About the Eta–Invariants of Berger Spheres

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Abstract

The integral of the top dimensional term of the multiplicative sequence of Pontryagin forms associated to an even formal power series is calculated for special Riemannian metrics on the unit ball of a hermitean vector space. Using this result we calculate the generating function of the reduced Dirac and signature η–invariants for the family of Berger metrics on the odd dimensional spheres.

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1 Introduction

Originally η–invariants of Dirac operators were introduced by Atiyah, Patodi and Singer in the course of establishing an index theorem for compact manifolds with boundary. In essence the η–invariant is a kind of error term arising from the geometry of the boundary, heuristically it can be interpreted as an expectation value for how many sections in the kernel and cokernel of the Dirac operator can be extended as L^2–integrable sections to the two ends of an infinite cylinder based on the boundary. Although the η–invariants are a spectral invariant of the underlying Dirac operator and vary with the Riemannian geometry of boundary, it appears that they somehow capture rather subtle $\mathbb{Q}/\mathbb{Z}$–valued information of the diffeomorphism type of the boundary manifold as exemplified by Kreck–Stolz invariants for 7–manifolds. Despite their genesis as an error term η–invariants have thus become an object of interest of their own, whether or not the given manifold is a boundary.

In this article we will focus on the calculation of the η–invariants of the untwisted Dirac operator and the signature operator for the family of Berger metrics on the odd dimensional spheres S^{2n-1}. Instead of persuing the original definition of the η–invariant as a spectral invariant we will solve the index formula of Atiyah–Patodi–Singer for the η–invariant. Needless to say this approach leads the original raison d'être of η–invariants ad adsurdum, moreover it only allows us to recover the η–invariant modulo $\mathbb{Z}$. Interestingly the spectrum of the

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untwisted Dirac operator is known explicitly for the Berger metrics due to work of Hitchin [Hit] in dimension 3 and Bär [B] in general. Nevertheless it seems almost impossible to find the analytic continuation necessary to determine the η–invariant directly from the spectrum.

Consider a hermitean vector space $\mathfrak{p}$ of dimension n. The real part of the hermitean form is a scalar product g on $\mathfrak{p}$ making $\mathfrak{p}$ a euclidean vector space endowed with an orthogonal complex structure $I : \mathfrak{p} \longrightarrow \mathfrak{p}$, $X \longmapsto iX$, satisfying $g(IX, IY) = g(X, Y)$ for $X, Y \in \mathfrak{p}$. Rescaling the round metric on the sphere S^{2n-1} of unit vectors in $\mathfrak{p}$ differently on the distinguished subspaces $\mathbb{R}IP$ and $\{ P, IP \}^\perp$ of $T_P S^{2n-1} = \{P\}^\perp$ defines the two–parameter family of Berger metrics on S^{2n-1}. Proportional Riemannian metrics share the same Levi–Civita connection and are thus virtually indistinguishable for the purpose of this article, so we are left with one geometrically relevant parameter, the ratio T between the radii of Hopf and “normal” great circles. Instead of T we prefer to use the equivalent $\rho := T^2 - 1$ so that a Berger metric of parameter $\rho > -1$ is a Riemannian metric on S^{2n-1} proportional to

$$g^\rho \;:=\; g \,+\, \rho\,(\,\gamma \otimes \gamma\,)$$

where $\gamma_P(X) := g(IP, X)$ is the contact form of the natural CR–structure induced on the real hypersurface S^{2n-1} in the flat Kähler manifold $\mathfrak{p}$. In order to solve the index formula of Atiyah–Patodi–Singer for manifolds with boundary for the η–invariants we need to know the complementary integral of the index density over the unit ball $B^{2n} \subset \mathfrak{p}$ with boundary S^{2n-1}. However the index densities of the untwisted Dirac operator and the signature operator are multiplicative sequences of Pontryagin forms and the complementary integrals are given by:

Theorem 1.1 (Special Values of Multiplicative Sequences)
For every smooth metric $g^{\mathrm{collar},\rho}$ on the closed unit ball $B^{2n} \subset \mathfrak{p}$, which is a product of the standard metric on $]-\varepsilon, 0]$ with a Berger metric of parameter $\rho > -1$ in a collar neighborhood $]-\varepsilon, 0] \times S^{2n-1}$ of the boundary, the multiplicative sequence $F(\,TB^{2n}, \nabla^{\mathrm{collar},\rho}\,)$ of Pontryagin forms associated to an even formal power series $F(z) = 1 + O(z^2)$ integrates to

$$\int_{B^{2n}} F(\,TB^{2n}, \nabla^{\mathrm{collar},\rho}\,) \;=\; \rho^n \operatorname{res}_{z=0}\left[\,\frac{F(z)^n}{z^{n+1}}\,dz\,\right] \;=\; \rho^n \operatorname{res}_{z=0}\left[\,\frac{(\log\,\phi)'(z)}{z^n}\,dz\,\right]$$

where $\phi(z) = z + O(z^3)$ is the composition inverse of the formal power series $\frac{z}{F(z)} = z + O(z^3)$.

