clifford multiplication $\star:\ (H\otimes E)\times\mathbf{S} \longrightarrow \mathbf{S}$. Besides the Clifford identity there is another crucial property of this multiplication, namely the compatibility condition with the action of the Lie algebra $\mathfrak{sp}(1)\oplus\mathfrak{sp}(n)$ on $\mathbf{S}$. The representation $\bullet$ of the complexified Lie algebra $\mathrm{Sym}^2H\oplus\mathrm{Sym}^2E$ of the group $\mathbf{Sp}(1)\times\mathbf{Sp}(n)$ on $\mathbf{S}$ has to agree with the representation implicitly defined by Clifford multiplication via $(X\wedge Y)\bullet := \frac{1}{2}(X\star Y\star+\langle X,Y\rangle)$. Choosing dual pairs of bases $\{de_\mu\}$, $\{e_\nu\}$ for E^*, E with $\langle de_\mu,\, e_\nu\rangle = \delta_{\mu\nu}$ and $\{dh_\alpha\}$, $\{h_\beta\}$ for H^*, H we can check that

$$(e\,\tilde e) \;\longmapsto\; \sum_\alpha (dh_\alpha^\flat\otimes e)\wedge(h_\alpha\otimes\tilde e) \qquad\qquad (h\,\tilde h) \;\longmapsto\; \sum_\mu (h\otimes de_\mu^\flat)\wedge(\tilde h\otimes e_\mu) \tag{8}$$

is a Lie algebra homomorphism $\mathrm{Sym}^2H\oplus\mathrm{Sym}^2E \longrightarrow \Lambda^2(TM\otimes_{\mathbb{R}}\mathbb{C})$ intertwining the given representations of Sym^2H, Sym^2E and $\Lambda^2(TM\otimes_{\mathbb{R}}\mathbb{C})$ on $H\otimes E = TM\otimes_{\mathbb{R}}\mathbb{C}$. Consequently the following two operator identities on the spinor representation $\mathbf{S}$ are at the heart of Proposition 3.1:

$$(e\,\tilde e)\bullet \;=\; \frac{1}{2}\sum_\alpha\Big((dh_\alpha^\flat\otimes e)\star(h_\alpha\otimes\tilde e)\star+\sigma(e,\,\tilde e)\Big) \tag{9}$$

$$(h\,\tilde h)\bullet \;=\; \frac{1}{2}\sum_\mu\Big((h\otimes de_\mu^\flat)\star(\tilde h\otimes e_\mu)\star+\sigma(h,\,\tilde h)\Big) \tag{10}$$

The most important point in our present discussion of quaternionic Kähler holonomy is of course the discussion of the curvature tensor of a quaternionic Kähler manifold and of the associated element $q(R)$ in the universal enveloping algebra of the Lie algebra $\mathfrak{sp}(1)\oplus\mathfrak{sp}(n)$ of the holonomy group $\mathbf{Sp}\,(1)\cdot\mathbf{Sp}\,(n)$. In fact compared to other holonomy groups quaternionic Kähler holonomy is rather rigid. This is mainly due to the fact that the curvature tensor of a quaternionic Kähler manifold has to satisfy very stringent constraints and can be described completely by the scalar curvature κ and a section $\mathfrak{R}$ of $\mathrm{Sym}^4 E$. This decomposition was first derived by D. V. Alekseevskii (cf.: [Al68] or[Sal82]) and can be made explicit in the following way (cf.: [KSW97a]):

Lemma 3.2. *(The curvature tensor)*
A quaternionic Kähler manifold M is Einstein with constant scalar curvature κ. Its curvature tensor depends only on κ and a section $\mathfrak{R}$ of $\mathrm{Sym}^4 E$, this dependence reads

$$R \quad = \quad -\frac{\kappa}{8n(n+2)}(R^H + R^E) + R^{hyper} \tag{11}$$

where the endomorphism valued two forms R^H, R^E and R^{hyper} are defined by:

$$\begin{aligned} R^H_{h_1\otimes e_1, h_2\otimes e_2} &= \sigma_E(e_1,e_2)(h_1 h_2\bullet\otimes \mathrm{id}_E) \\ R^E_{h_1\otimes e_1, h_2\otimes e_2} &= \sigma_H(h_1,h_2)(\mathrm{id}_H\otimes e_1 e_2\bullet) \\ R^{hyper}_{h_1\otimes e_1, h_2\otimes e_2} &= \sigma_H(h_1,h_2)(\mathrm{id}_H\otimes (e_2^\sharp \lrcorner\, e_1^\sharp \lrcorner\, \mathfrak{R})\bullet) \end{aligned} \tag{12}$$

At the end of this section we want to describe the action of the element $q(R)$ of the universal enveloping algebra $\mathcal{U}(\,\mathfrak{sp}(1)\oplus\mathfrak{sp}(n)\,)$ on some representations. In particular we will see that for a large class of representations of $\mathbf{Sp}\,(1)\times\mathbf{Sp}\,(n)$ the element $q(R)$ acts by scalar multiplication, because the contributions from the hyperkähler part R^{hyper} of the curvature tensor drop out. Observe first that $q(R)$ depends linearly on R and so we may write:

$$q(R) \quad = \quad -\frac{\kappa}{8n(n+2)}\,\big(\,q(R^H)\,+\,q(R^E\,)\big)\;+\;q(R^{hyper})$$

Using equation (8) we can write down the terms appearing in this sum more explicitly:

Lemma 3.3.

$$\begin{aligned} q(R^H) &= \frac{1}{4}\sum_{\alpha\beta}(dh^\flat_\alpha\, dh^\flat_\beta)\bullet(h_\alpha\, h_\beta)\bullet \\ q(R^E) &= \frac{1}{4}\sum_{\mu\nu}(de^\flat_\mu\, de^\flat_\nu)\bullet(e_\mu\, e_\nu)\bullet \\ q(R^{hyper}) &= \frac{1}{4}\sum_{\mu\nu}(de^\flat_\mu\, de^\flat_\nu)\bullet(e^\sharp_\mu\lrcorner\, e^\sharp_\nu\lrcorner\,\mathfrak{R})\bullet \end{aligned}$$

Proof: Converting the sum over a local orthonormal base $\{E_i\}$ into the sum

$$\sum_i E_i\otimes E_i \quad = \quad \sum_{\alpha\mu}(dh^\flat_\alpha\otimes de^\flat_\mu)\otimes(h_\alpha\otimes e_\mu)$$

over dual pairs $\{de_\mu\}$, $\{e_\mu\}$ and $\{dh_\alpha\}$, $\{h_\alpha\}$ of bases we calculate say for $q(R^{hyper})$

$$\begin{aligned}\frac{1}{4}\sum_{ij}(E_i\wedge E_j)\bullet R^{hyper}_{E_i,E_j} &= \frac{1}{4}\sum_{\alpha\beta\mu\nu}(dh^\flat_\alpha\otimes de^\flat_\mu\wedge dh^\flat_\beta\otimes de^\flat_\nu)\bullet\sigma(h_\alpha,h_\beta)(e^\sharp_\mu\lrcorner\, e^\sharp_\nu\lrcorner\,\mathfrak{R})\bullet\\ &= \frac{1}{4}\sum_{\alpha\mu\nu}(dh^\flat_\alpha\otimes de^\flat_\mu\wedge h_\alpha\otimes de^\flat_\nu)\bullet(e^\sharp_\mu\lrcorner\, e^\sharp_\nu\lrcorner\,\mathfrak{R})\bullet\end{aligned}$$

which is equivalent to the stated equality in view of equation (8). □

Evidently $2q(R^H)$ and $2q(R^E)$ respectively are the Casimir operators for $\mathfrak{sp}(1)$ and $\mathfrak{sp}(n)$ in σ–normalization, i. e. the defining invariant symmetric form on the Lie algebra Sym^2H or Sym^2E is not the Killing form itself but the natural extension of σ to the second symmetric powers using Gram's permanent. Hence the Casimir eigenvalues of $q(R^H)$ and $q(R^E)$ are easily calculated directly for the simplest representations of $\mathbf{Sp}(n)$:

Lemma 3.4. *(Casimir eigenvalues)*
For the irreducible representations Sym^lE *and* $\Lambda^d_\circ E$ *the Casimir eigenvalues for* $q(R^E)$ *are:*

$$q(R^E)_{\mathrm{Sym}^lE} = -l\,(n+\frac{l}{2})\qquad\qquad q(R^E)_{\Lambda^d_\circ E} = -d\,(n-\frac{d}{2}+1)$$

