Differential Forms on Quaternionic Kähler Manifolds

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Abstract

The bundle of differential forms on a hyperkähler manifold is a representation of a natural Lie algebra $\mathfrak{so}_{4,1}\mathbb{R}$ of parallel operators. After constructing this algebra from the point of view of representation theory we calculate the minimal parallel decomposition of the bundle of differential forms on hyperkähler and quaternionic Kähler manifolds.

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1 Introduction

Quaternionic Kähler manifolds and hyperkähler manifolds have long been studied for their rich geometry and their marvellous implications on the topology of the underlying manifold. In the context of these studies the holonomy decomposition of the bundle of complex valued differential forms into minimal parallel subbundles has been studied extensively for hyperkähler manifolds generalizing ideas developped in Kähler geometry. Interestingly there exists an intimate relationship between this construction and a strange property of general representation theory, the existence of a dual Lie algebra gouverning the decomposition of tensor products of exterior or symmetric algebras. For the complex valued differential forms on hyperkähler manifolds this dual Lie algebra is isomorphic to $\mathfrak{so}_{4,1}\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$.

In this article we will define the action of the Lie algebra $\mathfrak{so}_{4,1}\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$ in Section 2 entirely from the representation theoretic point of view with only a passing reference on how this algebra can be constructed using wedge products and contractions with Kähler forms. One merit of this approach is that it highlights the close relationship to Weyl's construction of the irreducible representations of the classical matrix groups showing in particular that for a generic hyperkähler manifold the resulting decomposition is minimal. Moreover the representation theoretic approach allows us to use the comparatively simple branching formulas from $\mathfrak{so}_{4,1}\mathbb{R}$ to $\mathbb{R} \oplus \mathfrak{sp}(1)$ instead of the branching formulas from $\mathfrak{so}_{4n}\mathbb{R}$ to $\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$ to calculate the parallel decomposition of differential forms on quaternionic Kähler and hyperkähler manifolds.

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Before presenting the final decomposition formula let us recall that a quaternionic Kähler manifold M carries quite a few geometric vector bundles besides the usual tensor bundles. Eventually this plentitude is due to the fact that the complexified tangent bundle of a quaternionic Kähler manifold M of dimension $4n$ factorizes parallely into a tensor product $TM \otimes_{\mathbb{R}} \mathbb{C} \cong HM \otimes EM$ of two (locally defined) complex symplectic vector bundles HM and EM of dimensions 2 and $2n$ respectively. All minimal parallel subbundles of the differential forms bundle $\Lambda^\bullet T^*M \otimes_{\mathbb{R}} \mathbb{C}$ on a generic quaternionic Kähler manifold M are of the form

$$\mathrm{Sym}^s HM \otimes \Lambda^{r,\bar{r}}_\circ EM$$

with $n \geq r \geq \bar{r} \geq 0$ and $s \equiv r + \bar{r}$ modulo 2 compare [SW], where $\Lambda^{r,\bar{r}}_\circ EM$ is the joint kernel of all possible contractions with the parallel symplectic form on EM inside the Schur functor bundle $\Lambda^{r,\bar{r}} EM := \ker(\,\overline{\mathrm{Pl}} : \Lambda^r EM \otimes \Lambda^{\bar{r}} EM \longrightarrow \Lambda^{r+1} EM \otimes \Lambda^{\bar{r}-1} EM\,)$:

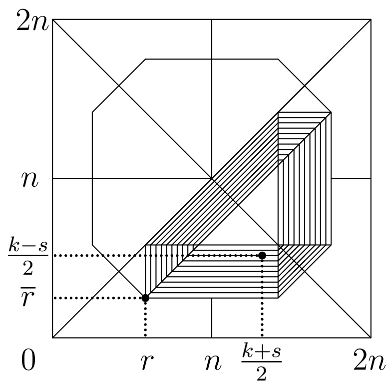
Theorem 1.1 (Parallel Decomposition of Differential Forms)
On a generic quaternionic Kähler manifold M of dimension $4n$ the holonomy decomposition of the bundle of complex valued differential forms into minimal parallel subbundles reads

$$\Lambda^k T^*M \otimes_{\mathbb{R}} \mathbb{C} \;=\; \bigoplus_{\substack{n \geq r \geq \bar{r} \geq 0 \\ s \geq 0}} m_{s,\,r,\,\bar{r}}(\,k\,)\;\; \mathrm{Sym}^s HM \otimes \Lambda^{r,\bar{r}}_\circ EM$$

where the multiplicities $m_{s,\,r,\,\bar{r}}(\,k\,)$ vanish unless $k \equiv s \equiv r + \bar{r}$ modulo 2 and:

$$s \;\leq\; 2n - r - \bar{r} \qquad\qquad \bar{r} \;\leq\; \frac{k-s}{2} \;\leq\; 2n - r \qquad\qquad r \;\leq\; \frac{k+s}{2} \;\leq\; 2n - \bar{r}$$

A precise and ready to use formula for the multiplicities $m_{s,r,\bar{r}}(k)$ is given in the specification Theorem 3.1 of Theorem 1.1 in Section 3, it appears inadequate for an introduction to reproduce the threefold case distinction necessary. Suffice it to say at this point that under the conditions of Theorem 1.1 the multiplicities $m_{s,r,\bar{r}}(k) > 0$ are strictly positive unless $\min\{r - \bar{r},\, s\} = 0$ and $r \not\equiv \frac{k-s}{2}$ modulo 2. It is useful to think of $m_{s,r,\bar{r}}(k)$ as depending on the position of the point $(\frac{k+s}{2}, \frac{k-s}{2})$ in the hexagonal pattern determined by the point $(r, \bar{r})$



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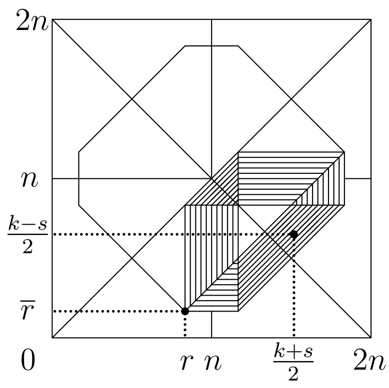


(1)

the different shapes correspond to $3r - \bar{r} \leq 2n$ and $3r - \bar{r} \geq 2n$. Essentially the multiplicities $m_{s,r,\bar{r}}(k)$ vanish outside the hexagon and increase from the value 1 on the edges towards the central triangle by steps of 1 every second parallel to their maximal value $\lceil \frac{r-\bar{r}}{2} \rceil$ or $1+n-r$ respectively taken in the triangle, of course only points $(\frac{k+s}{2}, \frac{k-s}{2})$ satisfying $k \equiv s \equiv r+\bar{r}$ modulo 2 ought to be considered. If r and $\bar{r}$ share the same parity, then this geometric interpretation of the formula for $m_{s,r,\bar{r}}(k)$ given in Theorem 3.1 is not entirely correct, the actual multiplicities may be one less than their geometrically deduced values.

2 The Lie Algebra $\mathfrak{so}_{4,1}\mathbb{R}$ of Operators on Forms

Usually an orthogonal quaternionic structure on a euclidian vector space T with scalar product g is defined as the choice of three anticommuting, skew–symmetric complex structures I, J and K on T. In this article however we prefer an alternative weaker notion, an orthogonal quaternionic structure on T is to be the choice of a subalgebra $Q \subset \mathrm{End}\, T$ containing the identity of T and isomorphic to $\mathbb{H}$ as an associative algebra with unit over $\mathbb{R}$, such that $g(AX, Y) = g(X, \overline{A}Y)$ for all $A \in Q$ and all $X, Y \in T$. Fixing an isomorphism $Q \cong \mathbb{H}$ brings us back to the standard notion of orthogonal quaternionic structures, this innocuous difference for euclidian vector spaces becomes important for Riemannian manifolds.

In order to discuss orthogonal quaternionic structures on euclidian vector spaces in more detail we recall that a positive quaternionic structure on a complex symplectic vector space E with symplectic form σ is a conjugate linear map $C : E \longrightarrow E$ satisfying $C^2 = -1$ as well as $\sigma(Ce_1, Ce_2) = \overline{\sigma(e_1, e_2)}$ and $\sigma(Ce, e) > 0$ for all $e \neq 0$. Quite remarkably the complexification $T \otimes_{\mathbb{R}} \mathbb{C}$ of a euclidian vector space T of dimension $4n$ with scalar product g and orthogonal quaternionic structure $Q \subset \mathrm{End}\, T$ turns out to be the tensor product

$$H \otimes E \quad \cong \quad T \otimes_{\mathbb{R}} \mathbb{C} \tag{2}$$

of two complex symplectic vector spaces H and E of dimension 2 and $2n$ endowed with positive quaternionic structures, such that every $A \in Q$ acts as a Kronecker product $A \otimes \mathrm{id}_E$ on $H \otimes E$, while the complex bilinear extension of g and the real structure on $T \otimes_{\mathbb{R}} \mathbb{C}$ agree with $\sigma \otimes \sigma$ and $C \otimes C$ respectively. One way to justify this convenient description of $T \otimes_{\mathbb{R}} \mathbb{C}$ as a tensor product is to use the representation theory of the unitary symplectic group $\mathbf{Sp}(n)$, $n \geq 1$, which can be interpreted alternatively as the group of orthogonal maps $T \longrightarrow T$ commuting with Q or as the group of symplectic maps $E \longrightarrow E$ commuting with C. A somewhat more direct approach is to note that $Q \cong \mathrm{Cl}\, \mathbb{R}^{2,0}$ is a Clifford algebra with a unique complex module H, which is a complex symplectic vector space of dimension 2 with a positive quaternionic structure, forcing the multiplicity space $E := \mathrm{Hom}_Q(H, T \otimes_{\mathbb{R}} \mathbb{C})$ to have a symplectic form and positive quaternionic structure as well [W1].

A particular merit of the description of $T \otimes_{\mathbb{R}} \mathbb{C} = H \otimes E$ is that the algebra Q acts on T essentially via its action on its module H. In particular the choice of a complex structure in Q corresponds to the choice of decomposition $H = L \oplus CL$ into two conjugated complex lines L and CL, in fact for any such decomposition we can find $I \in Q$ such that L and CL are the eigenspaces for the eigenvalues i and $-i$ respectively. In the same vein every unit vector $q \in H$ with $\sigma(Cq, q) = 1$ determines a unique algebra isomorphism $\varphi_q : \mathbb{H} \longrightarrow Q$

such that the images I, J of $i, j \in \mathbb{H}$ satisfy $Iq = iq$ and $Jq = Cq$, more precisely the matrices of the images $I, J, K \in Q$ with respect to the canonical basis Cq, q of H read:

$$I \;=\; \begin{pmatrix} -i & 0 \\ 0 & +i \end{pmatrix} \qquad J \;=\; \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \qquad K \;=\; \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \tag{3}$$

Consider now a Riemannian manifold M with scalar product g together with a smooth choice of orthogonal quaternionic structures $Q_pM \subset \mathrm{End}\, T_pM$ on every tangent space T_pM, $p \in M$. Such a Riemannian manifold M is called a quaternionic Kähler manifold, if the subalgebra bundle $QM \subset \mathrm{End}\, TM$ is parallel with respect to the Levi–Civita connection ∇, in other words if the parallel transport along every curve γ from p to q conjugates Q_pM with Q_qM. Hyperkähler manifolds are special quaternionic Kähler manifolds, their subalgebra bundle $QM \subset \mathrm{End}\, TM$ is not only parallel, but trivial for the Levi–Civita connection. Hence we can choose a parallel isomorphism $\mathbb{H}M \cong QM$ to obtain a smooth choice of an orthogonal quaternionic structure on every tangent space T_pM in the stronger standard sense.

In order to discuss the decomposition of the bundle of differential forms on a hyperkähler or quaternionic Kähler manifold M into minimal parallel subbundles we fix once and for all a point $p \in M$ and decompose the vector space $\Lambda^\bullet T^* \otimes_{\mathbb{R}} \mathbb{C}$ of complex–valued alternating forms on the tangent space $T := T_pM$ in p into irreducible subspaces under the holonomy group of parallel transports of curves beginning and ending in p, parallel transport along arbitrary curves will deliver this decomposition in p to all of M. The holonomy group in p is a group of orthogonal transformations γ normalizing the given orthogonal quaternionic structure $Q \subset \mathrm{End}\, T$ of the euclidian vector space T. In terms of decomposition (2) the group of all such orthogonal transformations agrees with $\mathbf{Sp}\, H \cdot \mathbf{Sp}\, E \subset \mathbf{SO}\, T$, where $\mathbf{Sp}\, E \cong \mathbf{Sp}(n)$ say is the group of all symplectic transformations of E commuting with C. Of course on a hyperkähler manifold the holonomy group actually centralizes the subalgebra $Q \subset \mathrm{End}\, T$ and the holonomy group becomes a subgroup of $\mathbf{Sp}\, E$.