In essence Theorem 1.1 is a consequence of a striking integrability condition enjoyed by the Berger spheres S^{2n-1} thought of as the boundary of geodesic distance balls in $\mathbb{C}P^n$. Hypersurfaces in Riemannian manifolds satisfying this integrability condition are called permeable in the sequel, they are discussed in more detail in Section 2, in particular we will show that all totally geodesic hypersurfaces and all hypersurfaces in a space form are permeable. Section 3 is devoted to a study of multiplicative sequences of Pontryagin forms and their associated (logarithmic) transgression forms, moreover we will calculate the values of multiplicative sequences of Pontryagin forms on $\mathbb{C}P^n$ and $\mathbb{H}P^n$. The subsequent Section 4 is the technical core of this article, its main result Corollary 4.3 provides us with a closed formula for the logarithmic transgression form of a permeable hypersurface. The preceeding Lemma 4.2 may be of independent interest, because it gives a managable formula for the logarithmic transgression form of arbitrary hypersurfaces. The calculations proving Theorem 1.1 are relegated to the last Section 5.

Our main motivation for studying multiplicative sequences on Berger spheres is the relationship between η–invariants and indices of twisted Dirac operators as formulated in the Atiyah–Patodi–Singer Index Theorem [APS] for manifolds with boundary. For the convenience of the reader a concise, self–contained formulation of the Index Theorem of Atiyah–Patodi–Singer is provided in Appendix A to this article. In order to avoid any technicalities in this introduction let us consider a $\mathbb{Z}_2$–graded Clifford module bundle $EM^{\rm out}$ on an even–dimensional oriented Riemannian manifold $M^{\rm out}$ and a totally geodesic hypersurface M in $M^{\rm out}$, which can be realized as the boundary hypersurface $M := \partial W$ of a compact subset $W \subset M^{\rm out}$ with non–trivial interior.

In this general setup the $\mathbb{Z}_2$–graded Clifford module bundle $EM^{\rm out}$ induces a Clifford module bundle E_+M on M and under some additional, rather severe restriction on the geometry of $EM^{\rm out}$ in a collar neighborhood of M in $M^{\rm out}$ the Index Theorem of Atiyah–Patodi–Singer relates the η–invariant of the Dirac operator D associated to E_+M to the integral of the usual index density of the $\mathbb{Z}_2$–graded Clifford module bundle $EM^{\rm out}$ over W:

$$\eta_D \;\equiv\; 2\int_W \widehat{A}(\,TM^{\rm out},\nabla^{\rm out}\,)\,{\rm ch}\,(\,EM^{\rm out}\;:\;\$M^{\rm out}\,) \qquad {\rm mod}\;\;\mathbb{Z} \tag{1}$$

With this congruence in mind we define the reduced η–invariant of D to be any representative $\overline{\eta}_D \in \mathbb{R}$ of the class of η_D in $\mathbb{R}/\mathbb{Z}$ in the sense $\overline{\eta}_D \equiv \eta_D$ modulo $\mathbb{Z}$.

In general the index density $\widehat{A}(TM^{\rm out},\nabla^{\rm collar})\,{\rm ch}(EM^{\rm out} : \$M^{\rm out})$ is not a multiplicative sequence of Pontryagin forms and so we can not apply Theorem 1.1 to calculate the interior integral in (1). However there are two well–known exceptions, the untwisted Dirac operator $D^{\rm out}_\$$ on an even–dimensional spin manifold $M^{\rm out}$ and the signature operator $D^{\rm out}_{\rm sign} = d + d^*$ on an oriented manifold $M^{\rm out}$ of dimension divisible by 4. Referring to Appendix A for the details of the ensuing calculations we recall that in the case of the untwisted Dirac operator $D^{\rm out}_\$$ on the spinor bundle $EM^{\rm out} = \$M^{\rm out}$ the associated Clifford bundle $E_+M = \$M$ is the spinor bundle of M with associated untwisted Dirac operator D. At the same time the index density for the untwisted Dirac operator is the multiplicative sequence of Pontryagin forms associated to the formal power series $\widehat{A}(z) := \frac{\frac{z}{2}}{\sinh\frac{z}{2}}$ and hence we obtain:

Corollary 1.2 (Eta Invariants of the Untwisted Dirac Operator)
The generating function for the reduced η–invariants $\overline{\eta}_D(S^{2n-1}, g^\rho)$ of the untwisted Dirac operator for the Berger metric $g^\rho := g + \rho\gamma\otimes\gamma$ on the odd dimensional spheres S^{2n-1} reads:

$$1 \;+\; \frac{1}{2}\sum_{n>0}\overline{\eta}_D(\;S^{2n-1},\;g^\rho\;)\;z^n \;\;=\;\; z\frac{d}{dz}\;\log(\;2\,{\rm arsinh}\,\frac{\rho\,z}{2}\;)$$

Observing that the composition inverse of a solution f to a differential equation $f' = p(f)$ is an antiderivative of $\frac{1}{p(z)}$ we employ the identity $\frac{d}{dz} \sinh z = (1 + \sinh^2 z)^{\frac{1}{2}}$ to calculate

$$\frac{d}{dz} \operatorname{arsinh} z \quad = \quad (\,1 \,+\, z^2\,)^{-\frac{1}{2}} \quad = \quad \sum_{k \geq 0} (\,-\frac{1}{4}\,)^k \binom{2k}{k} z^{2k}$$

via Newton's expansion $(1 + z)^s = \sum_{k \geq 0} \binom{s}{k} z^k$. Moreover the differential operator $z \frac{d}{dz}$ commutes with rescalings like $z \rightsquigarrow \frac{\rho z}{2}$, in this way we obtain the more managable formula

$$1 \,+\, \frac{1}{2} \sum_{n>0} \overline{\eta}_D(\, S^{2n-1},\, g^\rho\,)\, z^n \quad = \quad \frac{\sum_{k \geq 0} \;\; (\,-\frac{1}{16}\,)^k \binom{2k}{k} (\rho\, z)^{2k}}{\sum_{k \geq 0} \; \frac{1}{2k+1} \,(\,-\frac{1}{16}\,)^k \binom{2k}{k} (\rho\, z)^{2k}} \tag{2}$$

for the η–invariants of the untwisted Dirac operator on Berger spheres. It should be pointed out that Habel [Hab] has calculated these particular η–invariants for $n = 3, \ldots, 16$ using the explicit description of the spectrum of D given by Bär [B] and analytic continuation techniques. Computer experiments suggest that the general combinatorial formula conjectured in [Hab] agrees with (2) at least for say $n \leq 500$, although the complexity of the conjectured formula does not look too inviting to try a direct proof of equivalence.

Turning from the untwisted Dirac operator to the signature operator we consider the $\mathbb{Z}_2$–graded Clifford bundle $EM^{\mathrm{out}} = \Lambda T^*M^{\mathrm{out}}$ of differential forms on M^{out} with grading operator given by Clifford multiplication with the complex volume form. Again referring the reader to Appendix A for the details of the ensuing calculations we recall that the Clifford module bundle E_+M on M can be identified with the differential forms bundle ΛT^*M in this case in such a way that the associated Dirac operator becomes $D = \Gamma(d + d^*)$, where Γ is Clifford multiplication with the complex volume element of M. Up to constants Γ agrees with the Hodge $*$–isomorphism and thus changes the parity of forms on the odd–dimensional manifold M. By construction Γ however commutes with D so that D becomes the sum of two conjugated partial operators on even and odd differential forms on M respectively:

$$D \quad = \quad [\,\Gamma\,(\,d \,+\, d^*\,)\,]^{\mathrm{ev}} \;\oplus\; [\,\Gamma\,(\,d \,+\, d^*\,)\,]^{\mathrm{odd}} \quad \cong \quad 2\,\Gamma\,(\,d \,+\, d^*\,)^{\mathrm{ev}}$$

On the other hand the index density for the signature operator is the multiplicative sequence of Pontryagin forms associated to the formal power series $L(z) := \frac{z}{\tanh z}$ and we conclude:

Corollary 1.3 (Eta Invariants of the Signature Operator)
The reduced η–invariants $\overline{\eta}_{\Gamma\,(d+d^*)^{\mathrm{ev}}}(S^{2n-1}, g^\rho)$ of the signature operator $\Gamma\,(d + d^*)^{\mathrm{ev}}$ with respect to the Berger metric $g^\rho := g + \rho\gamma \otimes \gamma$ have the following generating function

$$1 \,+\, \sum_{n>0} \overline{\eta}_{\Gamma\,(d+d^*)^{\mathrm{ev}}}(\, S^{2n-1},\, g^\rho\,)\, z^n \quad = \quad z\frac{d}{dz}\, \log(\, \operatorname{artanh}(\rho z)\,) \quad = \quad \frac{\sum_{k \geq 0} \;\; (\rho z)^{2k}}{\sum_{k \geq 0} \; \frac{1}{2k+1}\, (\rho\, z)^{2k}}$$

whose power series expansion is implied by the differential equation $\frac{d}{dz} \tanh z = 1 - \tanh^2 z$.