The eigenvalues of $q(R^H)$ are given by the same formulas with $n=1$. Setting $l=2$ we get the Casimir eigenvalues for $q(R^E)$ and $q(R^H)$ in the adjoint representations Sym^2E and Sym^2H of $\mathfrak{sp}(n)$ and $\mathfrak{sp}(1)$. Since by definition the Casimir eigenvalue of the adjoint representation is always one for Casimirs in the Killing normalization we see in particular:

$$q(R^E) = -2\,(n+1)\,\mathrm{Cas}_{\mathfrak{sp}(n)}\qquad\qquad q(R^H) = -4\,\mathrm{Cas}_{\mathfrak{sp}(1)}$$

Now we claim that the hyperkähler contribution $q(R^{hyper})$ to the element $q(R)$ acts trivially on every irreducible representation occurring in the representation ΛE, i. e. on all representations $\Lambda^d_\circ E$ with $d=0,\ldots,n$. Because $q(R^{hyper})$ depends linearly on $\mathfrak{R}\in\mathrm{Sym}^4E$ we are allowed to expand $\mathfrak{R}$ into a sum of fourth powers $\frac{1}{24}e^4$, $e\in E$, to calculate $q(R^{hyper})$. It is thus sufficient to prove that the action of $q(\frac{1}{24}e^4)$ on ΛE is trivial for all $e\in E$. According to Lemma 3.3 the element $q(\frac{1}{24}e^4)$ acts on ΛE as:

$$q(\frac{1}{24}e^4) = \frac{1}{2}(\frac{1}{2}e^2)\bullet(\frac{1}{2}e^2)\bullet = \frac{1}{2}(e\wedge e^\sharp\lrcorner)(e\wedge e^\sharp\lrcorner) = -\frac{1}{2}e\wedge e\wedge e^\sharp\lrcorner\, e^\sharp\lrcorner = 0\,.$$

Consequently the curvature tensor $q(R)$ will act by scalar multiplication on all representations $\mathcal{R}^{l,d} := \mathrm{Sym}^lH\otimes\Lambda^d_\circ E$. From equation (6) we conclude that the squares $D^2_{\mathcal{R}^{l,d}}$ of the twisted Dirac operators with these particular twists have properties similar to the Hodge–Laplacian Δ and the square D^2 of the untwisted Dirac operator:

Proposition 3.5. *(Global decomposition principle)*
The restriction $D^2_{\mathcal{R}^{l,d}}\big|_\pi$ *of the square of a twisted Dirac operator* $D^2_{\mathcal{R}^{l,d}}$ *with twisting bundle* $\mathcal{R}^{l,d} := (\mathrm{Sym}^lH\otimes\Lambda^d_\circ E)(M)$ *to a parallel subbundle* $\pi(M)\subset(\mathbf{S}\otimes\mathcal{R}^{l,d})(M)$ *does not depend on the specific embedding of this subbundle and equation (6) becomes in this case:*

$$\Delta_\pi = D^2_{\mathcal{R}^{l,d}}\Big|_\pi + \frac{\kappa}{8n(n+2)}(l+d-n)(l-d+n+2)$$

4 Classification of minimal and maximal twists

In this section we will focus attention on the technicalities necessary to draw conclusions from Proposition 3.5. The irreducible representations occurring in the twisted spinor representations $\mathbf{S}\otimes\mathcal{R}^{l,d}$ are all of the form $\mathrm{Sym}^k H\otimes\Lambda^{a,b}_{\mathrm{top}}E$, where $\Lambda^{a,b}_{\mathrm{top}}E$ is the irreducible representation in the tensor product $\Lambda^a_\circ E\otimes\Lambda^b_\circ E$ corresponding to the sum of highest weights. Alternatively we see from Weyl's construction of the irreducible representations of the classical matrix groups that $\Lambda^{a,b}_{\mathrm{top}}E$ is the common kernel of the diagonal contraction with the symplectic form $\sigma:\ \Lambda^a_\circ E\otimes\Lambda^b_\circ E\longrightarrow\Lambda^{a-1}_\circ E\otimes\Lambda^{b-1}_\circ E$ and the Plücker differential:

$$\sum_\mu e_\mu\wedge\otimes de_\mu\lrcorner:\quad \Lambda^a_\circ E\otimes\Lambda^b_\circ E\longrightarrow\Lambda^{a+1}E\otimes\Lambda^{b-1}_\circ E$$

In particular, we will characterize the twists $\mathcal{R}^{l,d}$ with $\mathrm{Sym}^k H\otimes\Lambda^{a,b}_{\mathrm{top}}E\subset\mathbf{S}\otimes\mathcal{R}^{l,d}$. Moreover, for each representation $\mathrm{Sym}^k H\otimes\Lambda^{a,b}_{\mathrm{top}}E$ in this class and will classify the special twists maximizing the curvature expression

$$-\frac{\kappa}{8n(n+2)}\,(\,l+d-n\,)(\,l-d+n+2\,)$$

of Proposition 3.5 for $\kappa>0$ and $\kappa<0$. This classification is the most important step used in the applications of the ideas encoded in Proposition 3.5. Global questions are postponed to the next sections. Hence, we will deal with representations of $\mathbf{Sp}\,(1)\times\mathbf{Sp}\,(n)$ only.

Theorem 4.1. *(Characterization of admissible twists)*
A representation $\mathcal{R}^{l,d}:=\mathrm{Sym}^l H\otimes\Lambda^d_\circ E$ *with* $l\geq 0$ *and* $n\geq d\geq 0$ *is called an admissible twist for the irreducible representation* $\mathrm{Sym}^k H\otimes\Lambda^{a,b}_{\mathrm{top}}E$*, if there exists a non–trivial, equivariant homomorphism from* $\mathrm{Sym}^k H\otimes\Lambda^{a,b}_{\mathrm{top}}E$ *to the twisted spinor representation* $\mathbf{S}\otimes\mathcal{R}^{l,d}$*, i. e.*

$$\mathrm{Hom}_{\mathbf{Sp}\,(1)\times\mathbf{Sp}\,(n)}(\,\mathrm{Sym}^k H\otimes\Lambda^{a,b}_{\mathrm{top}}E,\ \mathbf{S}\otimes\mathcal{R}^{l,d}\,)\quad\neq\quad\{0\}\,.$$

A twist $\mathcal{R}^{l,d}$ *is admissible in this sense if and only if* $k+a+b\equiv n+l+d\ \mathit{mod}\,2$ *and:*

$$b\ \leq\ d \tag{13}$$
$$|\,k-l\,|+|\,a-d\,|\ \leq\ n-b \tag{14}$$
$$|\,n-a+b-d\,|\ \leq\ k+l\,. \tag{15}$$

A simple consequence of Theorem 4.1 is that all the representations $\mathrm{Sym}^k H\otimes\Lambda^{a,b}_{\mathrm{top}}E$ occur in twisted spinor representations, e. g. in $\mathbf{S}\otimes\mathcal{R}^{k+n-b,\,a}$ and $\mathbf{S}\otimes\mathcal{R}^{|n-a-k|,\,b}$. These two twists are the prototype examples of maximal and minimal twists to be defined below.

Proof: For the proof we recall a well–known fusion rule for the tensor product $\Lambda^c_\circ E\otimes\Lambda^d_\circ E$ of the two irreducible $\mathbf{Sp}\,(n)$–representations $\Lambda^c_\circ E$ and $\Lambda^d_\circ E$ (cf. [OnVi90]):

$$\Lambda^c_\circ E\otimes\Lambda^d_\circ E\quad=\bigoplus_{\substack{a+b\,\equiv\,c+d\ \mathrm{mod}\,2\\ a+b\,\leq\,c+d\\ |c-d|\,\leq\,a-b\,\leq\,2n-c-d}}\Lambda^{a,b}_{\mathrm{top}}E$$

Note in particular that each irreducible representation $\Lambda^{a,b}_{\rm top}E$ occurs at most once in the tensor product $\Lambda^{c}_{\circ}E\otimes\Lambda^{d}_{\circ}E$. Using this fusion rule together with the Clebsch–Gordan formula for irreducible $\mathbf{Sp}\,(1)$–representations and the decomposition of the spinor representation $\mathbf{S}$ under $\mathbf{Sp}\,(1)\times\mathbf{Sp}\,(n)$ given in Proposition 3.1 we can formally write down the decomposition

$$(16)\quad \bigoplus_{c=0}^{n}(\mathrm{Sym}^{\,n-c}H\otimes\Lambda^{c}_{\circ}E)\otimes(\mathrm{Sym}^{\,l}H\otimes\Lambda^{d}_{\circ}E) \;=\; \sum_{\substack{k\geq 0\\ n\geq a\geq b\geq 0}}\sharp\,\mathfrak{M}_{k,\,a,\,b}(l,\,d)\cdot\mathrm{Sym}^{\,k}H\otimes\Lambda^{a,b}_{\rm top}E$$

of $\mathbf{S}\otimes\mathcal{R}^{l,\,d}$, where $\mathfrak{M}_{k,\,a,\,b}(l,\,d)$ is the set of all $n\geq c\geq 0$ satisfying the set of constraints:

$$(17)\qquad \begin{array}{rcl} k&\equiv& n+c+l\mod 2\\ k&\leq& n-c+l\\ k&\geq&|n-c-l|\end{array}\qquad \begin{array}{rcl} a+b&\equiv& c+d\mod 2\\ a+b&\leq& c+d\\ a-b&\geq&|c-d|\\ a-b&\leq&2n-c-d\end{array}$$

It is clear from these constraints that $\mathfrak{M}_{k,\,a,\,b}(l,\,d)$ is empty unless $k+a+b\;\equiv\;n+l+d\mod 2$ reflecting in a way the consistency of the action of $(-1,\,-1)\;\in\;\mathbf{Sp}\,(1)\times\mathbf{Sp}\,(n)$. In particular, $k+a+b\;\equiv\;n+l+d\mod 2$ is a necessary condition for the twist $\mathcal{R}^{l,\,d}$ to be admissible.