Choosing a unit vector $q \in H$ satisfying $\sigma(Cq, q) = 1$ we get a canonical basis p, q of H with $p := Cq$, the resulting algebra isomorphism $Q \cong \mathbb{H}$ allows to replace $\mathbf{Sp}\, H$ by $\mathbf{Sp}(1)$ in the holonomy group and the complex linear forms on T by $E^* \oplus E^*$ via the isomorphism:

$$E^* \oplus E^* \stackrel{\cong}{\longrightarrow} T^* \otimes_{\mathbb{R}} \mathbb{C}, \qquad \eta \oplus \tilde{\eta} \longmapsto (\, dq \otimes \eta\,) + (\, dp \otimes \tilde{\eta}\,)$$

This isomorphism extends to an $\mathbf{Sp}\, E$–equivariant decomposition of the alternating forms

$$\Phi_q : \quad \Lambda^\bullet E^* \otimes \Lambda^\bullet E^* \stackrel{\cong}{\longrightarrow} \Lambda^\bullet(\, T^* \otimes_{\mathbb{R}} \mathbb{C}\,) \;\cong\; \Lambda^\bullet T^* \otimes_{\mathbb{R}} \mathbb{C} \tag{4}$$

on T (with the product grading on the left indicated by the repeated grading symbol) via:

$$\Phi_q(\, \eta_1 \wedge \ldots \wedge \eta_k \otimes \tilde{\eta}_1 \wedge \ldots \wedge \tilde{\eta}_{\overline{k}}\,) \;:=\; (dq \otimes \eta_1) \wedge \ldots \wedge (dq \otimes \eta_k) \wedge (dp \otimes \tilde{\eta}_1) \wedge \ldots \wedge (dp \otimes \tilde{\eta}_{\overline{k}})$$

Evidently the isomorphism Φ_q is an isomorphism of algebras for the twisted multiplication on $\Lambda\, E^* \otimes \Lambda\, E^*$ sometimes denoted $\Lambda\, E^* \widehat{\otimes} \Lambda\, E^*$. On the other hand the algebra isomorphism $Q \cong \mathbb{H}$ coming along with the choice of unit vector q is characterized by $Iq = iq$ and $Jq = p$.

The finer bigrading on $\Lambda^\bullet E^* \otimes \Lambda^\circ E^*$ thus agrees with the Hodge bigrading $\Lambda^{\bullet,\circ} T^* \otimes_{\mathbb{R}} \mathbb{C}$ with respect to the complex structure I, in other words Φ_q maps $\Lambda^k E^* \otimes \Lambda^{\bar{k}} E^*$ to the space $\Lambda^{k,\bar{k}} T^* \otimes_{\mathbb{R}} \mathbb{C}$ of $(k, \bar{k})$–forms on T. Interestingly the natural real structure on the algebra $\Lambda^\bullet T^* \otimes_{\mathbb{R}} \mathbb{C}$ of complex valued alternating forms on T by conjugation of values

$$\overline{\Omega}(\, h_1 \otimes e_1, \, \ldots, \, h_r \otimes e_r \,) \;\; := \;\; \overline{\Omega(\, Ch_1 \otimes Ce_1, \, \ldots, \, Ch_r \otimes Ce_r \,)}$$

on real arguments interchanges $\Lambda^{\bullet,\circ} T^* \otimes_{\mathbb{R}} \mathbb{C}$ with $\Lambda^{\circ,\bullet} T^* \otimes_{\mathbb{R}} \mathbb{C}$. In order to describe its precise relation to Φ_q we extend the quaternionic structures on E and H to quaternionic structures on E^* and H^* via $(C\eta)(e) := -\overline{\eta(Ce)}$. The musical isomorphism $\sharp : \; E \longrightarrow E^*, \; e \longmapsto \sigma(e, \cdot)$, and its inverse $\flat$ are then both real due to $C(e^\sharp) = -\overline{\sigma(e, C\cdot)} = (Ce)^\sharp$ and the identity

$$(\, C\alpha \, \otimes \, C\eta \,)(\, h \, \otimes \, e \,) \;\; = \;\; \overline{\alpha(\, Ch \,)\, \eta(\, Ce \,)} \;\; = \;\; \overline{(\, \alpha \, \otimes \, \eta \,)(\, Ch \, \otimes \, Ce \,)}$$

shows that the real structure $C \otimes C$ on $H^* \otimes E^*$ coincides with the natural real structure. In consequence $Cdp = -dq$ and $Cdq = dp$ tell us that on $\Lambda^k E^* \otimes \Lambda^{\bar{k}} E^*$ we have as expected:

$$\Phi_q^{-1}(\, \overline{\Phi_q(\, \eta \, \otimes \, \tilde{\eta} \,)} \,) \;\; = \;\; (-1)^{\bar{k}(k+1)} \, C\tilde{\eta} \, \otimes \, C\eta$$

Remark 2.1 (Real Structure on $\Lambda^\bullet E^* \otimes \Lambda^\circ E^*$)
The bigraded algebra isomorphism $\Phi_q : \Lambda^\bullet E^* \otimes \Lambda^\circ E^* \longrightarrow \Lambda^{\bullet,\circ} T^* \otimes_{\mathbb{R}} \mathbb{C}$ is real with respect to:

$$\Lambda^\bullet E^* \, \otimes \, \Lambda^\circ E^* \;\longrightarrow\; \Lambda^\circ E^* \, \otimes \, \Lambda^\bullet E^*, \qquad \eta \, \otimes \, \tilde{\eta} \;\longmapsto\; (-1)^{|\tilde{\eta}|\,(\,|\eta|\,+\,1\,)} \, C\tilde{\eta} \, \otimes \, C\eta$$

For the moment we want to quit discussion of $\Lambda^\bullet E^* \otimes \Lambda^\circ E^*$ and study the simpler algebra $\Lambda^\bullet E^*$ instead. Note first that for every pair $\{e_\mu\}$ and $\{de_\mu\}$ of dual bases for E and E^* respectively the pairs $\{Ce_\mu\}$ and $\{Cde_\mu\}$ as well as $\{de_\mu^\flat\}$ and $\{-e_\mu^\sharp\}$ are dual bases, too. Bilinear sums over pairs of dual bases are independent of the pair, so we can replace one of these pairs by another in such sums whenever convenient. For example the symplectic form $\sigma = \frac{1}{2} \sum_\mu de_\mu \wedge e_\mu^\sharp$ and $\sigma^\flat = \frac{1}{2} \sum_\mu de_\mu^\flat \wedge e_\mu$ are real, in consequence the two operators

$$\sigma \wedge \;\; := \;\; \frac{1}{2} \sum_\mu de_\mu \, \wedge \, e_\mu^\sharp \, \wedge \qquad\qquad \sigma^\flat \, \lrcorner \;\; := \;\; \frac{1}{2} \sum_\mu e_\mu \, \lrcorner \; de_\mu^\flat \, \lrcorner$$

on $\Lambda^\bullet E^*$ of bidegree $+2$ and -2 respectively both commute with C. The calculation

$$[\, \sigma \wedge, \, e \, \lrcorner \,] \;\; = \;\; \frac{1}{2} \sum_\mu \Big(\, (de_\mu^\flat)^\sharp \, \wedge \, \{\, e_\mu^\sharp \wedge, \, e \, \lrcorner \,\} \; - \; \{\, de_\mu \wedge, \, e \, \lrcorner \,\} \, e_\mu^\sharp \, \wedge \,\Big) \;\; = \;\; -\, e^\sharp \wedge$$

and its analogue for $[\, \sigma^\flat \lrcorner \,, \, \eta \wedge \,]$ provide us with the fundamental commutation relations:

$$\begin{array}{rclcrcl} [\, \sigma \wedge, \, \eta \wedge \,] & = & 0 & \qquad & [\, \sigma^\flat \lrcorner \,, \, \eta \wedge \,] & = & -\, \eta^\flat \lrcorner \\ [\, \sigma \wedge, \, e \, \lrcorner \,] & = & -\, e^\sharp \wedge & \qquad & [\, \sigma^\flat \lrcorner \,, \, e \, \lrcorner \,] & = & 0 \end{array} \qquad (5)$$

On a symplectic vector space E of dimension $2\,n$ the commutator of $\sigma\wedge$ and $\sigma^\flat\lrcorner$ reads

$$\begin{aligned}[\,\sigma\wedge,\;\sigma^\flat\lrcorner\,] \;&=\; \frac{1}{2}\sum_\mu\Big(\,[\,\sigma\wedge,\;e_\mu\lrcorner\,]\,de^\flat_\mu\lrcorner\;+\;e_\mu\lrcorner\,[\,s\wedge,\;de^\flat_\mu\lrcorner\,]\,\Big)\\ &=\;-\,\frac{1}{2}\sum_\mu(\,-\,de_\mu\wedge e_\mu\lrcorner\;+\;e_\mu\lrcorner\;de_\mu\wedge\,)\;=:\;N\;-\;n\end{aligned}$$

where $N\;:=\;\sum_\mu de_\mu\wedge e_\mu\lrcorner$ is the so called (fermionic) number operator, which multiplies forms in $\Lambda^k E^*$ by k. In other words there is a canonical $\mathfrak{sl}_2$–triple of real operators on $\Lambda^\bullet E^*$

$$H\;:=\;n\;-\;N\qquad\qquad X\;:=\;\sigma^\flat\lrcorner\qquad\qquad Y\;:=\;\sigma\wedge\tag{6}$$

satisfying the classical commutation relations $[H,X]\;=\;2\,X$, $[H,Y]\;=\;-2\,Y$, $[X,Y]\;=\;H$.

Coming back to the description of the alternating forms we note that the factor $\mathbf{Sp}\,H$ of the quaternionic Kähler holonomy group $\mathbf{Sp}\,H\,\cdot\,\mathbf{Sp}\,E$ does not act on the source $\Lambda^\bullet E^*\otimes\Lambda^\circ E^*$ of the isomorphism Φ_q, although it acts on the target $\Lambda^{\bullet,\circ}T^*\otimes_{\mathbb{R}}\mathbb{C}$. Nevertheless there is a well–defined action of the group $\mathbf{Sp}(1)\;\subset\;\mathbb{H}$ of unit quaternions on $\Lambda^\bullet E^*\otimes\Lambda^\circ E^*$ such that for all unit vectors $q\in H$ the isomorphism Φ_q is actually equivariant over the isomorphism

$$\varphi_q\cdot\mathrm{id}:\;\;\mathbf{Sp}(1)\,\cdot\,\mathbf{Sp}\,E\;\stackrel{\cong}{\longrightarrow}\;\mathbf{Sp}\,H\cdot\,\mathbf{Sp}\,E$$

induced by the algebra isomorphism $\varphi_q:\;\mathbb{H}\longrightarrow Q$. The characterization $\varphi_q(i)q\;=\;iq$ and $\varphi_q(j)q\;=\;Cq$ of the isomorphism φ_q implies that the isomorphism φ_{Aq} associated to another unit vector $Aq\in H$ with $A\in\mathbf{Sp}\,H\;\subset\;Q$ is conjugated to φ_q in the sense $\varphi_{Aq}\;:=\;A\,\varphi_q\,A^{-1}$. On the other hand the automorphisms $A\;\in\;\mathbf{GL}\,T$ of T act on the differential forms by their inverse adjoint A^{-*}. For the special isomorphisms $A\;\in\;\mathbf{Sp}\,H$ this representation satisfies

$$A^{-*}\,\Phi_q(\,\eta\,\otimes\,\tilde\eta\,)\;\;=\;\;\Phi_{A\,q}(\,\eta\,\otimes\,\tilde\eta\,)$$

because $C(Aq)\;=\;A(Cq)\;=\;Ap$ and $\{\,A^{-*}dp,\,A^{-*}dq\,\}$ is the basis dual to $Ap,\,Aq$. In consequence the representation $\star$ of $\mathbf{Sp}(1)$ on $\Lambda^\bullet E^*\otimes\Lambda^\circ E^*$ defined to make Φ_q equivariant

$$z\;\star\;(\,\eta\otimes\tilde\eta\,)\;\;:=\;\;\Phi_q^{-1}\Big(\,\varphi_q(\,z\,)^{-*}\,\Phi_q(\,\eta\otimes\tilde\eta\,)\,\Big)\tag{7}$$

over $\varphi_q\,\cdot\,\mathrm{id}$ is actually independent of the choice of the unit vector $q\in H$, because

$$\Phi_{Aq}^{-1}\Big(\,\varphi_{Aq}(\,z\,)^{-*}\,\Phi_{Aq}(\,\eta\otimes\tilde\eta\,)\,\Big)\;\;=\;\;\Phi_q^{-1}\Big(\,A^*\,(\,A\,\varphi_q(\,z\,)\,A^{-1}\,)^{-*}\,A^{-*}\,\Phi_q(\,\eta\otimes\tilde\eta\,)\,\Big)$$

and $(A\varphi_q(z)A^{-1})^{-*}\;=\;A^{-*}\varphi_q(z)^{-*}A^*$. For the time being we will only make the infinitesimal representation associated to (7) explicit. Recall that the infinitesimal representation for the representation of $\mathbf{GL}\,T$ on alternating forms by inverse adjoints lets the endmorphism $A\;\in\;\mathrm{End}\;T$ act by minus the derivation extension of the adjoint endomorphism A^* with:

$$(\,\mathrm{Der}_{A^*}\Omega\,)(\,X_1,\,X_2,\,\ldots,\,X_r\,)\;\;=\;\;\Omega(\,AX_1,\,X_2,\,\ldots,\,X_r\,)\;+\;\ldots\;+\;\Omega(\,X_1,\,X_2,\,\ldots,\,AX_r\,)$$