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2 Geometry of Permeable Hypersurfaces

In differential geometry the Berger metrics on odd dimensional spheres arise naturally as the Riemannian metrics induced on the distance spheres $S^{2n-1}_r \subset \mathbb{C}P^n$ of radius $r \in]\,0, \frac{\pi}{2}\,[$ in complex projective space. The identification of Berger spheres with distance spheres in $\mathbb{C}P^n$ endow S^{2n-1} with a family of second fundamental forms or more precisely Weingarten maps $\mathbb{I} \in \Gamma(\,\mathrm{End}\, TS^{2n-1}\,)$, which satisfy the strong integrability condition characterizing permeable hypersurfaces. Permeable hypersurfaces as introduced in this section are solutions $M \subset M^{\mathrm{out}}$ to a quasilinear second order differential equation weaker than the second order equation $\mathbb{I} = 0$ characterizing totally geodesic hypersurfaces. The permeability of the distance spheres in $\mathbb{C}P^n$ is to be seen as a rare exception to the generic situation, because in sufficiently wrinkled manifolds M^{out} all permeable hypersurface are probably totally geodesic.

Every hypersurface $M \subset M^{\mathrm{out}}$ in a Riemannian manifold M^{out} is naturally a Riemannian manifold itself with metric g induced from the metric g^{out} of M^{out}. The restriction of the tangent bundle TM^{out} to the hypersurface M is a euclidian vector bundle $TM^{\mathrm{out}}|_M$, which splits orthogonally into the tangent bundle TM and the normal bundle of M in M^{out}:

$$TM^{\mathrm{out}}\big|_M \;=\; TM \,\oplus\, \mathrm{Norm}\, M$$

The hypersurface M is called coorientable if the line bundle $\mathrm{Norm}\, M$ is trivial or equivalently if the monodromy of the unique metric (and thus flat) connection on $\mathrm{Norm}\, M$ vanishes

$$\pi_1(\,M\,) \;\longrightarrow\; \mathbb{Z}_2, \qquad [\,\gamma\,] \;\longmapsto\; \mathrm{or}_M(\,[\,\gamma\,]\,)\,\mathrm{or}_{M^{\mathrm{out}}}(\,\iota_*[\,\gamma\,]\,)$$

where or_M and $\mathrm{or}_{M^{\mathrm{out}}}$ are the orientation homomorphisms of M and M^{out} respectively. In this article we are eventually interested in boundary hypersurfaces $M = \partial W$ of compact subsets $W \subset M^{\mathrm{out}}$ in an oriented Riemannian manifold M^{out}, which are automatically oriented and cooriented by the outward pointing normal field $N \in \Gamma(\,\mathrm{Norm}\, M\,)$, nevertheless some of the arguments presented below are valid in greater generality.

By construction the euclidian vector bundle $TM^{\mathrm{out}}|_M$ is endowed with two metric connections, the restriction $\nabla^{\mathrm{out}}|_M$ of the Levi–Civita connection of M^{out} and the direct sum connection ∇ of the Levi–Civita connection of M with the unique metric connection on $\mathrm{Norm}\, M$. The difference between these two metric connections is governed by the second fundamental form $\mathbb{I}$ of M in M^{out} with respect to a (local) normal field N defined by

$$\nabla^{\mathrm{out}}_X Y \;=\; \nabla_X Y \;+\; \mathbb{I}(\,X,\,Y\,)\,N$$

for two vector fields X, Y on M. Alternatively we can think of the second fundamental form as a vector valued 1–form on M, the shape operator or Weingarten map $\mathbb{I} : TM \longrightarrow TM$, by means of the Riemannian metric or $g(\mathbb{I}_X, Y) := \mathbb{I}(X, Y)$. With $\nabla^{\mathrm{out}}|_M$ and ∇ being metric their difference is a 1–form on M with values in the skew–symmetric endomorphisms

$$\nabla^{\mathrm{out}}\big|_M \;=\; \nabla \;+\; \mathbb{I} \wedge N \tag{3}$$

on $TM^{\text{out}}|_M$, which faithfully reflects the second fundamental form through the definition:

$$(\mathrm{I\!I} \wedge N)_X Z \;=\; (\mathrm{I\!I}_X \wedge N)\, Z \;:=\; g^{\text{out}}(\mathrm{I\!I}_X,\, Z)\, N \;-\; g^{\text{out}}(N,\, Z)\, \mathrm{I\!I}_X$$