In view of this congruence we can drop one of the two constraints $a+b\;\equiv\;c+d\mod 2$ or $k\;\equiv\;n+c+l\mod 2$ and solve the inequalities (17) for c. After a little manipulation we arrive at an equivalent description of $\mathfrak{M}_{k,\,a,\,b}(l,\,d)$ as the set of all $c\;\equiv\;a+b+d\mod 2$ satisfying:

$$(18)\qquad \max\{\,b+|a-d|,n-k-l\,\}\;\leq\;c\;\leq\;n\;-\;\max\{\,|k-l|,|n-a+b-d|\}$$

Under the standing hypothesis $k+a+b\;\equiv\;n+l+d\mod 2$ we evidently have

$$\max\{\,b+|a-d|,n-k-l\,\}\;\equiv\;a\;+\;b\;+\;d\;\equiv\;n\;-\;\max\{\,|k-l|,|n-a+b-d|\}\mod 2$$

so that $\mathfrak{M}_{k,\,a,\,b}(l,\,d)$ will be non–empty if and only if the inequality (18) is consistent. Indeed the congruence $c\;\equiv\;a+b+d\mod 2$ will be fulfilled by either end of the resulting interval. However, the consistency condition for (18) is given by four inequalities in $l,\,d$ depending of course on $k,\,a,\,b$. The first $n-k-l\;\leq\;n-|k-l|$ is trivial for $k,\,l\geq 0$ and the next two become inequalities (14) and (15), whereas the last $b+|a-d|\;\leq\;n-|n-a+b-d|$ is equivalent to inequality (13) for all $b\leq a\leq n$ and $d\leq n$. □

Note that if the set $\mathfrak{M}_{k,\,a,\,b}(l,\,d)$ is non-empty all its elements will have the same parity as $a+b+d$. Of course their number $\sharp\,\mathfrak{M}_{k,\,a,\,b}(l,\,d)$ is just the multiplicity of the representation $\mathrm{Sym}^{\,k}H\otimes\Lambda^{a,b}_{\rm top}E$ in $\mathbf{S}\otimes\mathcal{R}^{l,\,d}$, which we will need below as index multiplicity:

Definition 4.2. *(The index of an admissible twist)*
The index of an admissible twist $\mathcal{R}^{l,\,d}$ for an irreducible representation $\mathrm{Sym}^{\,k}H\otimes\Lambda^{a,b}_{\rm top}E$ is the index multiplicity of $\mathrm{Sym}^{\,k}H\otimes\Lambda^{a,b}_{\rm top}E$ in the twisted spinor representation $\mathbf{S}^{\pm}\otimes\mathcal{R}^{l,\,d}$:

$$\begin{array}{rcl}\mathrm{index}\;(k,a,b;\,l,d)&:=&\dim\;\mathrm{Hom}_{\mathbf{Sp}\,(1)\times\mathbf{Sp}\,(n)}(\mathrm{Sym}^{\,k}H\otimes\Lambda^{a,b}_{\rm top}E,\;\mathbf{S}^{+}\otimes\mathcal{R}^{l,\,d})\\ &&-\quad\dim\;\mathrm{Hom}_{\mathbf{Sp}\,(1)\times\mathbf{Sp}\,(n)}(\mathrm{Sym}^{\,k}H\otimes\Lambda^{a,b}_{\rm top}E,\;\mathbf{S}^{-}\otimes\mathcal{R}^{l,\,d})\end{array}$$

From the proof of Theorem 4.1 we can easily read off an explicit formula for this index:

$$\text{index}\,(k,a,b;\,l,d)\;:=\;\frac{(-1)^{a+b+d}}{2}\Big(n\,+\,2\,-\,\max\{|k-l|,|n-a+b-d|\}\,-\,\max\{b+|a-d|,n-k-l\}\Big)$$

Although we have calculated the index multiplicity of the representation $\mathrm{Sym}^{\,k}H\otimes\Lambda^{a,\,b}_{\mathrm{top}}E$ for an arbitrary twisted spinor representation $\mathbf{S}\otimes\mathcal{R}^{l,\,d}$, it will turn out below that only very few representations actually contribute to the index of a particular twisted Dirac operator. These representations are characterized by the following extremality condition:

Definition 4.3. *(Minimal and maximal twists)*
An admissible twist $\mathcal{R}^{l,\,d}:=\mathrm{Sym}^{\,l}H\otimes\Lambda^{\,d}_{\circ}E$ *for the irreducible representation* $\mathrm{Sym}^{\,k}H\otimes\Lambda^{a,b}_{\mathrm{top}}E$ *is called a minimal or maximal twist, if the curvature term of Proposition 3.5, or equivalently the function* $\phi(\tilde{l},\tilde{d})\;:=\;(\tilde{l}+\tilde{d}-n)(\tilde{l}-\tilde{d}+n+2)$*, assumes its minimum or maximum among all admissible twists* $\mathcal{R}^{\tilde{l},\,\tilde{d}}$ *in the twist* $\mathcal{R}^{l,\,d}$*.*

To determine the index of a twisted Dirac operator in terms of the dimension of the eigenspaces of the operators Δ_π, all we will further need is a classification of all minimal twists for negative scalar curvature $\kappa<0$ and similarly of all maximal twists for $\kappa>0$:

Theorem 4.4. *(Classification of maximal twists)*
All representations $\mathrm{Sym}^{\,k}H\otimes\Lambda^{a,b}_{\mathrm{top}}E$ *with* $k>0$ *or* $a>b$ *have unique maximal twists:*

$$\mathcal{R}^{k+n-b,\,a}\;=\;\mathrm{Sym}^{\,k+n-b}H\otimes\Lambda^{\,a}_{\circ}E\qquad\qquad \text{index}\,(k,a,b;\,k+n-b,a)\;=\;(-1)^b$$

For the special representations $\Lambda^{a,a}_{\mathrm{top}}E$ *with* $k=0$ *and* $a=b$ *all admissible twists* $\mathcal{R}^{n-d,d}$ *with* $d=a,\,\ldots,\,n$ *have* $\phi(n-d,d)=0$ *and are thus automatically maximal and minimal:*

$$\mathcal{R}^{n-d,\,d}\;=\;\mathrm{Sym}^{\,n-d}H\otimes\Lambda^{\,d}_{\circ}E\qquad\qquad \text{index}\,(0,a,a;\,n-d,d)\;=\;(-1)^d$$

The classification of all minimal twists splits into more cases:

Theorem 4.5. *(Classification of minimal twists)*
According to their minimal twists the irreducible representations $\mathrm{Sym}^{\,k}H\otimes\Lambda^{a,b}_{\mathrm{top}}E$ *are divided into four classes. In the first class we have* $k>(n-a)+(n-b)$ *and a unique minimal twist:*

$$\mathcal{R}^{k-n+b,\,a}\;=\;\mathrm{Sym}^{\,k-n+b}H\otimes\Lambda^{\,a}_{\circ}E\qquad\qquad \text{index}\,(k,a,b;\,k-n+b,a)\;=\;(-1)^b$$