For the images I, J and K of the imaginary unit quaternions i, j, $k \in \mathbb{H}$ under φ_q we find the special values $I^*dp = -i\,dp$, $I^*dq = i\,dq$, as well as $J^*dp = dq$, $J^*dq = -dp$, and $K^*dp = -i\,dq$, $K^*dq = -i\,dp$, using (3) and conclude for the infinitesimal representation

$$\begin{aligned} i\star &:= \Phi_q^{-1} \circ (-\operatorname{Der}_{I^*}) \circ \Phi_q = i\,\overline{N} - i\,N \\ j\star &:= \Phi_q^{-1} \circ (-\operatorname{Der}_{J^*}) \circ \Phi_q = \mathrm{Pl} + \overline{\mathrm{Pl}} \\ k\star &:= \Phi_q^{-1} \circ (-\operatorname{Der}_{K^*}) \circ \Phi_q = i\,\mathrm{Pl} - i\,\overline{\mathrm{Pl}} \end{aligned} \tag{8}$$

where the Pücker differential Pl say is defined by (as usual $\widehat{\eta_s}$ denotes an omitted factor):

$$\mathrm{Pl}(\,\eta_1 \wedge \ldots \wedge \eta_k \otimes \tilde{\eta}_1 \wedge \ldots \wedge \tilde{\eta}_{\overline{k}}\,) = \sum_{s=1}^{k} (-1)^{k-s}\,\eta_1 \wedge \ldots \wedge \widehat{\eta_s} \wedge \ldots \wedge \eta_k \otimes \eta_s \wedge \tilde{\eta}_1 \wedge \ldots \wedge \tilde{\eta}_{\overline{k}}$$

The definition of $\overline{\mathrm{Pl}}$ reverses the role of the two sides while keeping $(-1)^{k-s}$, equivalently:

$$\mathrm{Pl} := (-1)^N \circ \left(\sum_\mu e_\mu \lrcorner \otimes de_\mu \wedge \right) \qquad \overline{\mathrm{Pl}} := (-1)^N \circ \left(\sum_\mu de_\mu \wedge \otimes e_\mu \lrcorner \right)$$

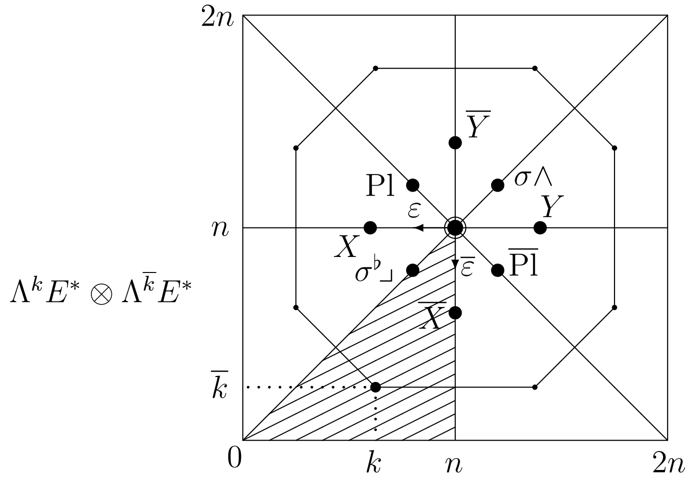
Defnition 2.2 (Natural Operators on $\Lambda^\bullet E^* \otimes \Lambda^\circ E^*$)
Consider the tensor product $\Lambda^\bullet E^* \otimes \Lambda^\circ E^*$ of two copies of the exterior algebra the alternating forms on a complex symplectic vector space E of dimension $2n$ with symplectic form σ and a pair of dual bases $\{e_\mu\}$ and $\{de_\mu\}$. Using the number operators N and $\overline{N}$ of the two tensor factors we can define ten natural, bigraded operators on $\Lambda^\bullet E^* \otimes \Lambda^\circ E^*$, namely two copies

$$\begin{aligned} H &:= n - N & \overline{H} &:= n - \overline{N} \\ X &:= \frac{1}{2} \sum_\mu e_\mu \lrcorner\; de_\mu^\flat \lrcorner \otimes \mathrm{id} & \overline{X} &:= \frac{1}{2} \sum_\mu \mathrm{id} \otimes e_\mu \lrcorner\; de_\mu^\flat \lrcorner \\ Y &:= \frac{1}{2} \sum_\mu de_\mu \wedge e_\mu^\sharp \wedge \otimes \mathrm{id} & \overline{Y} &:= \frac{1}{2} \sum_\mu \mathrm{id} \otimes de_\mu \wedge e_\mu^\sharp \wedge \end{aligned}$$

of $\mathfrak{sl}_2\mathbb{C}$ acting on the left and right tensor factor respectively, and four diagonal operators:

$$\begin{aligned} \mathrm{Pl} &:= (-1)^N \circ \Big(\sum_\mu e_\mu \lrcorner \otimes de_\mu \wedge \Big) & \sigma\wedge &:= (-1)^N \circ \Big(\sum_\mu de_\mu \wedge \otimes e_\mu^\sharp \wedge \Big) \\ \overline{\mathrm{Pl}} &:= (-1)^N \circ \Big(\sum_\mu de_\mu \wedge \otimes e_\mu \lrcorner \Big) & \sigma^\flat\lrcorner &:= (-1)^N \circ \Big(\sum_\mu de_\mu^\flat \lrcorner \otimes e_\mu \lrcorner \Big) \end{aligned}$$

In due course we are going about to prove that the subspace of endomorphisms of $\Lambda^\bullet E^* \otimes \Lambda^\circ E$ spanned by these ten natural operators is actually a Lie algebra isomorphic to $\mathfrak{so}_5\mathbb{C}$ or equivalently $\mathfrak{sp}_4\mathbb{C}$ with maximal torus spanned by H and $\overline{H}$. At this point let us simply point out that H and $\overline{H}$ are linear in the number operators N and $\overline{N}$ so that every bigraded endomorphism is an eigenvector for $\mathrm{ad}\, H$ and $\mathrm{ad}\, \overline{H}$. In terms of the basis $\{\varepsilon, \overline{\varepsilon}\}$ dual to the basis $\{H, \overline{H}\}$ of the maximal torus the eigenvalues of the other operators and the weight spaces $\Lambda^k E^* \otimes \Lambda^{\overline{k}} E^*$ of the representation $\Lambda^\bullet E^* \otimes \Lambda^\circ E^*$ can be read off from the diagram



(9)

where the dashed region indicates a preferred Weyl chamber and the octogon the support of the character of the irreducible representation of $\mathfrak{so}_5\mathbb{C}$ of highest weight $(n - \overline{k})\, \varepsilon + (n - k)\, \overline{\varepsilon}$.

Lemma 2.3 (Action of $\mathfrak{so}_{4,1}\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$ on $\Lambda^\bullet E^* \otimes \Lambda^\circ E^*$)
The subspace of operators on $\Lambda^\bullet E^* \otimes \Lambda^\circ E^*$ spanned linearly over $\mathbb{C}$ by the ten operators of Definition 2.2 is closed under brackets and thus a complex Lie algebra isomorphic to $\mathfrak{so}_5\mathbb{C}$. The real structure on $\Lambda^\bullet E^* \otimes \Lambda^\circ E^*$ induces the real structure on this Lie algebra $\mathfrak{so}_5\mathbb{C}$ of operators indicated by the choice of notation, both $\overline{\sigma\wedge} = -\sigma\wedge$ and $\overline{\sigma^\flat\lrcorner} = -\sigma^\flat\lrcorner$ are imaginary. The real subalgebra fixed by this real structure is isomorphic to $\mathfrak{so}_{4,1}\mathbb{R}$.

Proof: According to the calculation (6) of the commutators of the operators $\sigma\wedge$ and $\sigma^\flat\lrcorner$ on $\Lambda^\bullet E^*$ the two $\mathfrak{sl}_2$–triples $\langle\, H : X : Y \,\rangle$ and $\langle\, \overline{H} : \overline{X} : \overline{Y} \,\rangle$ span commuting $\mathfrak{sl}_2\mathbb{C}$–subalgebras of operators acting on different tensor factors. All operators in the direct sum $\mathfrak{sl}_2\mathbb{C} \oplus \mathfrak{sl}_2\mathbb{C}$ preserve the parity of the bigrading and so commute with the operator $(-1)^N$, hence brackets with this subalgebra leaves the subspace spanned by Pl, $\overline{\mathrm{Pl}}$, $\sigma\wedge$ and $\sigma^\flat\lrcorner$ invariant. More precisely the fundamental commutation relations (5) tell us that:

$$
\begin{array}{rclrclrcl}
[\, H,\ \mathrm{Pl}\,] &=& \mathrm{Pl} & [\, X,\ \mathrm{Pl}\,] &=& 0 & [\, Y,\ \mathrm{Pl}\,] &=& \sigma\wedge \\
[\, H,\ \overline{\mathrm{Pl}}\,] &=& -\overline{\mathrm{Pl}} & [\, X,\ \overline{\mathrm{Pl}}\,] &=& -\sigma^\flat\lrcorner & [\, Y,\ \overline{\mathrm{Pl}}\,] &=& 0 \\
[\, H,\ \sigma\wedge\,] &=& -\sigma\wedge & [\, X,\ \sigma\wedge\,] &=& \mathrm{Pl} & [\, Y,\ \sigma\wedge\,] &=& 0 \\
[\, H,\ \sigma^\flat\lrcorner\,] &=& \sigma^\flat\lrcorner & [\, X,\ \sigma^\flat\lrcorner\,] &=& 0 & [\, Y,\ \sigma^\flat\lrcorner\,] &=& -\overline{\mathrm{Pl}}
\end{array}
$$

Instead of verifying the missing commutation relations with $\overline{H}$, $\overline{X}$ and $\overline{Y}$ directly we can infer these relations using the reality $\overline{[A, B]} = [\overline{A}, \overline{B}]$ of the commutator. For this reason we skip this point and proceed to determine the real structure induced by the real structure on $\Lambda^\bullet E^* \otimes \Lambda^\circ E^*$ made explicit in Remark 2.1. This real structure interchanges the tensor factors and hence interchanges the corresponding endomorphisms $H \leftrightarrow \overline{H}$, $X \leftrightarrow \overline{X}$, $Y \leftrightarrow \overline{Y}$, say:

$$X\, (\, \overline{\eta \otimes \tilde{\eta}}\,) \;=\; (-1)^{|\tilde{\eta}|\,(\,|\eta|\,+\,1\,)}\, C(\, \sigma^\flat\lrcorner\; \tilde{\eta}\,) \otimes C\eta \;=\; \overline{\overline{X}\, (\, \eta \otimes \tilde{\eta}\,)}$$

The main point of the argument concerning the operators Pl, $\overline{\text{Pl}}$, $\sigma\wedge$ and $\sigma^\flat\lrcorner$ is that these operators change the parity of both (sic!) factors, with this in mind we find for example:

$$\sigma\wedge(\overline{\eta\otimes\tilde{\eta}}) \;=\; (-1)^{|\tilde{\eta}|\,|\eta|+1}\sum_\mu (\,Cde_\mu\,)\wedge(\,C\tilde{\eta}\,)\otimes(\,Ce^\sharp_\mu\,)\wedge(\,C\eta\,) \;=\; -\,\overline{\sigma\wedge(\eta\otimes\tilde{\eta})}$$

Last but not least we have to calculate the commutators of the operators Pl, $\overline{\text{Pl}}$, $\sigma\wedge$ and $\sigma^\flat\lrcorner$. To begin with we note that these operators anticommute with $(-1)^N$, hence the $(-1)^N$–factor of square 1 conveniently turns into a minus sign in all the calculations:

$$\begin{aligned}[\,\text{Pl},\,\overline{\text{Pl}}\,] &= -\sum_{\mu\nu}\Big(\{\,e_\mu\lrcorner\,,\,de_\nu\wedge\}\otimes de_\mu\wedge e_\nu\lrcorner \;-\; de_\nu\wedge e_\mu\lrcorner\otimes\{\,de_\mu\wedge,\,e_\nu\lrcorner\,\}\Big)\\ &= -\,\text{id}\otimes\overline{N}\,+\,N\otimes\text{id} \;=\; \overline{H}\,-\,H\end{aligned}$$