The standard formula $R^{\nabla+A} = R^\nabla + d^\nabla A + A^2$ becomes the Gauß–Codazzi–Mainardi equation

$$R^{\text{out}}\big|_M \;=\; R \;+\; (d^\nabla \mathrm{I\!I}) \wedge N \;+\; \mathrm{I\!I}^\sharp \otimes \mathrm{I\!I} \tag{4}$$

where $(\mathrm{I\!I}^\sharp \otimes \mathrm{I\!I})_{X,Y} Z := g^{\text{out}}(\mathrm{I\!I}_X, Z)\mathrm{I\!I}_Y - g^{\text{out}}(\mathrm{I\!I}_Y, Z)\mathrm{I\!I}_X$ is a 2–form on M with values in the skew–symmetric endomorphisms of $TM^{\text{out}}|_M$. The Gauß–Codazzi–Mainardi equation (4) describes the restriction of the curvature tensor R^{out} of M^{out} to M as an endomorphism–valued 2–form in terms of the curvature tensor R of M and the second fundamental form. Alternatively we can consider the complete restriction of the curvature tensor R^{out} to M:

Definition 2.1 (Intermediate Curvature of a Hypersurface)
The intermediate curvature of a hypersurface M in a Riemannian manifold M^{out} is the section R^{inter} of the bundle of algebraic curvature tensors on M defined by the complete restriction of the curvature tensor R^{out} to M using the orthogonal projection $\mathrm{pr}^\perp_{TM}$ to TM:

$$R^{\text{inter}}_{X,Y} Z \;=\; \mathrm{pr}^\perp_{TM}(\, R^{\text{out}}_{X,Y} Z \,) \;=\; R_{X,Y} Z \;+\; (\mathrm{I\!I}^\sharp \otimes \mathrm{I\!I})_{X,Y} Z$$

In terms of the Gauß–Codazzi–Mainardi equation (4) this definition reads $R^{\text{inter}} := R + \mathrm{I\!I}^\sharp \otimes \mathrm{I\!I}$.

Like every section of the bundle $\mathrm{Kr}\, TM$ of algebraic curvature tensors on M the intermediate curvature R^{inter} in its covariant incarnation $g(R^{\text{inter}}_{X,Y} Z, W)$ satisfies all symmetries of the curvature tensor R of M itself. In particular we can define the intermediate Ricci curvature

$$\mathrm{Ric}^{\text{inter}}(\, X,\, Y \,) \;:=\; \mathrm{tr}_{TM}(\, Z \longmapsto R^{\text{inter}}_{Z,X} Y \,) \;=\; \sum_\mu g(\, R^{\text{inter}}_{E_\mu, X} Y,\, E_\mu \,)$$

as a symmetric bilinear form relating via $\mathrm{Ric}^{\text{inter}} = \mathrm{Ric} + \mathrm{I\!I}^2 - (\mathrm{tr}_g \mathrm{I\!I})\, \mathrm{I\!I}$ to the actual Ricci curvature Ric of M. The difference $\Delta\mathrm{Ric} := \mathrm{Ric}^{\text{out}}\big|_M - \mathrm{Ric}^{\text{inter}}$ between the intermediate Ricci curvature and the restriction of the Ricci curvature $\mathrm{Ric}^{\text{out}}$ of M^{out} to M provides the missing piece of information in the Gauß–Codazzi–Mainardi equation (4) to describe the full curvature tensor R^{out} of M^{out} along M. In fact representation theory tells us that the bundle $\mathrm{Kr}\, TM^{\text{out}}$ of algebraic curvature tensors on M^{out} decomposes upon restriction to M as:

$$\begin{array}{ccccccc} \mathrm{Kr}\, TM^{\text{out}}|_M & \cong & \mathrm{Kr}\, TM & \oplus & \Lambda^{2,1} T^*M & \oplus & \mathrm{Sym}^2 T^*M \\ R^{\text{out}} & \widehat{=} & R^{\text{inter}} & \oplus & (\, d^\nabla \mathrm{I\!I}\,)^\sharp & \oplus & \Delta\mathrm{Ric} \end{array}$$

where $\Lambda^{2,1} T^*M$ is the kernel of the exterior multiplication $\Lambda^2 T^*M \otimes T^*M \longrightarrow \Lambda^3 T^*M$. Put differently the difference $\Delta\mathrm{Ric}(X, Y) = g^{\text{out}}(R^{\text{out}}_{N,X} Y, N)$ parametrizes all second order partial derivatives of the outer metric g^{out} off the hypersurface M up to change of coordinates.