In the second class with $k=(n-a)+(n-b)$ *the minimal twist is no longer unique. All minimal twists for representations in this class are given by*

$$\mathcal{R}^{n-d,\,d}\;=\;\mathrm{Sym}^{\,n-d}H\otimes\Lambda^{\,d}_{\circ}E\qquad\qquad \text{index}\,(k,a,b;\,n-d,d)\;=\;(-1)^{k+d}$$

with $d=b,\,\ldots,\,a$*. The special representations* $\Lambda^{a,a}_{\mathrm{top}}E$ *with* $k=0$ *and* $a=b$ *form the third class overlapping in* $k=0$ *and* $a=b=n$ *with the second. All admissible twists* $\mathcal{R}^{n-d,\,d}$ *with* $d=a,\,\ldots,\,n$ *for these special representations are minimal and maximal at the same time:*

$$\mathcal{R}^{n-d,\,d}\;=\;\mathrm{Sym}^{\,n-d}H\otimes\Lambda^{\,d}_{\circ}E\qquad\qquad \text{index}\,(0,a,a;\,n-d,d)\;=\;(-1)^d$$

The remaining representations are characterized by $k < (n-a)+(n-b)$ *and* $k+(a-b) > 0$. *The minimal twists of the representations in this fourth class are all unique:*

$$\mathcal{R}^{|n-a-k|,\,b} \quad = \quad \mathrm{Sym}^{|n-a-k|}H \otimes \Lambda^{b}_{\circ}E \qquad\qquad \mathrm{index}\,(k,a,b;\,|n-a-k|,b) \;=\; (-1)^{a}$$

It should not be too difficult for the reader to prove Theorems 4.4 and 4.5, but it is advisable to have a geometric picture in mind in order to help intuition. The set of solutions to the inequality (14) in the $(l,\,d)$–space is a ball in the L^1–norm, i. e. a diamond, with center $(k,\,a)$ and radius $n-b$. On the other hand the set of solutions to the inequality (15) is a cone opening diagonally to the right from its vertex in the point $(-k,\,n-a+b)$. We want to extremize the function $\phi(l,\,d) \;=\; (l+d-n)\,(l-d+n+2)$, whose level sets are hyperbolas with diagonal axes $l+d \;=\; n$ and $l-d \;=\; -n-2$. Eventually we only care for points $l \;\geq\; 0$ and $n \;\geq\; d \;\geq\; 0$ in its first $l+d \;\geq\; n,\; l-d \;\geq\; -n-2$ or second quadrant $l+d \;\leq\; n,\; l-d \;\geq\; -n-2$, where ϕ is positive or negative respectively.
Proof: We will only consider the last case of Theorem 4.5 characterized by $k+(a-b) \;>\; 0$ and $k \;<\; (n-a)+(n-b)$, which anyhow is the most difficult to prove. Observing that these two inequalities together are equivalent to $|n-a-k|+b \;<\; n$ we conclude that the point $(|n-a-k|,\,b)$ will lie in the strict interior of the second quadrant. The twist $\mathcal{R}^{|n-a-k|,\,b}$ corresponding to this point is certainly an admissible twist, because $|n-a| \;\leq\; k+|n-a-k|$ and $||-k|-|n-a-k|| \;\leq\; n-a$ by the contraction property of $x \longmapsto |x|$. Turning to the geometric picture we see that the bottom corner of the intersection rectangle of cone and diamond will be either $(k,\,a-n+b)$ for $k \;\geq\; n-a$ or $(n-a,\,b-k)$ for $k \;\leq\; n-a$, i. e. whatever point has larger l and d–coordinate. In particular this bottom corner fails in general to satisfy inequality (13) chopping off a triangle from the rectangle. The resulting face runs from the point $(|n-a-k|,\,b)$ to $(n-a+k,\,b)$ independent of whether $k \;\geq\; n-a$ or $k \;\leq\; n-a$. Note that the geometry may become even more complicated, but as the point $(|n-a-k|,\,b)$ solves all inequalities (13),(14),(15) and lies in the strict interior of the second quadrant it is the unique point, where the function ϕ assumes its minimum. □

5 Eigenvalue estimates

The potential applications of Proposition 3.5 include eigenvalue estimates for the Laplace and for twisted Dirac operators. The general procedure is described in this section and carried out in some particularly interesting cases. Our first example are the irreducible $\mathbf{Sp}\,(1)\cdot\mathbf{Sp}\,(n)$–representations $\mathrm{Sym}^{r}H \otimes \Lambda^{r}_{\circ}E$ defining parallel subbundles in the bundle of r–forms (cf. [Sal86]). On these subbundles we have the following lower bound for the spectrum of the Laplace operator.

Proposition 5.1. *(Eigenvalue estimate on* $\mathrm{Sym}^{r}H \otimes \Lambda^{r}_{\circ}E$*)*
Let $(M^{4n},\,g)$ *be a compact quaternionic Kähler manifold of positive scalar curvature* $\kappa > 0$. *Then any eigenvalue* λ *of the Laplace operator restricted to* $\mathrm{Sym}^{r}H \otimes \Lambda^{r}_{\circ}E$ *satisfies*

$$\lambda \quad \geq \quad \frac{r(n+1)}{2n(n+2)}\,\kappa\,.$$

Proof: It follows from Theorem 4.4 that $\mathrm{Sym}^{n+r}H \otimes \Lambda^r_\circ E$ is a maximal twist for the representation $\mathrm{Sym}^r H \otimes \Lambda^r_\circ E$. Using Proposition 3.5 with $l = n + r$ and $d = r$ we obtain:

$$\Delta_{\mathrm{Sym}^r H\otimes\Lambda^r_\circ E} \quad = \quad D^2_{\mathcal{R}^{n+r,r}}\Big|_{\mathrm{Sym}^r H\otimes\Lambda^r_\circ E} \; + \; \frac{r(n+1)}{2n(n+2)}\,\kappa \quad \geq \quad \frac{r(n+1)}{2n(n+2)}\,\kappa\,. \quad \square$$

An interesting special case is $H \otimes E = TM \otimes_{\mathbb{R}} \mathbb{C}$ for $r = 1$, leading to an eigenvalue estimate for the Laplace operator on 1–forms. In particular, the first Betti number has to vanish. Since the differential of any eigenfunction of the Laplace operator is an eigenform for the same eigenvalue we also obtain an estimate on functions (cf. [AlMa95] and [LeB95]). Replacing maximal by minimal twists to compensate the sign of the scalar curvature the same argument provides eigenvalue estimates on $\mathrm{Sym}^r H \otimes \Lambda^r_\circ E$ on manifolds with $\kappa < 0$. Again the first Betti number has to vanish reproving the result of [Ho96]. In Theorem 6.6 we will prove a stronger vanishing result for the odd Betti numbers.

Our next aim is to derive properties of twisted Dirac operators. For doing so we make the following crucial observation. If π is any representation with admissible twists $\mathcal{R}^{l,d}$ and $\mathcal{R}^{\tilde{l},\tilde{d}}$ then we can apply Proposition 3.5 twice to obtain

$$D^2_{\mathcal{R}^{l,d}}\Big|_\pi \;=\; D^2_{\mathcal{R}^{\tilde{l},\tilde{d}}}\Big|_\pi \;+\; \frac{\kappa}{8n(n+2)}\,\Big(\phi(\tilde{l},\,\tilde{d}) \;-\; \phi(l,\,d)\Big)\;, \tag{19}$$

with $\phi(l,\,d) = (l + d - n)(l - d + n + 2)$. We first use this observation to give a short proof of the eigenvalue estimate for the untwisted Dirac operator:

Proposition 5.2. *(Eigenvalue estimate for the untwisted Dirac operator [KSW97a])* *Let $(M^{4n},\,g)$ be a compact quaternionic Kähler spin manifold of positive scalar curvature κ. Then any eigenvalue λ of the untwisted Dirac operator satisfies*

$$\lambda^2 \quad \geq \quad \frac{n+3}{n+2}\,\frac{\kappa}{4}\;.$$

Proof: According to Proposition 3.1 the spinor bundle decomposes into the parallel subbundles $\mathbf{S} \;=\; \oplus_{r=0}^n\, \mathbf{S}_r$ with $\mathbf{S}_r \;=\; \mathrm{Sym}^r H \otimes \Lambda^{n-r}_\circ E$. To estimate the square of the Dirac operator on $\mathrm{Sym}^r H\otimes\Lambda^{n-r}_\circ E$ we observe that the unique maximal twist for $\mathrm{Sym}^r H\otimes\Lambda^{n-r}_\circ E$ is $\mathcal{R}^{n+r,n-r}$ and for $l = d = 0$ and $\tilde{l} = n + r$, $\tilde{d} = n - r$ equation (19) reads:

$$D^2\Big|_{\mathbf{S}_r} \quad = \quad D^2_{\mathcal{R}^{n+r,n-r}}\Big|_{\mathbf{S}_r} \;+\; \frac{\kappa}{8n(n+2)}\,\Big(n(2r+n+2) \;+\; n(n+2)\Big) \quad \geq \quad \frac{n+2+r}{n+2}\,\frac{\kappa}{4}\;.$$