We omit the analoguous calculation of the remaining commutators and tabulate the result:

$$\begin{array}{lll} [\,\text{Pl},\ \overline{\text{Pl}}\,] \;=\; \overline{H}-H & [\,\text{Pl},\,\sigma\wedge\,] \;=\; -2\,\overline{Y} & [\,\text{Pl},\,\sigma^\flat\lrcorner\,] \;=\; -2\,X\\ [\,\sigma\wedge,\,\sigma^\flat\lrcorner\,] \;=\; \overline{H}+H & [\,\overline{\text{Pl}},\,\sigma\wedge\,] \;=\; 2\,Y & [\,\overline{\text{Pl}},\,\sigma^\flat\lrcorner\,] \;=\; 2\,\overline{X}\end{array}$$

In order to prove that the algebra of operators is isomorphic to $\mathfrak{so}_5\mathbb{C}$ we observe that the ten operators of Definition 2.2 all occur in at least one $\mathfrak{sl}_2\mathbb{C}$–subalgebra of operators, more precisely the relevant $\mathfrak{sl}_2\mathbb{C}$–subalgebras are spanned by the $\mathfrak{sl}_2$–triples of operators:

$$\langle\,H:X:Y\,\rangle \qquad \langle\,\overline{H}:\overline{X}:\overline{Y}\,\rangle \qquad \langle\,H-\overline{H}:\text{Pl}:-\overline{\text{Pl}}\,\rangle \qquad \langle\,H+\overline{H}:\sigma^\flat\lrcorner\,:-\sigma\wedge\,\rangle$$

On the other hand the trace form of every finite–dimensional representation R of $\mathfrak{sl}_2\mathbb{C}$ satisfies $\text{tr}_R(H^2) = 2\,\text{tr}_R(XY)$, it is actually sufficient to verify this for the symmetric powers $\text{Sym}^k\mathbb{C}^2$ of the defining representation $\mathbb{C}^2$ of $\mathfrak{sl}_2\mathbb{C}$. Consulting the weight diagram (9) for the eigenvalues of the diagonalizable operators ad H and ad $\overline{H}$ we find immediately

$$B(\,H,\,H\,) \;=\; 12 \;=\; B(\,\overline{H},\,\overline{H}\,) \qquad\qquad B(\,H,\,\overline{H}\,) \;=\; 0$$

for the trace form B of the adjoint representation, the so called Killing form, and conclude $B(H+\overline{H},H+\overline{H}) = 24 = B(H-\overline{H},H-\overline{H})$. The remaining non–zero values of B are

$$B(\,X,\,Y\,) \;=\; 6 \;=\; B(\,\overline{X},\,\overline{Y}\,) \qquad\qquad B(\,\text{Pl},\,\overline{\text{Pl}}\,) \;=\; -12 \;=\; B(\,\sigma^\flat\lrcorner\,,\,\sigma\wedge\,)$$

all other tuples of basis vectors are orthogonal/isotropic by equivariance alone. Having a non–degenerate Killing form the Lie algebra of operators on $\Lambda^\bullet E^*\otimes\Lambda^\circ E^*$ spanned by the ten operators of Definition 2.2 is semisimple and the weight diagram (9) tells us that it is isomorphic to $\mathfrak{so}_5\mathbb{C}$. An interesting twist to this argument allows us to determine the signature of the Killing form B restricted to the real subalgebra and thus its isomorphism type as well. Evidently the real structure preserves the orthogonal decomposition

$$\text{span}_{\mathbb{C}}\{\,H,\,\overline{H}\,\}\,\oplus\,\text{span}_{\mathbb{C}}\{\,X,\,\overline{X},\,Y,\,\overline{Y}\,\}\,\oplus\,\text{span}_{\mathbb{C}}\{\,\text{Pl},\,\overline{\text{Pl}}\,\}\,\oplus\,\text{span}_{\mathbb{C}}\{\,\sigma\wedge,\,\sigma^\flat\lrcorner\,\}$$

into subspaces of signature $(1,1)$, $(2,2)$, $(0,2)$ and $(1,1)$. Restricted to the real subalgebra the Killing form thus has signature $(4,6)$ characterizing the real form $\mathfrak{so}_{4,1}\mathbb{R}$ of $\mathfrak{so}_5\mathbb{C}$. □

Remark 2.4 (Explicit Isomorphism with $\mathfrak{so}_{4,1}\mathbb{R}$)
Consider the realization $\mathfrak{so}_r\mathbb{C} = \{ A \in \mathrm{Mat}_{r\times r}\mathbb{C} \mid A^T S + SA = 0 \}$ *of the complex Lie algebra* $\mathfrak{so}_r\mathbb{C}$ *associated to a non–degenerate, real, symmetric* $r \times r$*–matrix* S*. The signature of* S *determines the isomorphism type of the standard real structure* $\mathfrak{so}_r\mathbb{C} \longrightarrow \mathfrak{so}_r\mathbb{C}$, $A \longmapsto \overline{A}$, *given by conjugation of coefficients. In particular the symmetric matrix of signature* $(4, 1)$

$$S \;:=\; \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

defines a real structure with real form $\mathfrak{so}_{4,1}\mathbb{R}$ *on* $\mathfrak{so}_5\mathbb{C}$*. Sending the operators* H, X, Y *to*

$$\begin{pmatrix} 0 & i & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & -i & 0 & 0 & 0 \end{pmatrix}$$

respectively and the operators Pl, $\sigma\wedge$ *and* $\sigma^\flat\lrcorner$ *similarly to the matrices*

$$\begin{pmatrix} 0 & 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -i & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \sqrt{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 \end{pmatrix} \qquad \sqrt{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

defines a real isomorphism between the algebra of operators on $\Lambda^\bullet E^* \otimes \Lambda^\circ E^*$ *and* $\mathfrak{so}_5\mathbb{C}$.

There are other ways to construct the algebra $\mathfrak{so}_{4,1}\mathbb{R}$ besides Definition 2.2 closer in spirit to the standard procedure in Kähler geometry. Research on this idea and the resulting algebra goes back at least to the extensive work of Bonan [B] [V], who introduced the algebra $\mathfrak{so}_{4,1}\mathbb{R}$ by considering all ways to wedge and contract with Kähler forms. An orthogonal quaternionic structure $Q \subset \mathrm{End}\, T$ gives rise to three linearly independent Kähler forms $\omega_I = g(I\cdot, \cdot)$ etc. associated to the images I, J, K of the unit quaternions i, j, $k \in \mathbb{H}$ under an isomorphism $\mathbb{H} \longrightarrow Q$. Every such isomorphism is induced by the choice of a unit vector $q \in H$ satisfying $\sigma(Cq, q) = 1$ and is in this way related to a canonical basis p, q of H with $p := Cq$. In terms of this basis the scalar product on $T \otimes_\mathbb{R} \mathbb{C} \cong H \otimes E$ can be written:

$$g \;=\; \sigma \otimes \sigma \;=\; \sum_\mu \Big((\, dp \otimes de_\mu \,) \otimes (\, dq \otimes e^\sharp_\mu \,) \;-\; (\, dq \otimes de_\mu \,) \otimes (\, dp \otimes e^\sharp_\mu \,) \Big)$$

The equivalent formulation $\omega_I = (I^* \otimes \mathrm{id})(\, g \,)$ of the definition of the Kähler form together with the explicit values $I^* dp = -i\, dp$, $I^* dq = i\, dq$ as well as $J^* dp = dq$, $J^* dq = -dp$ and

$K^*dp = -i\,dq$, $K^*dq = -i\,dp$ allows us to expand the three Kähler forms according to:

$$\begin{aligned}
\omega_I &= -\,i\sum_\mu (\,dp\otimes de_\mu\,)\wedge(\,dq\otimes e^\sharp_\mu\,)\\
\omega_J &= \frac{1}{2}\sum_\mu \Big(\,(\,dp\otimes de_\mu\,)\wedge(\,dp\otimes e^\sharp_\mu\,)\;+\;(\,dq\otimes de_\mu\,)\wedge(\,dq\otimes e^\sharp_\mu\,)\,\Big)\\
\omega_K &= \frac{i}{2}\sum_\mu \Big(\,(\,dp\otimes de_\mu\,)\wedge(\,dp\otimes e^\sharp_\mu\,)\;-\;(\,dq\otimes de_\mu\,)\wedge(\,dq\otimes e^\sharp_\mu\,)\,\Big)
\end{aligned}$$

Some additional considerations based on these expansions lead to the description

$$\begin{array}{rclrcl}
\Phi_q^{-1}\circ(\,\omega_I\wedge\,)\circ\Phi_q &=& i\,\sigma\wedge & \Phi_q^{-1}\circ(\,\omega_I^\flat\,\lrcorner\,)\circ\Phi_q &=& i\,\sigma^\flat\lrcorner\\
\Phi_q^{-1}\circ(\,\omega_J\wedge\,)\circ\Phi_q &=& \overline{Y}\;+\;Y & \Phi_q^{-1}\circ(\,\omega_J^\flat\,\lrcorner\,)\circ\Phi_q &=& X\;+\;\overline{X}\\
\Phi_q^{-1}\circ(\,\omega_J\wedge\,)\circ\Phi_q &=& i\,\overline{Y}\;-\;i\,Y & \Phi_q^{-1}\circ(\,\omega_K^\flat\,\lrcorner\,)\circ\Phi_q &=& i\,X\;-\;i\,\overline{X}
\end{array}$$

of wedge products and contractions with Kähler forms in terms of the Lie algebra $\mathfrak{so}_{4,1}\mathbb{R}$, conversely these six operators evidently suffice to generate all $\mathfrak{so}_{4,1}\mathbb{R}$. Recall now that the isomorphism Φ_q conjugates the representation of $\mathbf{Sp}\,H\subset\mathbf{SO}\,T$ on the alternating forms $\Lambda^{\bullet,\circ}T^*\otimes_\mathbb{R}\mathbb{C}$ on the euclidian vector space T to a representation of the group $\mathbf{Sp}(1)$ on $\Lambda^\bullet E^*\otimes\Lambda^\circ E^*$, whose infinitesimal generators we have calculated in equation (8):

$$\begin{aligned}
i\,\star &= \Phi_q^{-1}\circ(\,-\mathrm{Der}_{I^*}\,)\circ\Phi_q &&= i\,H\;-\;i\,\overline{H}\\
j\,\star &= \Phi_q^{-1}\circ(\,-\mathrm{Der}_{J^*}\,)\circ\Phi_q &&= \mathrm{Pl}\;+\;\overline{\mathrm{Pl}}\\
k\,\star &= \Phi_q^{-1}\circ(\,-\mathrm{Der}_{K^*}\,)\circ\Phi_q &&= i\,\mathrm{Pl}\;-\;i\,\overline{\mathrm{Pl}}
\end{aligned}$$

Somewhat more general the infinitesimal representation of the Lie algebra $\mathfrak{so}_{4,1}\mathbb{R}$ of operators on $\Lambda^\bullet E^*\otimes\Lambda^\circ E^*$ integrates to an actual represention of the unique connected and simply connected Lie group $\mathbf{Spin}^+_{4,1}\mathbb{R}$ with Lie algebra $\mathfrak{so}_{4,1}\mathbb{R}$. The weight diagram (9) tells us that the central element $-1\in\mathbf{Spin}^+_{4,1}\mathbb{R}$ acts as $(-1)^{k+\bar{k}}$ on $\Lambda^{k,\bar{k}}T^*\otimes_\mathbb{R}\mathbb{C}\cong\Lambda^kE^*\otimes\Lambda^{\bar{k}}E^*$, hence this representation does not descend to $\mathbf{SO}^+_{4,1}\mathbb{R}$. Nevertheless the invariant subspace $\Lambda^{\mathrm{ev}}T^*\otimes_\mathbb{R}\mathbb{C}$ of forms of even total degree is a genuine representation of $\mathbf{SO}^+_{4,1}\mathbb{R}$.

Besides the subgroup $\mathbf{Sp}(1)\subset\mathbf{Spin}^+_{4,1}\mathbb{R}$ conjugated to the factor $\mathbf{Sp}\,H\subset\mathbf{SO}\,T$ of the holonomy group we want to mention two other interesting subgroups of $\mathbf{Spin}^+_{4,1}\mathbb{R}$, namely the double cover $\mathbf{SL}_2\mathbb{C}\cong\mathbf{Spin}^+_{3,1}\mathbb{R}$ of the Lorentz group and the maximal compact subgroup $\mathbf{Spin}_4\mathbb{R}$. Interestingly there seems to be no natural choice for the latter, and it is tempting to try to relate this oddity to geometrical and topological properties of hyperkähler or quaternionic Kähler manifolds. Concerning the former we note that the involutive automorphism of $\mathfrak{so}_5\mathbb{C}$ given by conjugation with $(-1)^N$ fixes the subalgebra $\mathfrak{sl}_2\mathbb{C}\oplus\mathfrak{sl}_2\mathbb{C}\subset\mathfrak{so}_5\mathbb{C}$ spanned by the operators H, X, Y and $\overline{H}$, $\overline{X}$, $\overline{Y}$. The resulting symmetric pair of real Lie algebras

$$\mathfrak{so}_{4,1}\mathbb{R}\;=\;\mathfrak{so}_{3,1}\mathbb{R}\oplus\mathbb{R}^{3,1}\;\supset\;\mathfrak{so}_{3,1}\mathbb{R}$$

defines a Lorentz symmetric space, the de Sitter space $\mathbf{Spin}^+_{4,1}\mathbb{R}/\mathbf{SL}_2\mathbb{C}$. Incidentally we note that the inclusion of the factor $(-1)^N$ in the definition of the operators Pl, $\overline{\mathrm{Pl}}$, $\sigma\wedge$ and $\sigma^\flat\lrcorner$ has the same effect on the symmetric pair $\mathfrak{so}_{4,1}\mathbb{R} \supset \mathfrak{so}_{3,1}\mathbb{R}$ as the multiplication by i in the eigenspace of eigenvalue -1 in the classical construction of the dual symmetric pair.