Definition 2.2 (Permeable Hypersurface)
Consider a hypersurface $M \subset M^{\text{out}}$ in a Riemannian manifold M^{out} with metric g^{out}. The curvature tensor R of the induced metric g on M considered as a 2–form on M with values in End TM *and the second fundamental form $\mathrm{I\!I}$ with respect to a (local) normal field N considered as a vector–valued 1–form combine into a vector–valued 3–form on M:*

$$d^\nabla(\, d^\nabla \mathrm{I\!I}\,)(X, Y, Z) \;=\; (\, R\,\mathrm{I\!I}\,)(X, Y, Z) \;:=\; R_{X,Y}\,\mathrm{I\!I}_Z \;+\; R_{Y,Z}\,\mathrm{I\!I}_X \;+\; R_{Z,X}\,\mathrm{I\!I}_Y$$

The hypersurface M is called permeable if $R\,\mathrm{I\!I} \in \Gamma(\Lambda^3 T^*M \otimes TM)$ vanishes identically, i. e. if the covariant exterior derivative $d^\nabla \mathrm{I\!I}$ of the second fundamental form is covariantly closed.

Of course this definition makes sense even for non–coorientable hypersurfaces $M \subset M^{\text{out}}$, because the vanishing of $R\,\mathrm{I\!I} = 0$ is independent of the change of local normal field $N \rightsquigarrow -N$. Similarly it is possible to replace the curvature R of M in Definition 2.2 by the intermediate curvature R^{inter}, because the difference $R^{\text{inter}}\mathrm{I\!I} - R\,\mathrm{I\!I} = (\mathrm{I\!I}^\sharp \otimes \mathrm{I\!I})\,\mathrm{I\!I} = 0$ vanishes identically, after all its definition involves the skew–symmetrization of the symmetric 2–form $g(\mathrm{I\!I}, \mathrm{I\!I})$. The resulting alternative definition of permeability is in a sense more natural than Definition 2.2 in that the intermediate curvature R^{inter} depends only on the first order jet $T_pM \subset T_pM^{\text{out}}$ of the hypersurface M in p, whereas the curvature R involves the second order jet of M in the guise of the second fundamental form $\mathrm{I\!I}$. Permeable hypersurfaces are thus solutions to a quasilinear, second order differential equation, whose symbol in $T_pM \subset T_pM^{\text{out}}$ is the kernel of the linear map $\mathrm{Sym}^2 T_p^*M \longrightarrow \Lambda^3 T_p^*M \otimes T_pM$, $\mathrm{I\!I} \longmapsto R^{\text{inter}}\mathrm{I\!I}$. The definition of permeability adopted in this article however has the advantage of being directly applicable to the calculation of transgressions forms.

Recall now that a hypersurface $M \subset M^{\text{out}}$ is called totally geodesic if every geodesic starting with a vector tangent to M stays in M for all times, equivalently M is totally geodesic if and only if its second fundamental form $\mathrm{I\!I} = 0$ vanishes identically. Clearly every totally geodesic hypersurface is permeable as is every hypersurface in flat space. Slightly more general every hypersurface in a space form M^{out} is permeable by the following argument:

Lemma 2.3 (Permeability and Ricci Curvature)
The second fundamental form $\mathrm{I\!I} : TM \longrightarrow TM$ of a permeable hypersurface $M \subset M^{\text{out}}$ commutes with the Ricci and the intermediate Ricci endomorphism Ric *and* $\mathrm{Ric}^{\text{inter}}$ *of TM. The necessary condition $[\,\mathrm{Ric}^{\text{inter}}, \mathrm{I\!I}\,] = 0$ for permeability is already sufficient for 3–dimensional hypersurfaces $M \subset M^{\text{out}}$ in* $\dim M^{\text{out}} = 4$ *or for hypersurfaces in a conformally flat Riemannian manifold M^{out}. In particular all hypersurfaces in a space form are permeable.*

Proof: Necessity of the condition $[\,\mathrm{Ric}, \mathrm{I\!I}\,] = 0$ is a straightforward calculation using the definition of $R\,\mathrm{I\!I}$. The trace $\sum g(R_{X,Y}\mathrm{I\!I}_{E_\mu}, E_\mu) \;=\; -\sum \mathrm{I\!I}(E_\mu, R_{X,Y}E_\mu)$ vanishes and so

$$\begin{aligned}
\sum_\mu g(\,(R\,\mathrm{I\!I})(E_\mu, X, Y),\, E_\mu\,) \;&=\; \sum_\mu \Big(\, g(R_{E_\mu,X}\mathrm{I\!I}_Y,\, E_\mu) \;+\; g(R_{Y,E_\mu}\mathrm{I\!I}_X,\, E_\mu)\,\Big) \\
&=\; g((\mathrm{Ric} \circ \mathrm{I\!I})\,Y, X) \;-\; g((\mathrm{Ric} \circ \mathrm{I\!I})\,X, Y)
\end{aligned}$$