Consequently some hypothetical eigenspinor $\phi \in \Gamma(\mathbf{S}\,)$ of D^2 with eigenvalue $\lambda^2 \;<\; \frac{n+3}{n+2}\frac{\kappa}{4}$ would have to be localized in the subbundle $\mathbf{S}_0 \subset \mathbf{S}$. But the Dirac operator on a manifold of positive scalar curvature has trivial kernel so that $D\phi \,\in\, \Gamma(\mathbf{S}_1)$ would be a nontrivial eigenspinor for D^2 again with eigenvalue λ^2 in contradiction to the estimate for $\mathbf{S}_1$. $\square$

We now use equation (19) for describing the kernels of twisted Dirac operators in the case of positive scalar curvature. If π is any representations which contributes to the kernel of

$D^2_{\mathcal{R}^{l,d}}$ then $\mathcal{R}^{l,d}$ has to be a maximal twist for π. In fact equation (19) implies that $D^2_{\mathcal{R}^{l,\,d}}$ is positive on π as soon as there is another admissible twist $\mathcal{R}^{\tilde{l},\,\tilde{d}}$ for π with $\phi(\tilde{l},\tilde{d}) > \phi(l,d)$. From this remark and Proposition 3.5 we conclude in the case of positive scalar curvature

$$\ker(D^2_{\mathcal{R}^{l,d}}) \;=\; \bigoplus_{\pi} \ker\left(\Delta_\pi \;-\; \frac{\kappa}{8n(n+2)}\phi(l,d)\right) \tag{20}$$

where the sum is over all representations π for which $\mathcal{R}^{l,\,d}$ is a maximal twist. Since $\frac{\kappa}{8n(n+2)}\,\phi(l,d)$ is the smallest possible eigenvalue of the operator Δ_π equation (20) is in essence a decomposition of $\ker(D^2_{\mathcal{R}^{l,d}})$ into a sum of minimal eigenspaces for the operators Δ_π.

If $\mathcal{R}^{l,\,d}$ is a maximal twist for a representation π then Theorem 4.4 also provides us with the information whether π occurs in $\mathbf{S}^+ \otimes \mathcal{R}^{l,\,d}$ or in $\mathbf{S}^- \otimes \mathcal{R}^{l,\,d}$. Hence a corollary of equation (20) is a formula for the index of the twisted Dirac operator $D_{\mathcal{R}^{l,\,d}}$ in terms of dimensions of certain minimal eigenspaces. We will describe this in two examples:

Proposition 5.3.
Let $(M^{4n},\, g)$ be a compact quaternionic Kähler manifold of positive scalar curvature $\kappa > 0$, then:

$$\ker\left(D^2_{\mathcal{R}^{l,d}}\right) \;=\; \{0\} \qquad \textit{for} \qquad l \,+\, d \,<\, n\ .$$

Proof: All maximal twists $\mathcal{R}^{l,d}$ satisfy $l + d \geq n$ by Theorem 4.4. □

An immediate consequence of this proposition is the vanishing of the index $\mathrm{index}\,(D_{\mathcal{R}^{l,\,d}})$ for $l + d < n$. This was also proved in [LeBSa94] by using the Akizuki–Nakano vanishing theorem on the twistor space. For the second example we consider the twisted Dirac operator $D_{\mathcal{R}^{n+2,\,0}}$. It easily follows from Theorem 4.4 that $\mathrm{Sym}^{\,2}H$ is the unique representation with maximal twist $\mathcal{R}^{n+2,\,0}$:

Proposition 5.4. *(Killing vector fields)*
On every compact quaternionic Kähler manifold (M^{4n}, g) of positive scalar curvature κ we have:

$$\ker\left(D^2_{\mathcal{R}^{n+2,0}}\right) \;=\; \ker\left(\Delta_{\mathrm{Sym}^{\,2}H} \;-\; \frac{\kappa}{2n}\right)\ .$$

The index of $D_{\mathcal{R}^{n+2,0}}$ equals the dimension of the isometry group of (M,g) (cf. [Sal82]). But since $\mathrm{Sym}^{\,2}H$ is the only representation contributing to $\ker(D^2_{\mathcal{R}^{n+2,0}})$ the index is just the dimension of the minimal eigenspace of $\Delta_{\mathrm{Sym}^{\,2}H}$. In fact, there is an explicit isomorphism from the space of Killing vector fields to $\mathrm{Sym}^{\,2}H$ (cf. [AlMa98]). It is given by projecting the covariant derivative of a Killing vector field onto its component in $\mathrm{Sym}^{\,2}H \subset \Lambda^{\,2}T^*M \otimes_{\mathbb{R}} \mathbb{C}$.

More generally it follows that the index of $D_{\mathcal{R}^{n+r,0}}$, coincides with the dimension of the minimal eigenspace for $\Delta_{\mathrm{Sym}^{\,r}H}$. Combining this insight with appropriate Weitzenböck formulas we will show in a forthcoming paper (c.f. [SW01]) that for even numbers $r \geq 0$ this index is always less or equal the corresponding value on the quaternionic projective space, i. e. we obtain a sharp upper bound for the Hilbert polynomial $P(r) := \mathrm{index}\,(D_{\mathcal{R}^{n+r,0}})$. This in turn provides an upper bound on the degree of the twistor space or in more geometric terms an upper bound for the volume of a quaternionic Kähler manifold of positive scalar curvature.

6 Harmonic forms and Betti numbers

This section contains the most important application of Proposition 3.5. We will determine which parallel subbundles of the differential forms may carry harmonic forms and thus prove vanishing theorems for Betti numbers both for positive and negative scalar curvature. These results will lead to quaternionic Kähler analogues of the weak and strong Lefschetz theorem in Kähler geometry. Recall that the weak Lefschetz theorem for Kähler manifolds M states the inequality $b_k \leq b_{k+2}$ of the Betti numbers for $k < \frac{1}{2}\dim M$, whereas the strong Lefschetz theorem asserts that the wedge product with the parallel 2–form descends to an injective map of the cohomology $H^k(M, \mathbb{R}) \longrightarrow H^{k+2}(M, \mathbb{R})$ in the same range.

Theorem 6.1. *(Representations and harmonic forms)*
Let (M^{4n}, g) be a compact quaternionic Kähler manifold of scalar curvature $\kappa \neq 0$ and let π be an irreducible representation of $\mathbf{Sp}(1)\cdot\mathbf{Sp}(n)$ occurring in the forms $\Lambda^\bullet(H\otimes E)$:

$$\mathrm{Hom}_{\mathbf{Sp}(1)\cdot\mathbf{Sp}(n)}(\pi, \Lambda^\bullet(H\otimes E)) \neq \{0\}$$

If the scalar curvature is positive then $\ker \Delta_\pi = \{0\}$ *unless $\pi = \Lambda^{a,a}_{\mathrm{top}}E$ for some a with $n \geq a \geq 0$. Similarly if the scalar curvature is negative then* $\ker(\Delta_\pi) = \{0\}$ *unless either $\pi = \Lambda^{a,a}_{\mathrm{top}}E$ as before or π is a representation of the form $\pi = \mathrm{Sym}^{2n-a-b}H \otimes \Lambda^{a,b}_{\mathrm{top}}E$ with $n \geq a \geq b \geq 0$.*

Although the representations $\mathrm{Sym}^{2n-a-b}H \otimes \Lambda^{a,b}_{\mathrm{top}}E$ form a larger class of representations they are still rather special among all the representations occurring in the forms. The appearance of these exceptional representations potentially carrying harmonic forms could have been foreseen from the difficulties encountered in the attempt to push Kraines original strong Lefschetz theorem ([Kra66]) for quaternionic Kähler manifolds beyond degree n. In higher degrees the given proofs fail precisely for these representations. It follows from Theorem 6.1 that this problem is absent in the positive scalar curvature case.