Interestingly the construction of the algebra $\mathfrak{so}_{4,1}\mathbb{R}$ acting on $\Lambda^\bullet E^* \otimes \Lambda^\circ E^*$ is a special case of a general property of representation theory related to Howe's Theory of dual pairs. On the tensor product of r copies of the exterior or symmetric algebra of a complex vector space V lives a natural simple Lie algebra of operators isomorphic to $\mathfrak{gl}_r\mathbb{C}$, consisting of the $r(r-1)$ possible generalizations of the Plücker differentials Pl and $\overline{\mathrm{Pl}}$ and r shifted number operators like H or $\overline{H}$. If either the factors are exterior algebras and the vector space V is symplectic or the factors are symmetric algebras and V is euclidian, then we can extend this algebra by $r(r+1)$ additional operators like X or $\sigma\wedge$ and the resulting algebra is isomorphic to $\mathfrak{sp}_{2r}\mathbb{C} = \mathfrak{sp}(r)\otimes_\mathbb{R}\mathbb{C}$. The remarkble thing about this construction is that Weyl's construction of the irreducible representations of the classical matrix groups using Schur functors [FH] tells us that under rather general assumptions on r and $\dim V$ the multiplicity spaces for the decomposition of the tensor product into irreducible representations of $\mathfrak{gl}_r\mathbb{C}$ or $\mathfrak{sp}_{2r}\mathbb{C}$ are irreducible representations of the automorphism group of V!

The algebra $\mathfrak{so}_{4,1}\mathbb{R}\otimes_\mathbb{R}\mathbb{C} \cong \mathfrak{sp}_4\mathbb{C}$ acting on $\Lambda^\bullet E^* \otimes \Lambda^\circ E^*$ is a special example of this construction. The space of highest weight vectors of $\mathfrak{so}_{4,1}\mathbb{R}$ in $\Lambda^\bullet E^*\otimes\Lambda^\circ E^*$ given a dominant weight $(n-\overline{k})\,\overline{\varepsilon} + (n-k)\,\varepsilon$ in the Weyl chamber indicated in diagram (9) is by definition the intersection of the kernels of the four operators X, $\sigma^\flat\lrcorner$, $\overline{X}$ and $\overline{\mathrm{Pl}}$ corresponding to the positive roots with the weight space $\Lambda^k E^* \otimes \Lambda^{\overline{k}} E^*$. Denoting the irreducible representation of $\mathfrak{so}_{4,1}\mathbb{R}$ of highest weight λ by $R^{\mathfrak{so}_{4,1}\mathbb{R}}_\lambda$ we can rewrite this definition in the following way

$$\mathrm{Hom}_{\,\mathfrak{so}_{4,1}\mathbb{R}}(\ R^{\mathfrak{so}_{4,1}\mathbb{R}}_{(n-\overline{k})\,\overline{\varepsilon}\,+\,(n-k)\,\varepsilon},\ \Lambda\,E^*\otimes\Lambda\,E^*\ ) \ = \ \ker X\ \cap\ \ker\overline{\mathrm{Pl}}\ \cap\ (\,\Lambda^k E^*\otimes\Lambda^{\overline{k}}E^*\,)$$

because the operators X and $\overline{\mathrm{Pl}}$ corresponding to the simple roots generate the subalgebra of operators corresponding to positive roots. According to Weyl's construction of the irreducible representations of the symplectic Lie groups as trace–free Schur functors however the right hand side is exactly the irreducible representation $\Lambda^{k,\overline{k}}_\circ E^* \cong \Lambda^{k,\overline{k}}_\circ E$ of $\mathbf{Sp}\,E$ of highest weight $2\varepsilon_1+\ldots+2\varepsilon_{\overline{k}}+\varepsilon_{\overline{k}+1}+\ldots+\varepsilon_k$, where $\pm\varepsilon_1,\ldots,\pm\varepsilon_n$ are the weights of the defining representation E of $\mathbf{Sp}\,E$. In consequence the complete decomposition of the tensor product $\Lambda\,E^*\otimes\Lambda\,E^*$ of two copies of the exterior algebra of a symplectic vector space E^* into irreducible representations reads for either member of the Howe dual pair $\mathbf{Sp}\,E$ and $\mathfrak{so}_{4,1}\mathbb{R}$:

$$\Lambda^{\bullet,\circ}T^*\otimes_\mathbb{R}\mathbb{C} \ = \ \Lambda^\bullet E^*\otimes\Lambda^\circ E^* \ = \bigoplus_{n\geq r\geq\overline{r}\geq 0} R^{\mathfrak{so}_{4,1}\mathbb{R}}_{(n-\overline{r})\,\overline{\varepsilon}\,+\,(n-r)\,\varepsilon}\ \otimes\ \Lambda^{r,\overline{r}}_\circ E \tag{10}$$

A detailed discussion of Weyl's construction is outside the scope of this article, the reader is referred to [FH] instead, nevertheless we want to give a very brief sketch of Weyl's argument for the irreducibility of $\ker X\cap\ker\overline{\mathrm{Pl}} \subset \Lambda^k E^*\otimes\Lambda^{\overline{k}}E^*$ as a representation of $\mathbf{Sp}\,E$. The most difficult aspect of Weyl's argument concerns proving the surjectivity of the restriction

$$\sigma^\flat\lrcorner\ :\quad \Lambda^k_\circ E^*\ \otimes\ \Lambda^{\overline{k}}_\circ E^*\ \longrightarrow\ \Lambda^{k-1}_\circ E^*\ \otimes\ \Lambda^{\overline{k}-1}_\circ E^*$$

of the operator $\sigma^\flat \lrcorner$ to the joint kernel $\Lambda^k_\circ E^* \otimes \Lambda^{\bar{k}}_\circ E^* \subset \Lambda^k E^* \otimes \Lambda^{\bar{k}} E^*$ of the two commuting operators X and $\overline{X}$ in bidegrees $n \geq k, \bar{k} \geq 0$. Because of surjectivity we can calculate the dimension of the joint kernel $\Lambda^\bullet_\circ E^* \otimes_\circ \Lambda^\circ_\circ E^*$ of the three operators X, $\overline{X}$ and $\sigma^\flat \lrcorner$ from the known dimension of $\Lambda^k_\circ E^* \otimes \Lambda^{\bar{k}}_\circ E^*$. In turn the surjectivity of the restriction

$$\overline{\mathrm{Pl}} : \quad \Lambda^k_\circ E^* \otimes_\circ \Lambda^{\bar{k}}_\circ E^* \longrightarrow \Lambda^{k+1}_\circ E^* \otimes_\circ \Lambda^{\bar{k}-1}_\circ E^*$$

of the operator $\overline{\mathrm{Pl}}$ to the joint kernel of X, $\overline{X}$ and $\sigma^\flat \lrcorner$ for bidegrees $k \geq \bar{k}$ is a direct consequence of representation theory of $\mathfrak{sl}_2$ so that we eventually end up calculating the dimension of the joint kernel $\Lambda^{k,\bar{k}}_\circ E^*$ of all four operators X, $\overline{X}$, $\sigma^\flat \lrcorner$ and $\overline{\mathrm{Pl}}$ in bidegrees $n \geq k \geq \bar{k} \geq 0$. Choosing on the other hand a complex basis $\eta_1, \ldots, \eta_n \in L$ of a maximal isotropic or Lagrangian subspace $L \subset E^*$ we note that for all $n \geq k \geq \bar{k} \geq 0$ the form

$$\eta_1 \wedge \ldots \wedge \eta_k \otimes \eta_1 \wedge \ldots \wedge \eta_{\bar{k}} \quad \in \quad \Lambda^{k,\bar{k}}_\circ E^*$$

is a highest weight vector in $\Lambda^{k,\bar{k}}_\circ E^*$ of weight $2\varepsilon_1 + \ldots + 2\varepsilon_{\bar{k}} + \varepsilon_{\bar{k}+1} + \ldots + \varepsilon_k$ for a suitable choice of a maximal torus of $\mathbf{Sp}\,E$ and a suitable ordering of weights. In consequence the joint kernel $\Lambda^{k,\bar{k}}_\circ E^*$ of X, $\overline{X}$, $\sigma^\flat \lrcorner$ and $\overline{\mathrm{Pl}}$ contains at least the irreducible representation of $\mathbf{Sp}\,E$ of this specific highest weight, by the dimension part of Weyl's Character Formula however the dimension of this irreducible summand agrees with the dimension of $\Lambda^{k,\bar{k}}_\circ E^*$ we have calculated before proving the irreducibility of the latter under $\mathbf{Sp}\,E$.

In concluding this section we want to formulate some straightforward consequences of the decomposition (10) into irreducible representations. Say the maximal torus $\mathbb{R} \oplus i\mathbb{R}$ of $\mathfrak{so}_{4,1}\mathbb{R}$ is generated by $\overline{H} + H$ and $i\,\overline{H} - i\,H$, which take the values $\bar{k} + k$ and $i\bar{k} - ik$ on the weight space $\Lambda^k E^* \otimes \Lambda^{\bar{k}} E^* \cong \Lambda^{k,\bar{k}} T^* \otimes_{\mathbb{R}} \mathbb{C}$. Branching from $\mathfrak{so}_{4,1}\mathbb{R}$ to $\mathbb{R} \oplus i\mathbb{R}$ thus calculates the decomposition of $\Lambda^{k,\bar{k}} T^* \otimes_{\mathbb{R}} \mathbb{C}$ under the holonomy group $\mathbf{Sp}(n)$ of hyperkähler manifolds:

Lemma 2.5 (Differential Forms on Hyperkähler Manifolds)
Under the action of $\mathbf{Sp}\,E \subset \mathbf{SO}\,T$ the space $\Lambda^{k,\bar{k}} T^* \otimes_{\mathbb{R}} \mathbb{C}$ of complex $(k, \bar{k})$–forms with respect to the complex structure I decomposes into a sum of representations $\Lambda^{r,\bar{r}}_\circ E$ with multiplicities

$$\Lambda^{k,\bar{k}} T^* \otimes_{\mathbb{R}} \mathbb{C} \;\cong\; \bigoplus_{n \geq r \geq \bar{r} \geq 0} \dim \mathrm{Hom}_{\,\mathbb{R}\oplus i\mathbb{R}} \Big(R^{\mathbb{R}\oplus i\mathbb{R}}_{(n-\bar{k})\,\bar{\varepsilon} + (n-k)\,\varepsilon},\; R^{\mathfrak{so}_{4,1}\mathbb{R}}_{(n-\bar{r})\,\bar{\varepsilon} + (n-r)\,\varepsilon} \Big)\, \Lambda^{r,\bar{r}}_\circ E$$

given by the multiplicity of the irreducible representation of highest weight $(n - \bar{k})\,\bar{\varepsilon} + (n - k)\,\varepsilon$ for $\mathbb{R} \oplus i\mathbb{R}$ in the irreducible representation of $\mathfrak{so}_{4,1}\mathbb{R}$ of highest weight $(n - \bar{r})\,\bar{\varepsilon} + (n - r)\,\varepsilon$.