for all $X, Y \in T_pM$. With Ric and $\mathrm{I\!I}$ being symmetric endomorphisms the condition $R\,\mathrm{I\!I} = 0$ enforces $[\,\mathrm{Ric},\,\mathrm{I\!I}\,] = 0$. Replacing the curvature tensor R by the intermediate curvature R^{inter} in this argument we get $[\,\mathrm{Ric}^{\mathrm{inter}},\,\mathrm{I\!I}\,] = 0$, alternatively we can infer $[\,\mathrm{Ric}^{\mathrm{inter}},\,\mathrm{I\!I}\,] = [\,\mathrm{Ric},\,\mathrm{I\!I}\,]$ from the explicit formula $\mathrm{Ric}^{\mathrm{inter}} = \mathrm{Ric} + \mathrm{I\!I}^2 - (\mathrm{tr}_g\mathrm{I\!I})\mathrm{I\!I}$.

Turning from necessity to sufficiency we note that common point of the additional assumptions is that the intermediate curvature R^{inter} of M can be written as a cross product[1]

$$\begin{aligned}(\,g\times h\,)(\,X,\,Y,\,Z,\,W\,) \;:=\; &+g(\,X,\,Z\,)\,h(\,Y,\,W\,) \;-\; g(\,X,\,W\,)\,h(\,Y,\,Z\,)\\ &+g(\,Y,\,W\,)\,h(\,X,\,Z\,) \;-\; g(\,Y,\,Z\,)\,h(\,X,\,W\,)\end{aligned}$$

of the Riemannian metric g and a suitable symmetric bilinear form $h \in \Gamma(\mathrm{Sym}^2T^*M)$. In fact in dimension 3 the intermediate curvature R^{inter} of M is a cross product with g like every other algebraic curvature tensor. In dimensions greater than 3 this is no longer true, but the curvature tensor of a conformally flat manifold M^{out} can still be written in the form $R^{\mathrm{out}} = g^{\mathrm{out}} \times h^{\mathrm{out}}$ for some $h^{\mathrm{out}} \in \Gamma(\mathrm{Sym}^2T^*M^{\mathrm{out}})$ so that $R^{\mathrm{inter}} = g \times h^{\mathrm{out}}|_M$.

Under the stated assumptions we can thus safely assume that $R^{\mathrm{inter}} = g \times h$ is a cross product with g for suitable h. The symmetric bilinear form h can be recovered from the intermediate Ricci curvature $\mathrm{Ric}^{\mathrm{inter}} = (2-m)h - (\mathrm{tr}_g h)g$ in dimensions $m := \dim M$ different from 2, more precisely $[\,\mathrm{Ric}^{\mathrm{inter}},\,\mathrm{I\!I}\,] = (2-m)[\,h,\,\mathrm{I\!I}\,]$ as g considered as a symmetric endomorphism is the identity. This equation confirms the triviality of Lemma 2.3 in dimension $m = 2$, in higher dimensions however the condition $[\,\mathrm{Ric}^{\mathrm{inter}},\,\mathrm{I\!I}\,] = 0$ for hypersurfaces M with intermediate curvature of the form $R^{\mathrm{inter}} = g \times h$ is equivalent to $[\,h,\,\mathrm{I\!I}\,] = 0$. Summing

$$\begin{aligned}&g(\,R^{\mathrm{inter}}_{X,Y}\mathrm{I\!I}_Z,\,W\,)\\ &= g(X,\mathrm{I\!I}_Z)h(Y,W) - g(X,W)h(Y,\mathrm{I\!I}_Z) - g(Y,\mathrm{I\!I}_Z)h(X,W) + g(Y,W)h(X,\mathrm{I\!I}_Z)\\ &= \mathrm{I\!I}(X,Z)h(Y,W) - \mathrm{I\!I}(Z,Y)h(X,W) + g(Y,W)g((h\circ\mathrm{I\!I})Z,X) - g(X,W)g((\mathrm{I\!I}\circ h)Y,Z)\end{aligned}$$

cyclically over $X,\,Y,\,Z \in T_pM$ we find eventually the formula

$$g(\,(R^{\mathrm{inter}}\mathrm{I\!I})_{X,Y,Z},W\,) \;=\; g(\,[\,h,\,\mathrm{I\!I}\,]\,X,\,Y\,)\,g(\,Z,\,W\,) \;+\; \text{cyclic permutations of } X,Y,Z$$

for all $X,\,Y,\,Z,\,W \in T_pM$, which allows us to conclude $R^{\mathrm{inter}}\mathrm{I\!I} = 0$ in case $[\,h,\,\mathrm{I\!I}\,] = 0$. □