Proof: For any manifold of even dimension the bundle of exterior forms is the tensor product of the spinor bundle $\mathbf{S}$ with its dual $\mathbf{S}^*$. In the quaternionic Kähler case $\mathbf{S} \cong \mathbf{S}^*$ is real or quaternionic and the decomposition given in Proposition 3.1 implies:

$$\Lambda^\bullet(H\otimes E) \quad = \quad \mathbf{S}\otimes\mathbf{S} \quad = \quad \bigoplus_{r=0}^{n} \mathbf{S}\otimes\mathcal{R}^{r,\,n-r}\,.$$

In particular, a representation π occurs in the forms if and only if it occurs in a twisted spinor bundle $\mathbf{S}\otimes\mathcal{R}^{r,\,n-r}$ for some r with $n \geq r \geq 0$. It is consequently of the form $\pi = \mathrm{Sym}^kH \otimes \Lambda^{a,b}_{\mathrm{top}}E$ for suitable $k \geq 0$ and $n \geq a \geq b \geq 0$. In this situation Proposition 3.5 becomes:

$$\Delta\Big|_\pi \quad = \quad \Delta_\pi \quad = \quad D^2_{\mathcal{R}^{r,\,n-r}}\Big|_\pi$$

A harmonic form in the parallel subbundle determined by π is thus identified with an harmonic twisted spinor for the twist $\mathcal{R}^{r,\,n-r}$. However, we have already expressed the kernel of the twisted Dirac operators $D^2_{\mathcal{R}^{r,\,n-r}}$ in formula (20) at least for positive scalar curvature.

The point in this formula is of course that only those representations π may contribute to the kernel of the twisted Dirac operator $D^2_{\mathcal{R}^{r,\,n-r}}$, for which the twist $\mathcal{R}^{r,\,n-r}$ is a maximal twist. Replacing maximal by minimal twists the same argument applies in the case of negative scalar curvature and we conclude that a representation π may carry harmonic forms in the case of negative or positive scalar curvature if and only if it has a minimal or maximal twist respectively of the form $\mathcal{R}^{r,\,n-r}$ for some r with $n \geq r \geq 0$. A look at the classification of maximal and minimal twists in Theorems 4.4 and 4.5 completes the proof. □

We now want to point out a remarkable property of minimal and maximal twists: If a twist $\mathcal{R}^{l,\,d}$ is minimal or maximal for a representation π then π always occurs with multiplicity one in the twisted spinor representation $\mathbf{S} \otimes \mathcal{R}^{l,\,d}$. Although this property seems very natural it is obtained only as a corollary of the calculation of the index multiplicities in Theorems 4.4 and 4.5 using all the rather technical calculations of that section. Surely it is tempting to search for a direct argument providing better insight into the nature of this property.

For us this property is very convenient counting the total multiplicity of those representations π in the differential forms, which may carry harmonic forms. In fact for any representation π this total multiplicity is given by:

$$(21)\quad \dim \mathrm{Hom}_{\mathbf{Sp}\,(1)\cdot\mathbf{Sp}\,(n)}(\,\pi,\,\Lambda^\bullet(H \otimes E)\,) \;=\; \sum_{r=0}^{n} \dim \mathrm{Hom}_{\mathbf{Sp}\,(1)\cdot\mathbf{Sp}\,(n)}(\,\pi,\,\mathbf{S} \otimes \mathcal{R}^{r,\,n-r}\,)\;.$$

However, in the course of the proof of Theorem 6.1 we characterized the representations π potentially carrying harmonic forms in negative or positive scalar curvature by their property of having a minimal or maximal twist respectively of the form $\mathcal{R}^{r,\,n-r}$, $n \geq r \geq 0$. For such a representation π a twist of the form $\mathcal{R}^{\tilde{r},\,n-\tilde{r}}$ is minimal or maximal respectively if and only if it is admissible, because in this case $\phi(r, n-r) = 0 = \phi(\tilde{r}, n-\tilde{r})$.

Consequently for any representation π which may carry harmonic forms the summands on the right hand side of equation (21) are all either 0 or 1 and the total multiplicity of π in the differential forms is just the number of different minimal or maximal twists respectively. This number is easily read off from Theorems 4.4 and 4.5 and is part of the following lemma:

Lemma 6.2. *(Embeddings of harmonic forms)*
The representation $\pi = \Lambda^{a,\,a}_{\mathrm{top}}E$, $n \geq a \geq 0$, occurs $n-a+1$ times in the forms: it occurs with multiplicity one in the forms of degree $2a$, $2a+4$, $2a+8$, $\ldots$, $4n-2a$. Similarly the representation $\pi = \mathrm{Sym}^{\,2n-a-b}H \otimes \Lambda^{a,\,b}_{\mathrm{top}}E$, $n \geq a \geq b \geq 0$, occurs in the forms of degree $2n-a+b$, $2n-a+b+2$, $2n-a+b+4$, $\ldots$, $2n+a-b$ with multiplicity 1 and $a-b+1$ times in total.

Proof: We have already calculated the total multiplicity of the representations $\Lambda^{a,\,a}_{\mathrm{top}}E$ and $\mathrm{Sym}^{\,2n-a-b}H \otimes \Lambda^{a,\,b}_{\mathrm{top}}E$ in the differential forms so that it is sufficient to prove the existence of embeddings of these representations into the forms of the claimed degrees. First let us recall the well known general decomposition of the exterior forms $\Lambda^{\,k}(\,H \otimes E\,)$ into Schur functors

$$\Lambda^{\,k}(\,H \otimes E\,) \quad = \quad \bigoplus_{\mathfrak{Y}} \mathrm{Schur}_{\,\mathfrak{Y}}H \otimes \mathrm{Schur}_{\,\overline{\mathfrak{Y}}}E$$

where the sum is over all Young tableaus $\mathfrak{Y}$ of size $|\mathfrak{Y}| = k$ and $\overline{\mathfrak{Y}}$ denotes the conjugated Young tableau ([FuHa]). All Schur functors have two preferred realizations as the images of Schur symmetrizers in iterated tensor products. Specifying the Young tableau $\mathfrak{Y}$ either by the length of its rows $(r_1, r_2, \ldots, r_{c_1})$ or of its columns $(c_1, c_2, \ldots, c_{r_1})$ satisfying $r_1 \geq r_2 \geq \ldots \geq r_{c_1}$ and $c_1 \geq c_2 \geq \ldots \geq c_{r_1}$ these two preferred realizations of the Schur functors

$$\begin{array}{lcl} \mathrm{Schur}_{\mathfrak{Y}} H & \subset & \Lambda^{c_1} H \otimes \Lambda^{c_2} H \otimes \ldots \otimes \Lambda^{c_{r_1}} H \\ \mathrm{Schur}_{\mathfrak{Y}} E & \subset & \mathrm{Sym}^{r_1} E \otimes \mathrm{Sym}^{r_2} E \otimes \ldots \otimes \mathrm{Sym}^{r_{c_1}} E \end{array}$$

are given as the intersection of the kernels of all possible Plücker differentials. In our case all Schur functors in H corresponding to Young tableaus of more than two rows vanish and since $\Lambda^2 H \cong \mathbb{C}$ is trivial the Schur functor in H for the Young tableau of size k with two rows $(k-s,\, s)$ is equivalent to $\mathrm{Sym}^{k-2s} H$:

$$\Lambda^k (H \otimes E) \quad = \quad \bigoplus_{s=0}^{\lfloor \frac{k}{2} \rfloor} \mathrm{Sym}^{k-2s} H \otimes \mathrm{Schur}_{\overline{(k-s,\, s)}} E\,.$$

Conjugation of Young tableaus is defined by exchanging rows and columns. Conjugated to the Young tableau with two rows $(k-s,\, s)$ is the tableau with two columns $\overline{(k-s,\, s)}$. Thus $\mathrm{Schur}_{\overline{(k-s,\, s)}} E$ can be defined as the kernel of the Plücker differential:

$$\sum_\mu e_\mu \wedge \otimes de_\mu \lrcorner : \quad \Lambda^{k-s} E \otimes \Lambda^s E \longrightarrow \Lambda^{k-s+1} E \otimes \Lambda^{s-1} E\,.$$

From Weyl's construction of the representation $\Lambda^{a,\,b}_{\mathrm{top}} E$ as the intersection of the kernel of the Plücker differential $\Lambda^a_\circ E \otimes \Lambda^b_\circ E \longrightarrow \Lambda^{a+1} E \otimes \Lambda^{b-1}_\circ E$ with the kernel of the diagonal contraction with the symplectic form we see that $\Lambda^{a,\,a}_{\mathrm{top}} E \subset \mathrm{Schur}_{\overline{(a,\,a)}} E$. Consider now the map

$$\Omega : \quad \Lambda^a E \otimes \Lambda^b E \longrightarrow \Lambda^{a+2} E \otimes \Lambda^{b+2} E$$

defined by

$$\Omega := \sum_{\mu,\nu} \left(de^\flat_\mu \wedge de^\flat_\nu \wedge \otimes e_\mu \wedge e_\nu \wedge + de^\flat_\mu \wedge e_\mu \wedge \otimes de^\flat_\nu \wedge e_\nu \wedge \right)\,,$$

which curiously enough commutes with the Plücker differential. Consequently we may extend the above embedding to a chain of $\mathbf{Sp}\,(n)$–equivariant linear maps:

$$\Lambda^{a,\,a}_{\mathrm{top}} E \longrightarrow \mathrm{Schur}_{\overline{(a,\,a)}} E \stackrel{\Omega}{\longrightarrow} \mathrm{Schur}_{\overline{(a+2,\,a+2)}} E \stackrel{\Omega}{\longrightarrow} \ldots \stackrel{\Omega}{\longrightarrow} \mathrm{Schur}_{\overline{(2n-a,\,2n-a)}} E\,.$$