On quaternionic Kähler manifolds we are interested instead in decomposing $\Lambda^\bullet T^* \otimes_{\mathbb{R}} \mathbb{C}$ under the holonomy group $\mathbf{Sp}\,H \cdot \mathbf{Sp}\,E$ or equivalently $\Lambda^\bullet E^* \otimes \Lambda^\circ E^*$ under $\mathbf{Sp}(1) \cdot \mathbf{Sp}\,E$ using the isomorphism Φ_q, where the factor $\mathbf{Sp}(1) \subset \mathbf{Spin}^+_{4,1}\mathbb{R}$ has Lie algebra $\mathfrak{sp}(1) \subset \mathfrak{so}_{4,1}\mathbb{R}$ generated by $iH - i\overline{H}$, $\mathrm{Pl} + \overline{\mathrm{Pl}}$ and $i\mathrm{Pl} - i\overline{\mathrm{Pl}}$, compare equation (8). It is convenient to consider the central extension $\mathbb{R} \times \mathbf{Sp}(1) \subset \mathbf{Spin}^+_{4,1}\mathbb{R}$ of $\mathbf{Sp}(1)$ with Lie algebra generated by $\mathfrak{sp}(1)$ and $H + \overline{H}$ in order to keep track of the total degree of a differential form. To wit $e^{t\,(H+\overline{H})} \in \mathbb{R} \subset \mathbf{Spin}^+_{4,1}\mathbb{R}$ acts by multiplication with $e^{t\,(2n-k)}$ on forms $\Lambda E^* \otimes \Lambda E^*$ of total degree k similar to its action on the irreducible representation $\mathbb{C}_{(n-\frac{k}{2})\,(\overline{\varepsilon}+\varepsilon)} \otimes \mathrm{Sym}^s\mathbb{H}$ of $\mathbb{R}\times\mathbf{Sp}(1)$ of highest weight $(n-\frac{k}{2})\,(\overline{\varepsilon}+\varepsilon) + \frac{s}{2}\,(\overline{\varepsilon}-\varepsilon) = (n-\frac{k-s}{2})\,\overline{\varepsilon} + (n-\frac{k+s}{2})\,\varepsilon$:

Lemma 2.6 (Differential Forms on Quaternionic Kähler Manifolds)
Under the action of $\mathbf{Sp}\,H\cdot\mathbf{Sp}\,E \subset \mathbf{SO}\,T$ *the space of complex valued differential forms on a euclidian vector space* T *of dimension* $4n$ *with orthogonal quaternionic structure* $Q \subset \mathrm{End}\,T$ *decomposes into a sum of irreducible representations* $\mathrm{Sym}^s H \otimes \Lambda^{r,\overline{r}}_\circ E^*$ *with multiplicities*

$$\Lambda^k T^* \otimes_{\mathbb{R}} \mathbb{C} \;\cong\; \bigoplus_{\substack{s\geq 0\\ n\geq r\geq\overline{r}\geq 0}} \dim \mathrm{Hom}_{\,\mathbb{R}\oplus\mathfrak{sp}(1)}\Big(R^{\mathbb{R}\oplus\mathfrak{sp}(1)}_{(n-\frac{k-s}{2})\,\overline{\varepsilon}+(n-\frac{k+s}{2})\,\varepsilon},\, R^{\mathfrak{so}_{4,1}\mathbb{R}}_{(n-\overline{r})\,\overline{\varepsilon}+(n-r)\,\varepsilon}\Big)\,\mathrm{Sym}^s H \otimes \Lambda^{r,\overline{r}}_\circ E$$

given by the multiplicity of the irreducible representation of $\mathbb{R}\oplus\mathfrak{sp}(1)$ *of highest weight* $(n-\frac{k-s}{2})\,\overline{\varepsilon} + (n-\frac{k+s}{2})\,\varepsilon$ *in the irreducible representation of* $\mathfrak{so}_{4,1}\mathbb{R}$ *for* $(n-\overline{r})\,\overline{\varepsilon} + (n-r)\,\varepsilon$.

3 Quaternionic Kähler Decomposition of Forms

Of course neither Lemma 2.5 nor Lemma 2.6 are really explicit decomposition formulas, but they reduce the problem of decomposing the differential forms on a hyperkähler manifold or quaternionic Kähler manifold of quaternionic dimension n under the holonomy group $\mathbf{Sp}(n)$ or $\mathbf{Sp}(1)\cdot\mathbf{Sp}(n)$ respectively to finding the branching rules from $\mathfrak{so}_{4,1}\mathbb{R}$ to $\mathbb{R}\oplus i\mathbb{R}$ or $\mathbb{R}\oplus\mathfrak{sp}(1)$. In this section we will describe a rather general strategy to solve this standard problem in representation theory for a pair $\mathfrak{g}\supset\mathfrak{h}$ of real reductive Lie algebras. Using this strategy we will readily turn Lemma 2.5 and Lemma 2.6 into effective decomposition formulas.

Consider for a moment the Lie algebra $\mathfrak{g}$ of a compact group G and for each $X\in\mathfrak{g}$ the adjoint endomorphism $\mathrm{ad}\,X:\,\mathfrak{g}\longrightarrow\mathfrak{g},\ Y\longmapsto[X,Y]$. With G being compact there exists a G–invariant negative definite scalar product B on $\mathfrak{g}$, for example we can take the Killing form $B(X,Y):=\mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}\,X\circ\mathrm{ad}\,Y)$ if G is semisimple. On the Zariski open subset of regular elements $X\in\mathfrak{g}$ the rank of $\mathrm{ad}\,X$ becomes maximal, in turn its kernel becomes minimal and defines a maximal abelian subalgebra of $\mathfrak{g}$ aka a maximal torus $\mathfrak{t}=\{\,H\in\mathfrak{g}\,|\,[X,H]=0\,\}$. With $\mathfrak{g}$ being the Lie algebra of a compact group G the kernel of the exponential map $\exp:\,\mathfrak{g}\longrightarrow G,\ X\longmapsto e^X$, restricted to $\mathfrak{t}$ is a lattice in $\mathfrak{t}$, whose dual lattice

$$\Lambda \;:=\; \{\,\lambda\in i\,\mathfrak{t}^*\,|\;\lambda(\,X\,)\in 2\pi i\,\mathbb{Z}\text{ for all }X\in\mathfrak{t}\text{ with }e^X=1\,\}\;\subset\; i\,\mathfrak{t}^*$$

is called the weight lattice of G. In general the weight lattice Λ becomes finer if we replace G by a compact covering group $\tilde{G}$, in other words the weight lattice encodes the global structure of the Lie group G and does not only depend on its Lie algebra $\mathfrak{g}$.

A very useful tool for the calculations to come is the group ring $\mathbb{Z}\,\Lambda$ of the weight lattice Λ considered as an additive group with coefficients in $\mathbb{Z}$. Elements of the group ring $\mathbb{Z}\,\Lambda$ are appropriately thought of as finite formal sums of terms $c\,e^\lambda$ with $c\in\mathbb{Z}$ and $\lambda\in\Lambda$, because the naive multiplication of such formal sums agrees with the standard convolution product in the group ring $\mathbb{Z}\Lambda$. In particular the character of every finite–dimensional representation V of the compact group G finds a natural home in the ring $\mathbb{Z}\Lambda$

$$\mathrm{ch}\,V \;:=\; \sum_{\lambda\in\Lambda} (\,\dim E_\lambda\,)\,e^\lambda$$

where E_λ is the generalized complex eigenspace of the action of the maximal torus on V:

$$E_\lambda \;:=\; \{\,v\,\in\,V\otimes_{\mathbb{R}}\mathbb{C}\,|\;\;H\star v \;=\; \lambda(H)\,v\,\}$$

The character ch defined this way is a ring homomorphism from the representation ring RG of the group G to the group ring $\mathbb{Z}\Lambda$ of the weight lattice, in other words the equalities $\mathrm{ch}\,V\oplus W \;=\; \mathrm{ch}\,V\;+\;\mathrm{ch}\,W$ and $\mathrm{ch}\,V\otimes W \;=\; (\mathrm{ch}\,V)\,(\mathrm{ch}\,W)$ hold true. For connected G the character ch is injective, moreover its image is precisely the subring $[\,\mathbb{Z}\Lambda\,]^{\mathfrak{W}}$ of elements invariant under the Weyl group $\mathfrak{W} \;=\; \mathrm{Norm}\,T/T$ of G acting on $\mathbb{Z}\Lambda$ by the linear extension of $(w,\,e^\lambda)\longmapsto e^{w\lambda}$ with $(w\lambda)(X)\;:=\;\lambda(\mathrm{Ad}_w^{-1}X)$. Incidentally we note that the automorphism $*\,:\;\mathbb{Z}\Lambda\longrightarrow\mathbb{Z}\Lambda,\,e^\lambda\longmapsto e^{-\lambda}$, is useful to "dualize" the character $\mathrm{ch}(\,V^*\,)\;=\;(\mathrm{ch}\,V)^*$ of a representation V. For group rings like $\mathbb{Z}\Lambda$ the analogue of the residue of complex analysis

$$\mathrm{ev}_\lambda:\;\;\mathbb{Z}\Lambda\;\longrightarrow\;\mathbb{Z},\qquad e^\mu\;\longmapsto\;\delta_{\lambda=\mu}$$

can be introduced to pick up the coefficient of the weight $\lambda\in\Lambda$ in a given element of $\mathbb{Z}\Lambda$.

The weights (in the support of the character) of the adjoint representation $\mathfrak{g}$ of G are of special importance and are called roots. Choosing once and for all a regular element $X\in\mathfrak{t}$ in a fixed maximal torus $\mathfrak{t}$ we can classify a root α according to the value of $i\alpha(X)\;\in\;\mathbb{R}\setminus\{0\}$ as positive or negative. In turn an imaginary valued linear form $\lambda\;\in\;i\mathfrak{t}^*$ on $\mathfrak{t}$ is called dominant, if it has non–negative scalar product $B(\lambda,\alpha)\geq 0$ with every positive root α. The basic example of a dominant weight is the half sum of positive roots:

$$\rho\;\;:=\;\;\frac{1}{2}\sum_{\alpha\text{ positive root}}\alpha\;\;\in\;\;\Lambda$$

For every connected compact Lie group G the dominant weights are in bijective correspondence to the isomorphism classes of irreducible representations, up to isomorphism there is thus a unique representation R_λ of G of "highest" dominant weight $\lambda\in\Lambda$. The character of this representation is determined by a very useful formula called Weyl's Character Formula

$$A\;\mathrm{ch}\,R_\lambda\;\;=\;\;\sum_{w\in\mathfrak{W}}(-1)^{|w|}\,e^{w(\,\lambda+\rho\,)} \tag{11}$$

where the denominator A can be defined in two ways due to Weyl and Kostant respectively:

$$A\;\;:=\;\;\sum_{w\in\mathfrak{W}}(-1)^{|w|}\,e^{w\rho}\;\;=\;\;\prod_{\alpha\text{ positive root}}\left(\,e^{\frac{\alpha}{2}}\;-\;e^{-\frac{\alpha}{2}}\,\right) \tag{12}$$

A direct consequence of Weyl's Character Formula is the following formula for the multiplicity

$$m_\lambda(R) \; := \; \dim \mathrm{Hom}_G(R_\lambda, \; R) \; = \; \mathrm{ev}_{\lambda+\rho}\Big[A \,\mathrm{ch}\, R \Big]$$

of the irreducible representation R_λ in an arbitrary finite–dimensional representation R.

Coming back to our branching problem we consider a pair $\mathfrak{g} \supset \mathfrak{h}$ of Lie algebras associated to a pair $G \supset H$ compact Lie groups. In this case we can find a regular element $X \in \mathfrak{h}$ for $\mathfrak{g}$ in $\mathfrak{h}$ and a fortiori get a pair $\mathfrak{t} \subset \mathfrak{s}$ of maximal tori for $\mathfrak{h}$ and $\mathfrak{g}$ respectively. By construction the restriction map $i\mathfrak{s}^* \longrightarrow i\mathfrak{t}^*$ preserves positivity of linear forms in the sense $i\lambda(X) > 0$ and maps the weight lattice $\Lambda_\mathfrak{g} \subset i\mathfrak{s}^*$ to the weight lattice $\Lambda_\mathfrak{h}$ of $\mathfrak{h}$. In consequence the restriction map induces a ring homomorphism $\mathrm{res} : \mathbb{Z}\Lambda_\mathfrak{g} \longrightarrow \mathbb{Z}\Lambda_\mathfrak{h}$, which maps the characters of representations of G to the characters of their restriction to H. For example the adjoint representations $\mathfrak{g}$ and $\mathfrak{h}$ of G and H both induce characters in $\mathbb{Z}\Lambda_\mathfrak{h}$ satisfying

$$\mathrm{res}(\mathrm{ch}\, \mathfrak{g}) - \mathrm{ch}\, \mathfrak{h} \; = \; \mathrm{ch}(\mathfrak{g}/\mathfrak{h}) \; = \; (\dim \mathfrak{s} - \dim \mathfrak{t}) + \sum_{\substack{\alpha \text{ positive} \\ \text{weight of } \mathfrak{g}/\mathfrak{h}}} \Big(e^\alpha + e^{-\alpha} \Big)$$

where the right hand side is to be understood as a summation with multiplicities. Using Kostant's Formula (12) for Weyl's denominator and the restriction homomorphism we find

$$e^{\rho_\mathfrak{h} - \mathrm{res}\,\rho_\mathfrak{g}} \frac{\mathrm{res}\, A_\mathfrak{g}}{A_\mathfrak{h}} \; = \; e^{\rho_\mathfrak{h} - \mathrm{res}\,\rho_\mathfrak{g}} \prod_{\substack{\alpha \text{ positive} \\ \text{weight of } \mathfrak{g}/\mathfrak{h}}} \Big(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}} \Big) \; = \; \prod_{\substack{\alpha \text{ positive} \\ \text{weight of } \mathfrak{g}/\mathfrak{h}}} \Big(1 - e^{-\alpha} \Big)$$

and this expression can be inverted in a suitable completion of the group ring $\mathbb{Z}\Lambda_\mathfrak{h}$ to define

$$B_{\mathfrak{g}/\mathfrak{h}} \; := \; \Big(e^{\mathrm{res}\,\rho_\mathfrak{g} - \rho_\mathfrak{h}} \frac{A_\mathfrak{h}}{\mathrm{res}\, A_\mathfrak{g}} \Big)^* \; = \; \prod_{\substack{\alpha \text{ positive} \\ \text{weight of } \mathfrak{g}/\mathfrak{h}}} \Big(1 + e^\alpha + e^{2\alpha} + e^{3\alpha} + \ldots \Big) \qquad (13)$$

the universal branching formula for the pair $\mathfrak{g} \supset \mathfrak{h}$. The standard branching problem to find the multiplicity of $R^\mathfrak{h}_\mu$ in $R^\mathfrak{g}_\lambda$ is solved in terms of this universal branching formula via:

$$\begin{aligned} \dim \mathrm{Hom}_H(R^\mathfrak{h}_\mu, \; R^\mathfrak{g}_\lambda) \; &= \; \mathrm{ev}_{\mu + \rho_\mathfrak{h}} \Big[A_\mathfrak{h} \,\mathrm{ch}\, R^\mathfrak{g}_\lambda \Big] \\ &= \; \mathrm{ev}_{\mu + \mathrm{res}\,\rho_\mathfrak{g}} \Big[e^{\mathrm{res}\,\rho_\mathfrak{g} - \rho_\mathfrak{h}} \frac{A_\mathfrak{h}}{\mathrm{res}\, A_\mathfrak{g}} \sum_{w \in \mathfrak{W}_\mathfrak{g}} (-1)^{|w|} e^{\mathrm{res}\, w(\lambda + \rho_\mathfrak{g})} \Big] \\ &= \; \sum_{w \in \mathfrak{W}_\mathfrak{g}} (-1)^{|w|} \mathrm{ev}_{\mu + \mathrm{res}\,\rho_\mathfrak{g} - \mathrm{res}\, w(\lambda + \rho_\mathfrak{g})} \Big[e^{\mathrm{res}\,\rho_\mathfrak{g} - \rho_\mathfrak{h}} \frac{A_\mathfrak{h}}{\mathrm{res}\, A_\mathfrak{g}} \Big] \end{aligned}$$

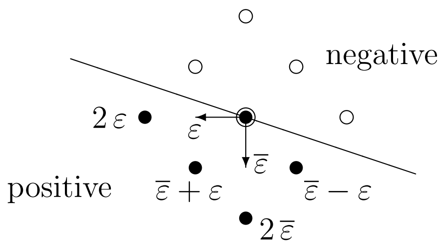
For latter reference we will write this branching formula in the following way:

$$\dim \mathrm{Hom}_H(R^\mathfrak{h}_\mu, \; R^\mathfrak{g}_\lambda) \; = \; \sum_{w \in \mathfrak{W}_\mathfrak{g}} (-1)^{|w|} \mathrm{ev}_{\mathrm{res}(w(\lambda + \rho_\mathfrak{g}) - \rho_\mathfrak{g}) - \mu} \Big[B_{\mathfrak{g}/\mathfrak{h}} \Big] \qquad (14)$$

According to the results of Section 2 we need to solve the branching problems for the pairs $\mathfrak{so}_{4,1}\mathbb{R} \supset \mathbb{R} \oplus i\mathbb{R}$ and $\mathfrak{so}_{4,1}\mathbb{R} \supset \mathbb{R} \oplus \mathfrak{sp}(1)$ respectively in order to turn Theorem 2.5 and Theorem 2.6 into effective decomposition formulas. Of course none of the algebras involved is the Lie algebra of a compact group, however the compact pairs $\mathbf{Sp}(2) \supset S^1 \times S^1$ and $\mathbf{Sp}(2) \supset S^1 \times \mathbf{Sp}(1)$ have on the level of Lie algebras isomorphic complexifications

$$\mathfrak{sp}(2) \otimes_{\mathbb{R}} \mathbb{C} \supset (\, i\mathbb{R} \oplus i\mathbb{R}\,) \otimes_{\mathbb{R}} \mathbb{C} \qquad\qquad \mathfrak{sp}(2) \otimes_{\mathbb{R}} \mathbb{C} \supset (\, i\mathbb{R} \oplus \mathfrak{sp}(1)\,) \otimes_{\mathbb{R}} \mathbb{C}$$

and this is sufficient to ensure that we can apply the branching formula (14) for the problems at hand as well. The maximal torus $S^1 \times S^1 \subset S^1 \times \mathbf{Sp}(1) \subset \mathbf{Sp}(2)$ is evidently the same for all groups and so the restriction is simply the identity. The root diagram (9) of $\mathfrak{so}_{4,1}\mathbb{R}$



tells us that the positive roots of the pair $\mathfrak{so}_{4,1}\mathbb{R} \supset \mathbb{R} \oplus \mathfrak{sp}(1)$ are $2\,\overline{\varepsilon}$, $\overline{\varepsilon} + \varepsilon$ and $2\,\varepsilon$, the positive root $\overline{\varepsilon} - \varepsilon$ corresponds to $\overline{\mathrm{Pl}} \in \mathfrak{sp}(1) \otimes_{\mathbb{R}} \mathbb{C}$. Hence the branching formula reads:

$$\begin{aligned} & B_{\mathfrak{so}_{4,1}\mathbb{R}/\mathbb{R}\oplus\mathfrak{sp}(1)} \\ &\quad = \big(\, 1 + e^{\overline{\varepsilon}+\varepsilon} + e^{2\overline{\varepsilon}+2\varepsilon} + \ldots \big) \big(\, 1 + e^{2\varepsilon} + e^{4\varepsilon} + \ldots \big) \big(\, 1 + e^{2\overline{\varepsilon}} + e^{4\overline{\varepsilon}} + \ldots \big) \\ &\quad = \sum_{k,\overline{k}\geq 0} \sharp\{\, r \mid 0 \leq r \leq \min\{k, \overline{k}\} \text{ and } r \equiv k \equiv \overline{k} \bmod 2 \,\}\, e^{k\,\varepsilon + \overline{k}\,\overline{\varepsilon}} \end{aligned}$$

Evaluation of this power series in e^{ε}, $e^{\overline{\varepsilon}}$ in the weight $\overline{k}\,\overline{\varepsilon} + k\,\varepsilon$ defines the function:

$$B_{\overline{k},k} \;:=\; \mathrm{ev}_{\overline{k}\,\overline{\varepsilon}+k\,\varepsilon}\Big[\, B_{\mathfrak{so}_{4,1}\mathbb{R}/\mathbb{R}\oplus\mathfrak{sp}(1)} \,\Big] \;=\; \begin{cases} 1 + \lfloor \frac{\min\{\,\overline{k},\,k\,\}}{2} \rfloor & \text{if } k, \overline{k} \in \mathbb{N}_0 \text{ and } k \equiv \overline{k} \bmod 2 \\ 0 & \text{for all other arguments} \end{cases}$$

Consider now a tuple $(r, \overline{r})$ of integers satisfying $0 \leq \overline{r} \leq r \leq n$ with associated dominant weight $\lambda := (n - \overline{r})\,\overline{\varepsilon} + (n - r)\,\varepsilon$ in the Weyl chamber indicated in diagram (9). For this choice of Weyl chamber the half sum of positive roots is $\rho_{\mathfrak{so}_{4,1}\mathbb{R}} = 2\,\overline{\varepsilon} + \varepsilon$ and the affine Weyl orbit of λ under the Weyl group $\mathfrak{W}_{\mathfrak{so}_{4,1}\mathbb{R}} = S_2 \ltimes \mathbb{Z}_2^2$ of $\mathbf{SO}_{4,1}\mathbb{R}$ is readily calculated:

$(-1)^{\lvert w \rvert}$	w	$w(\,\lambda + \rho_{\mathfrak{so}_{4,1}\mathbb{R}}\,) - \rho_{\mathfrak{so}_{4,1}\mathbb{R}}$
$+1$	id	$(+n - \overline{r}\,)\,\overline{\varepsilon} + (+n - r\,)\,\varepsilon$
-1	τ_1	$(+n - r - 1)\,\overline{\varepsilon} + (+n - \overline{r} + 1)\,\varepsilon$
-1	τ_2	$(+n - \overline{r}\,)\,\overline{\varepsilon} + (-n + r - 2)\,\varepsilon$
$+1$	$\tau_1\tau_2$	$(-n + r - 3)\,\overline{\varepsilon} + (+n - \overline{r} + 1)\,\varepsilon$
$+1$	$\tau_2\tau_1$	$(+n - r - 1)\,\overline{\varepsilon} + (-n + \overline{r} - 3)\,\varepsilon$
-1	$\tau_2\tau_1\tau_2$	$(-n + r - 3)\,\overline{\varepsilon} + (-n + \overline{r} - 3)\,\varepsilon$
-1	$\tau_1\tau_2\tau_1$	$(-n + \overline{r} - 4)\,\overline{\varepsilon} + (+n - r\,)\,\varepsilon$
$+1$	$\tau_1\tau_2\tau_1\tau_2$	$(-n + \overline{r} - 4)\,\overline{\varepsilon} + (-n + r - 2)\,\varepsilon$

Under the rather superficial assumption $0 \leq k \leq 2n$ on the degree of the differential forms considered the highest weight $\mu = (n - \frac{k-s}{2})\,\overline{\varepsilon} + (n - \frac{k+s}{2})\,\varepsilon$ of the corresponding representation $\mathbb{C}_{(n-\frac{k}{2})\,(\overline{\varepsilon}+\varepsilon)} \otimes \mathrm{Sym}^s\mathbb{H}$ of $\mathbb{R} \oplus \mathfrak{sp}(1)$ has $\overline{\varepsilon}$–coefficient and coefficient sum positive or zero. For this reason the five Weyl group elements w, which have $\overline{\varepsilon}$–coefficient or coefficient sum of $w(\,\lambda + \rho_{\mathfrak{so}_{4,1}\mathbb{R}}\,) - \rho_{\mathfrak{so}_{4,1}\mathbb{R}}$ negative, can not contribute to the summation in the branching formula (14) and we are left with the three summands corresponding to id and the reflections $\tau_1 : \overline{\varepsilon} \longmapsto \varepsilon,\ \varepsilon \longmapsto \overline{\varepsilon}$ and $\tau_2 : \overline{\varepsilon} \longmapsto \overline{\varepsilon},\ \varepsilon \longmapsto -\varepsilon$ along the simple roots:

$$\begin{aligned} \dim \mathrm{Hom}_{\,\mathbb{R}\oplus\mathfrak{sp}(1)}\Big(\, & R^{\mathbb{R}\oplus\mathfrak{sp}(1)}_{(n-\frac{k-s}{2})\,\overline{\varepsilon}+(n-\frac{k+s}{2})\,\varepsilon},\ R^{\mathfrak{so}_{4,1}\mathbb{R}}_{(n-\overline{r})\,\overline{\varepsilon}+(n-r)\,\varepsilon} \Big) \\ &= B_{\frac{k-s}{2}-\overline{r},\ \frac{k+s}{2}-r} - B_{\frac{k-s}{2}-r-1,\ \frac{k+s}{2}-\overline{r}+1} - B_{\frac{k-s}{2}-\overline{r},\ \frac{k+s}{2}+r-2n-2} \end{aligned} \tag{15}$$

The other branching problem from $\mathfrak{so}_{4,1}\mathbb{R}$ to $\mathbb{R} \oplus i\mathbb{R}$ can be treated similarly, the main difference is the positive root $\overline{\varepsilon} - \varepsilon$. The evaluation of the universal branching formula

$$B_{\mathfrak{so}_{4,1}\mathbb{R}/\mathbb{R}\oplus i\mathbb{R}} \;=\; \big(\,1 + e^{\overline{\varepsilon}-\varepsilon} + e^{2\overline{\varepsilon}-2\varepsilon} + \ldots\big)\; B_{\mathfrak{so}_{4,1}\mathbb{R}/\mathbb{R}\oplus\mathfrak{sp}(1)}$$

at a weight $\overline{k}\,\overline{\varepsilon} + k\,\varepsilon$ however defines a significantly more involved function

$$\widehat{B}_{\overline{k},k} \;=\; \begin{cases} \lfloor\frac{(\overline{k}+2)^2}{4}\rfloor & \text{if } k \geq \overline{k} \geq 0 \text{ and } k \equiv \overline{k} \bmod 2 \\ \lfloor\frac{(\overline{k}+2)^2}{4}\rfloor - \frac{(\overline{k}-k)(\overline{k}-k+2)}{8} & \text{if } \overline{k} \geq k \geq 0 \text{ and } k \equiv \overline{k} \bmod 2 \\ \qquad\qquad + \frac{(\overline{k}+k+2)(\overline{k}+k+4)}{8} & \text{if } 0 \geq k \geq -\overline{k} \text{ and } k \equiv \overline{k} \bmod 2 \\ \qquad\qquad 0 & \text{in all other cases} \end{cases}$$

which essentially describes the coefficients of the denominator of Weyl's Character Formula for $\mathfrak{so}_{4,1}\mathbb{R}$ and is closely related to Kostant's Polynomial [FH]. Note in particular that $\widehat{B}_{\overline{k},k}$ vanishes in case the sum $k + \overline{k}$ of its two arguments is negative. For a weight μ with non–negative coefficient sum we can thus ignore the four Weyl group elements w in the branching formula (14), which see $w(\,\lambda + \rho_{\mathfrak{so}_{4,1}\mathbb{R}}\,) - \rho_{\mathfrak{so}_{4,1}\mathbb{R}}$ having negative coefficient sum:

$$\begin{aligned} \dim \mathrm{Hom}_{\,\mathbb{R}\oplus i\mathbb{R}}\Big(\, & R^{\mathbb{R}\oplus i\mathbb{R}}_{(n-\overline{k})\,\overline{\varepsilon}+(n-k)\,\varepsilon},\ R^{\mathfrak{so}_{4,1}\mathbb{R}}_{(n-\overline{r})\,\overline{\varepsilon}+(n-r)\,\varepsilon} \Big) \\ &= \widehat{B}_{\overline{k}-\overline{r},\ k-r} - \widehat{B}_{\overline{k}-r-1,\ k-\overline{r}+1} - \widehat{B}_{\overline{k}-\overline{r},\ k+r-2n-2} + \widehat{B}_{\overline{k}+r-2n-3,\ k-\overline{r}+1} \end{aligned} \tag{16}$$