Explicit calculation shows that $\Omega^{n-a} = (2n-2a+1)!\,(\star \otimes \star)$ on $\Lambda^{a,\,a}_{\mathrm{top}} E$, where $\star$ denotes the Hodge isomorphism $\Lambda^a E \longrightarrow \Lambda^{2n-a} E$. Hence $\Lambda^{a,\,a}_{\mathrm{top}} E$ embeds into all the Schur functors $\mathrm{Schur}_{\overline{(a+2s,\,a+2s)}} E$ with $n-a \geq s \geq 0$ and further into the forms $\Lambda^{2a+4s}(\,H \otimes E\,)$ of degree $2a+4s$ with $n-a \geq s \geq 0$. The appearance of the map Ω is by no means an accident, it can be shown that it corresponds exactly to the wedge product with the parallel Kraines form Ω on the level of forms.

The construction of the different embeddings of the representations $\mathrm{Sym}^{2n-a-b}H \otimes \Lambda^{a,b}_{\mathrm{top}}E$ is simpler, although it is a dead end to start with the inclusion $\Lambda^{a,b}_{\mathrm{top}}E \subset \mathrm{Schur}_{\overline{(a,b)}}E$. Instead we have to use the Hodge isomorphism $(\star \otimes 1) : \Lambda^a E \otimes \Lambda^b E \longrightarrow \Lambda^{2n-a} E \otimes \Lambda^b E$, which interchanges in a sense the roles of the Plücker differential and the diagonal contraction with the symplectic form. The Hodge isomorphism can be extended to a chain of maps

$$\Lambda^{a,b}_{\mathrm{top}}E \longrightarrow \Lambda^{2n-a}E \otimes \Lambda^b E \stackrel{\sigma}{\longrightarrow} \Lambda^{2n-a+1}E \otimes \Lambda^{b+1}E \stackrel{\sigma}{\longrightarrow} \dots \stackrel{\sigma}{\longrightarrow} \Lambda^{2n-b}E \otimes \Lambda^a E$$

using the diagonal multiplication σ with the symplectic form. Since diagonal contraction and multiplication with the symplectic form generate an $\mathfrak{sl}_2$–algebra of operators the final map $\Lambda^{a,b}_{\mathrm{top}}E \longrightarrow \Lambda^{2n-b}E \otimes \Lambda^a E$ is injective and maps into the kernel of σ. In addition the commutator relations between the Plücker differential and σ imply that $\Lambda^{a,b}_{\mathrm{top}}E$ is mapped into the kernel $\mathrm{Schur}_{\overline{(2n-a+s,\, b+s)}}E$ of the Plücker differential at each step, so that

$$\mathrm{Sym}^{2n-a-b}H \otimes \Lambda^{a,b}_{\mathrm{top}}E \longrightarrow \mathrm{Sym}^{2n-a-b}H \otimes \mathrm{Schur}_{\overline{(2n-a+s,\, b+s)}}E \stackrel{\subset}{\longrightarrow} \Lambda^{2n-a+b+2s}(H \otimes E)$$

embeds into the forms of degree $2n - a + b + 2s$ for all $a - b \geq s \geq 0$. □

The weak Lefschetz Theorem for quaternionic Kähler manifolds of positive scalar curvature was proved by S. Salamon (cf. [Sal82]) by analyzing the cohomology of the twistor space. In the course of the proof of Lemma 6.2 we have sketched a proof of the strong Lefschetz Theorem for quaternionic Kähler manifolds of positive scalar curvature. The wedge product with the parallel Kraines form Ω is injective on the forms of type $\Lambda^{a,a}_{\mathrm{top}}E$ in all degrees $k < \frac{1}{2}\dim M$ and hence descends to an injective map of the cohomology.

A completely different argument can be given to show that the wedge product with the Kraines form is injective on forms of type $\mathrm{Sym}^{2n-a-b}H \otimes \Lambda^{a,b}_{\mathrm{top}}E$ in degrees $k < \frac{1}{2}\dim M - 1$, too. In contrast to the positive scalar curvature case however, the decomposition of the cohomology given in Theorem 6.1 for quaternionic manifolds of negative scalar curvature is finer than the decomposition into primitive cohomologies with respect to the Kraines form.

Theorem 6.3. *(Strong Lefschetz theorem for quaternionc Kähler manifolds with $\kappa > 0$) Let (M^{4n}, g) be a quaternionic Kähler manifold of positive scalar curvature $\kappa > 0$. Its odd Betti numbers vanish $b_{2k+1} = 0$ for all $0 \leq k < n$. The wedge product with the parallel Kraines form $\Omega \in \Gamma(\Lambda^4 T^*M)$ descends to an injective map on the level of cohomology*

$$\Omega\wedge : \quad H^{2k}(M, \mathbb{R}) \longrightarrow H^{2k+4}(M, \mathbb{R})$$

for all $k < n$. In particular the even Betti numbers of M satisfy the inequality:

$$b_{2k}(M) \leq b_{2k+4}(M)$$

for all $0 \leq k < n$ and the space of primitive forms of degree $2k$ agrees with the kernel of the operator Δ_{π_k} for the irreducible representation $\pi_k = \Lambda^{k,k}_{\mathrm{top}}E$ of $\mathbf{Sp}(1) \cdot \mathbf{Sp}(n)$.

Proof: For a compact quaternionic Kähler manifold of positive scalar curvature it follows from Proposition 6.1 that the only representations potentially carrying harmonic forms are $\Lambda^{a,\,a}_{\mathrm{top}}E$ with $n \geq a \geq 0$. But according to Lemma 6.2 all these representations embed into forms of even degree, i. e. all odd Betti numbers necessarily vanish. Moreover the representations $\Lambda^{a,\,a}_{\mathrm{top}}E$ occur in the forms of degree $2k$ if and only if $a = k,\ k-2,\ \ldots$ and in this case they occur with multiplicity one. $\square$

Remark 6.4. *(Associated twistor space and 3–Sasakian manifold [GaSa96])*
Let $\mathcal{S}$ be the 3–Sasakian manifold and $\mathcal{Z}$ the twistor space associated with the quaternionic Kähler manifold M^{4n}. The dimension of $\ker \Delta_{\Lambda^{k,\,k}_{\mathrm{top}}E}$ can be reinterpreted as the dimension of the cohomology of $\mathcal{S}$ and as the dimension of the primitive cohomology group of $\mathcal{Z}$:

$$\dim(\ker \Delta_{\Lambda^{k,\,k}_{\mathrm{top}}E}) \;=\; b_{2k}(\mathcal{S}) \;=\; b_{2k}(\mathcal{Z}) \;-\; b_{2k-2}(\mathcal{Z}) \qquad\qquad k \leq n \quad .$$

At this point it is easy to see that Theorem 6.3 provides the expression for the dimension of the kernel of the twisted Dirac operators $D_{\mathcal{R}^{n-d,\,d}}$ in terms of Betti numbers. Indeed we conclude formula (20) that in the case of positive scalar curvature a representation π may contribute to the kernel of $D^2_{\mathcal{R}^{l,\,d}}$ only if the twist $\mathcal{R}^{l,\,d}$ is maximal for π. On the other hand the twisted spinor representation $\mathbf{S} \otimes \mathcal{R}^{n-d,\,d}$ occurs in the forms so that a representation π contributes to the kernel of $D^2_{\mathcal{R}^{n-d,\,d}}$ if and only if it carries harmonic forms, i. e. π must be one of the representations $\Lambda^{a,\,a}_{\mathrm{top}}E$ for some a with $n \geq a \geq 0$. From inequality (13) of Theorem 4.1 it is evident that $\pi = \Lambda^{a,\,a}_{\mathrm{top}}E$ occurs in $\mathbf{S} \otimes \mathcal{R}^{n-d,\,d}$ if and only if $a \leq d$, hence we obtain

$$\ker(D^2_{\mathcal{R}^{n-d,\,d}}) \;=\; \bigoplus_{a\leq d} \ker(\Delta_{\Lambda^{a,\,a}_{\mathrm{top}}E}) \ .$$

An immediate consequence of this formula is then the well-known result of S. Salamon and C. LeBrun (cf. [LeBSa94]) on the index of the twisted Dirac operator $D_{\mathcal{R}^{l,\,d}}$ with $l + d \,=\, n$.