Since $\mathbb{R} \oplus i\mathbb{R}$ is essentially the maximal torus of $\mathfrak{so}_{4,1}\mathbb{R}$, this formula may be seen as a formula for the dimension of the weight spaces in an arbitrary representation of $\mathfrak{so}_{4,1}\mathbb{R}$. In the context of hyperkähler manifolds formula (16) together with Lemma 2.5 calculates the multiplicity of the irreducible representation $\Lambda^{r,\overline{r}}_{\circ}E$ of $\mathbf{Sp}\,E$ in the differential forms of bidegree $(k, \overline{k})$ provided $0 \leq k + \overline{k} \leq 2n$. Instead of persuing this idea any further we turn to:

Theorem 3.1 (Explicit Formulae for the Multiplicities)
The complete decomposition of the complexified differential forms $\Lambda^k T^* \otimes_{\mathbb{R}} \mathbb{C} \;\cong\; \Lambda^k(H^* \otimes E^*)$

up to middle dimension $2n \geq k \geq 0$ on the tangent representation T of the holonomy group $\mathbf{Sp}\,H \cdot \mathbf{Sp}\,E$ *of quaternionic Kähler geometry into irreducible subrepresentations reads*

$$\Lambda^k(\,H^* \otimes E^*\,) \;=\; \bigoplus_{\substack{s\geq 0\\ n\geq r\geq \overline{r}\geq 0}} m_{s,\,r,\,\overline{r}}(\,k\,)\;\mathrm{Sym}^s H \,\otimes\, \Lambda^{r,\,\overline{r}}_{\circ} E$$

where the multiplicities $m_{s,\,r,\,\overline{r}}(\,k\,)$ are zero unless both $s \equiv k \equiv r+\overline{r}$ modulo 2 and the parameters s, r, $\overline{r}$ satisfy the defining inequalities of at least one of the following three cases:

First Case:

The multiplicities in the first case defined by $\overline{r} \leq \frac{k-s}{2} \leq r \leq \frac{k+s}{2} < 2n+2-r$ are:

$$m_{s,\,r,\,\overline{r}}(\,k\,) \;=\; 1 + \left\lfloor \frac{k-s-2r+2\min\{s,\,r-\overline{r}\}}{4} \right\rfloor \;>\; 0$$

Second Case:

The second case is characterized by $\frac{k-s}{2} > r$, depending on whether the common parity of $k \equiv s \equiv r-\overline{r}$ mod 2 is even or odd the multiplicities in this case are given by

$$m_{s,\,r,\,\overline{r}}(\,k\,) \;=\; \frac{\min\{r-\overline{r},\,s\}}{2} + \delta_{\frac{k-s}{2}\equiv r\ (2)} \quad \text{or} \quad m_{s,\,r,\,\overline{r}}(\,k\,) \;=\; \frac{\min\{\,r-\overline{r},\,s\}+1}{2}$$

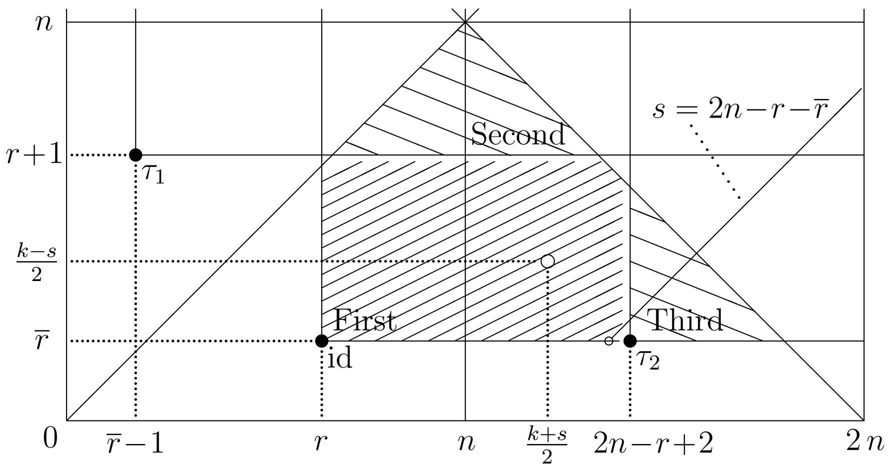
respectively, where the δ–summand equals 1 if $\frac{k-s}{2} \equiv r$ mod 2 and 0 otherwise. These multiplicities are strictly positive unless k is even, $s=0$ or $r=\overline{r}$ and $\frac{k-s}{2} \not\equiv r$ mod 2.

Third Case:

Characterized by $\frac{k+s}{2} \geq 2n-r \geq s+\overline{r}$ the third case has strictly positive multiplicities:

$$m_{s,\,r,\,\overline{r}}(\,k\,) \;=\; 1 + n - r + \frac{\min\{r-\overline{r},\,s\}-s}{2} \;>\; 0$$

The theorem is a good example of how a seemingly simple sum (15) of three terms may lead to a messy case distinction. Although it is natural to believe that there is a different, simpler way to present the result, no such simplification has been found by the author and the hexagonal pattern (1) of the multiplicities $m_{s,r,\overline{r}}(k)$ suggests that no simpler presentation exists. The reader interested in the details of the proof is invited to study the diagram first



(17)

which gives a geometric interpreation of the inequalities in Theorem 3.1. The black points represent the images of the highest weight $\lambda = (n-\overline{k})\,\overline{\varepsilon} + (n-k)\,\varepsilon$ under the Weyl group elements id, τ_1 and τ_2 contributing to (15). The support of the corresponding B–summand considered as functions of $(\frac{k+s}{2}, \frac{k-s}{2})$ is a translate of the first quadrant based at this point. The case distinction thus arises from sectors, where some parts of formula (15) vanish:

Proof: To begin with we note that the arguments of the B–summands in formula (15) are integers of the same parity if and only f $k \equiv s \equiv r + \overline{r}$ modulo 2, hence this condition is certainly necessary to have a positive multiplicity. Fixing a degree $0 \le k \le 2n$ and $s \ge 0$ we note that the corresponding point $(\frac{k+s}{2}, \frac{k-s}{2})$ lies in the big triangle of diagram (17) or below, hence this point lies in the support of some of the summands of formula (15) if and only if $\frac{k+s}{2} \ge r$ and $\frac{k-s}{2} \ge \overline{r}$. The stronger assumptions of the first case $r \le \frac{k+s}{2} < 2n - r + 2$ and $\overline{r} \le \frac{k-s}{2} \le r$ correspond exactly to the rectangle denoted "First" in diagram (17). In this rectangle only the summand corresponding to id contributes to (15) and we obtain:

$$B_{\frac{k-s}{2}-\overline{r},\frac{k+s}{2}-r} \;=\; 1+\left\lfloor\frac{\min\{k-s-2\,\overline{r}, k+s-2\,r\}}{4}\right\rfloor \;=\; 1+\left\lfloor\frac{k-s-2\,r+2\min\{r-\overline{r}, s\}}{4}\right\rfloor$$

Under our standing assumptions $0 \le k \le 2n$ and $s \ge 0$ the assumption $\frac{k-s}{2} > r$ characterizing the second case corresponds to the triangle denoted "Second" in diagram (17), which lies in the support of the two summands corresponding to id and τ_1 in formula (15):

$$\begin{aligned} & B_{\frac{k-s}{2}-\overline{r},\frac{k+s}{2}-r} \;-\; B_{\frac{k-s}{2}-r-1,\frac{k+s}{2}-\overline{r}+1} \\ &\quad = \left\lfloor\frac{k-s-2\,r+2\min\{r-\overline{r}, s\}}{4}\right\rfloor - \frac{k-s-2\,r-2-2\delta_{\frac{k-s}{2}\equiv r\;(2)}}{4} \\ &\quad = \left\lfloor\frac{\min\{r-\overline{r}, s\}+1+\delta_{\frac{k-s}{2}\equiv r\;(2)}}{2}\right\rfloor \end{aligned}$$

Evidently the δ–term will have no bearance on the result in case $s \equiv r - \overline{r}$ is odd, for even $s \equiv r - \overline{r}$ on the other hand it will take us to the next even integer or not. Changing the assumptions of the remaining third case slightly we observe that the conditions $0 \le k \le 2n$ and $\frac{k+s}{2} \ge 2n - r + 2$ characterize points in the triangle denoted "Third" or below. In this region the two non–trivial summands in formula (15) will cancel each other in points below the diagonal $s = 2\,n - r - \overline{r}$. According to the branching formula (15) the multiplicities in the points $(\frac{k+s}{2}, \frac{k-s}{2})$ on or above this diagonal $s \le 2\,n - r - \overline{r}$ can be calculated as:

$$\begin{aligned} & B_{\frac{k-s}{2}-\overline{r},\frac{k+s}{2}-r} \;-\; B_{\frac{k-s}{2}-\overline{r},\frac{k+s}{2}-2\,n+r-2} \\ &\quad = \left\lfloor\frac{k-s-2\,r+2\min\{r-\overline{r}, s\}}{4}\right\rfloor - \frac{k+s-4\,n+2\,r-4-2\,\delta_{\frac{k+s}{2}\not\equiv r\;(2)}}{4} \\ &\quad = n+1-r+\left\lfloor\frac{\min\{r-\overline{r}, s\}-s+\delta_{\frac{k+s}{2}\not\equiv r\;(2)}}{2}\right\rfloor \end{aligned}$$

However $s \equiv r - \overline{r}$ modulo 2 by our standing assumption, hence $\min\{r - \overline{r}, s\} - s$ is even and we can safely trade the δ–summand for skipping to round down. $\square$

Of course the stated positivity of the multiplicities in the different cases of Theorem 3.1 is of particular importance in applications, because in this way it is possibly to locate exactly those degrees, in which a representation $\mathrm{Sym}^s H \otimes \Lambda^{r,\overline{r}}_\circ E$ definitely does occur in the differential forms. In particular the exception k even, $s = 0$ or $r = \overline{r}$ and $\frac{k-s}{2} \not\equiv r \bmod 2$ in the second case has a direct bearance on the Betti numbers of quaternionic Kähler manifolds, in that it allows the Betti numbers to increase in steps of 4 only instead of the expected 2.

More precisely it has been shown in [SW] that on compact quaternionic Kähler manifolds of positive scalar curvature $\kappa > 0$ every harmonic form is a sum of harmonic forms of types $\Lambda^{r,r}_\circ E$ with $n \geq r \geq 0$, while on a compact quaternionic Kähler manifold with negative scalar curvature $\kappa < 0$ the basic harmonic forms can be of the two different types $\Lambda^{r,r}_\circ E$, $n \geq r \geq 0$, and $\mathrm{Sym}^{2n-r-\overline{r}} H \otimes \Lambda^{r,\overline{r}}_\circ E$ with $n \geq r \geq \overline{r} \geq 0$. In this context Theorem 3.1 confirms the central conclusion of [SW] about the degrees and the multiplicities of general differential forms and thus about the degrees of harmonic forms of type $\Lambda^{r,r}_\circ E$ and $\mathrm{Sym}^{2n-r-\overline{r}} H \otimes \Lambda^{r,\overline{r}}_\circ E$ in the differential forms. The representation $\Lambda^{r,r}_\circ E$ occurs with multiplicity 1 in the forms of degrees $k = 2r, 2r + 4, 2r + 8, \ldots, 4n - 2r$, while the representation $\mathrm{Sym}^{2n-r-\overline{r}} H \otimes \Lambda^{r,\overline{r}}_\circ E$ occurs with multiplicity 1 in the forms of degrees $k = 2n-r+\overline{r}, 2n-r+\overline{r}+2, \ldots, 2n+r-\overline{r}$.

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