In dealing with quaternionic Kähler manifolds of negative curvature it is convenient to decompose their cohomology into two direct summands with quite different behavior:

Definition 6.5. *($\mathfrak{sp}(1)$–invariant and exceptional cohomology)*
Let $(M^{4n},\, g)$ be a compact quaternionic Kähler manifold of negative scalar curvature. According to Theorem 6.1 two different series of representations contribute to harmonic forms on M, namely $\Lambda^{a,\,a}_{\mathrm{top}}E$, $n \geq a \geq 0$ and $\mathrm{Sym}^{2n-a-b}H \otimes \Lambda^{a,\,b}_{\mathrm{top}}E$, $n \geq a \geq b \geq 0$. In particular the de Rham cohomology of M splits into the direct sum

$$H^\bullet_{dR}(\,M,\,\mathbb{C}\,) \quad = \quad H^\bullet_{\mathfrak{sp}(1)}(\,M,\,\mathbb{C}\,) \,\oplus\, H^\bullet_{\mathrm{expt}}(\,M,\,\mathbb{C}\,)$$

of its $\mathfrak{sp}(1)$–invariant cohomology $H^\bullet_{\mathfrak{sp}(1)}(\,M,\,\mathbb{C}\,)$, which is the sum of all isotypical components corresponding to the representations $\Lambda^{a,\,a}_{\mathrm{top}}E$, $n \geq a \geq 0$, and its exceptional cohomology $H^\bullet_{\mathrm{expt}}(\,M,\,\mathbb{C}\,)$, which is the direct sum of all isotypical components corresponding to the remaining representations $\mathrm{Sym}^{2n-a-b}H \otimes \Lambda^{a,\,b}_{\mathrm{top}}E$, $n \geq a \geq b \geq 0$, $b \neq n$.

Because the curvature tensor of M is $\mathfrak{sp}(1)$–invariant the same is true for all its characteristic classes. Moreover $H^{\bullet}_{\mathfrak{sp}(1)}(\,M,\,\mathbb{C}\,)$ is closed under multiplication and the decomposition of the de Rham–cohomology into $\mathfrak{sp}(1)$–invariant and exceptional cohomology is respected by the induced modul structure. A deeper analysis of the ring structure of the cohomology ring of M will be given in a forthcoming paper (cf. [Wei00]).

Theorem 6.6. *(Weak Lefschetz theorem for negative scalar curvature)*
Let $(M^{4n},\,g)$ be a compact quaternionic Kähler manifold of negative scalar curvature $\kappa < 0$. Its $\mathfrak{sp}(1)$–invariant and exceptional Betti numbers $b_{\mathfrak{sp}(1),\,k}$ and $b_{\mathrm{expt},\,k}$ satisfy:

$$\begin{array}{lcll} b_{\mathfrak{sp}(1),\,k} & = & 0 & \textit{for } k \textit{ odd}, \\ b_{\mathrm{expt},\,k} & = & 0 & \textit{for } k \,\leq\, n-1\,, \\ b_{\mathfrak{sp}(1),\,k} & \leq & b_{\mathfrak{sp}(1),\,k+4} & \textit{for } k \,\leq\, 2n-2\,, \\ b_{\mathrm{expt},\,k} & \leq & b_{\mathrm{expt},\,k+2} & \textit{for } k \,\leq\, 2n-1\,. \end{array}$$

In particular, its Betti numbers $b_k = b_{\mathfrak{sp}(1),\,k} + b_{\mathrm{expt},\,k}$ satisfy:

$$\begin{array}{lcll} b_{2k+1} & = & 0 & \textit{for } 2k+1 \,\leq\, n-1\,, \\ b_k & \leq & b_{k+2} & \textit{for odd } k \,\leq\, 2n-1\,, \\ b_k & \leq & b_{k+4} & \textit{for } \quad k \,\leq\, 2n-2\,. \end{array}$$

Proof: Since the $\mathfrak{sp}(1)$–invariant Betti numbers correspond by definition to the representations $\Lambda^{a,\,a}_{\mathrm{top}}E$, $n \geq a \geq 0$, they have the same properties as Betti numbers of a quaternionic Kähler manifolds of positve scalar curvature given in Theorem 6.3. It follows from Lemma 6.2 that the remaining representations $\mathrm{Sym}^{\,2n-a-b}H \otimes \Lambda^{a,\,b}_{\mathrm{top}}E$ with $n \geq a \geq b \geq 0$ and $b \neq n$ corresponding to the exceptional Betti numbers embed into forms of degree $2n-a+b$, $2n-a+b+2$, $\ldots$, $2n+a-b$. For $a \not\equiv b \bmod 2$ these embeddings give rise to harmonic forms of odd degree. Nevertheless the odd Betti numbers of degree less than n have to vanish because of $2n-a+b \geq n$. □

References

[Al68] *Alekseevskii, D. V.,* Riemannian spaces with exceptional holonomy groups, Functional Anal. Appl. **2** (1968), 97–105;

[AlMa95] *Alekseevskii, D. V., Marchiafava, S.*, Transformations of a quaternionic Kähler manifold, C.R. Acad. Sci. Paris. Sér. I, **320** (1995), 703–708;

[AlMa98] *Alekseevskii, D. V., Marchiafava, S., Pontecorvo, M.*, Spectral properties of the twistor fibration of a quaternion Kähler manifold, Preprint ESI **628** (1998), Vienna;

[Be] *Besse, A.*,Einstein manifolds, Erg. Math. **10**, Springer-Verlag, New York, 1987;

[BaS83] *Barker, R. and Salamon, S.*, Analysis on a generalized Heisenberg group, J. London Math Soc. **28** (1983), 184–192;

[Ch57] *Chern, S.S.*, On a generalization of Kähler geometry, Algebraic geometry and topology. A symposium in honor of S. Lefschetz, pp. 103–121. Princeton University Press, Princeton, N. J., 1957;

[CGH99] *Calderbank, D.M.J., Gauduchon, P., Herzlich, M.* Refined Kato inequalities and conformal weights in Riemannian geometry preprint (1999), math/9909116, to appear in J. Funct. Anal.;

[FuHa] *Fulton, W., Harris, J.*, Representation Theory, Graduate Texts in Mathematics **129**, Springer 1991;

[GaSa96] *Galicki, K., Salamon, S.*, Betti numbers of 3–Sasakian manifolds, Geom. Dedicata **63** (1996), no.1, 45–68;

[Ho96] *Horan, R.*, A rigidity theorem for quaternionic Kähler manifolds, Differential Geom. Appl. **6** (1996), no.2, 189–196;

[Kra66] *Kraines, V.Y.*, Topology of quaternionic manifolds, Trans. Amer. Math. Soc. **122** (1966), 357–367;

[KSW97a] *Kramer, W., Semmelmann, U., Weingart, G.*, Eigenvalue estimates for the Dirac operator on quaternionic Kähler manifolds, Math. Z. **230** (1999), no.4, 727–751;

[LeBSa94] *LeBrun, C. and Salamon, S.*, Strong rigidity of positive quaternion–Kähler manifolds, Invent. Math. **118** (1994), no.1, 109–132;

[LeB95] *LeBrun, C.*, Fano manifolds, contact structures and quaternionic geometry, Int. J. of Math., **6**, no. 3 (1995), 419–437;

[Me71] *Meyer, D.*, Sur les variétés riemanniennes à opérateur de courbure positif, Conte Rendue de l'Académie des Sciences Paris Série A I, **272** (1971),482—485;

[OnVi90] *Onishchik, A.L. and Vinberg, E.B.*, Lie groups and algebraic groups, Springer Verlag, 1990;

[Po81] *Poor, W.*, Differential geometric structures, McGraw-Hill Book Co., New York, 1981;

[Sal82] *Salamon, S.M.*, Quaternionic Kähler manifolds, Invent. Math. **67** (1982), 143–171;

[Sal86] *Salamon, S.M.*, Differential geometry of quaternionic manifolds, Ann. scient.Éc. Norm. Sup. **19** (1986), 31–55;

[Sal89] *Salamon, S.M.*, Riemannian geometry and holonomy groups, Pitman Research Notes in Mathematics Series, **201**, Longman Scientific & Technical, Harlow (1989);

[SW01] *Semmelmann, U. and Weingart, G.*, Weitzenböck formulas on quaternionic Kähler manifolds, preprint (2001);

[Wan89] *Wang, M.Y.*, Parallel spinors and parallel forms, Anal. Global Anal. Geom. **7** (1989), 59–68;

[Wei00] *Weingart, G.*, Formal cohomology rings, preprint (2000);

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