3 Multiplicative Sequences of Pontryagin Classes

In essence multiplicative sequences are a method to construct meaningful "characteristic" classes in the de Rham cohomology of a differentiable manifold M starting with a connection ∇ on a vector bundle VM over the manifold M in question. In turn the top dimensional term of a multiplicative sequence can be integrated over a closed manifold M yielding a differentiable invariant of M. Although multiplicative sequences may not appear particularly interesting from the more general Chern–Weil point of view, in this section we will

[1] Sometimes called the "Nomizu–Kulkarni" product, although it is actually an instance of the isomorphism between the two standard presentations of a Schur functor, in this case the Schur functor associated to ⊞.

focus on their construction and evaluation on closed manifolds without reference to the Chern–Weil homomorphism. The advantage of staying with multiplicative sequences is that their associated transgression forms can be calculated directly from the simpler logarithmic transgression forms, which we will discuss in more detail in the next section.

Recall the definition of the total Chern and Pontryagin differential forms associated to a connection ∇ on a complex or real vector bundle VM respectively over a manifold M:

$$c(\,VM,\,\nabla\,) \;:=\; \det\Big(\;\mathrm{id}\;-\;\frac{R^\nabla}{2\pi i}\;\Big) \;:=\; \exp\Big(\;-\sum_{k>0}\frac{1}{k}\,\mathrm{tr}\Big(\frac{R^\nabla}{2\pi i}\Big)^k\;\Big)$$

$$p(\,VM,\,\nabla\,) \;:=\; \det{}^{\frac{1}{2}}\Big(\;\mathrm{id}\;-\;\Big(\frac{R^\nabla}{2\pi}\Big)^2\;\Big) \;:=\; \exp\Big(\;-\sum_{k>0}\frac{1}{2k}\,\mathrm{tr}\Big(\frac{R^\nabla}{2\pi}\Big)^{2k}\;\Big)$$

Evidently the homogeneous components of the total Chern form, the Chern forms $c_k(VM,\nabla)$, are complex valued differential forms of degree $2k$ on M, but actually they are real differential forms for every hermitean connection ∇ on VM, because a symmetrized product of hermitean matrices is again hermitean with real trace. The second Bianchi identity $d^\nabla R^\nabla = 0$ implies that all Chern forms $c_k(VM,\nabla)$, $k \geq 1$, are closed.

The reality of the homogeneous components of the total Pontryagin form, the Pontryagin forms $p_k(VM,\nabla)$, $k \geq 1$, of degree $4k$, goes without saying and again all Pontryagin forms are closed by the second Bianchi identity. For the sake of a uniform treatment of multiplicative sequences of Chern and Pontryagin forms however we need to spoil the real definition of the total Pontryagin form with the introduction of some spurious imaginary units i

$$p(\,VM,\,\nabla\,) \;=\; \det{}^{\frac{1}{2}}\Big(\;\mathrm{id}\;+\;\Big(\frac{R^\nabla}{2\pi i}\Big)^2\;\Big) \;=\; \exp\Big(\;\sum_{k>0}\frac{(-1)^{k-1}}{2k}\,\mathrm{tr}\Big(\frac{R^\nabla}{2\pi i}\Big)^{2k}\;\Big)$$

although strictly speaking this formula makes no sense unless we replace the real vector bundle VM by its complexification $VM\otimes_{\mathbb{R}}\mathbb{C}$. Forgetting about this nuisance we get

$$\log p(\,VM,\,\nabla\,) \;=\; \sum_{k>0}\frac{(-1)^{k-1}}{2k}\,\mathrm{tr}\Big(\frac{R^\nabla}{2\pi i}\Big)^{2k} \tag{5}$$

right from the definition of the Pontryagin form. Specifying to euclidian plane bundles EM over M we remark that the curvature of a metric connection ∇ can be written $R^\nabla = \omega^\nabla\otimes I$ for a (local) orthogonal complex structure I on EM and a suitable 2–form ω^∇ so that:

$$\frac{1}{2}\,\mathrm{tr}\Big(\frac{R^\nabla}{2\pi i}\Big)^{2k} \;=\; \frac{1}{2}\Big(\frac{\omega^\nabla}{2\pi}\Big)^{2k}\,\mathrm{tr}\,(\,-iI\,)^{2k} \;=\; \Big(\frac{\omega^\nabla}{2\pi}\Big)^{2k}$$

In consequence we find for the powers of the first Pontryagin form $p_1(EM,\nabla) := (\frac{\omega^\nabla}{2\pi})^2$

$$p_1(\,EM,\,\nabla\,)^k \;=\; \frac{1}{2}\,\mathrm{tr}\Big(\frac{R^\nabla}{2\pi i}\Big)^{2k} \tag{6}$$

and conclude that all higher Pontryagin forms $p_2(EM, \nabla)$, $p_3(EM, \nabla)$, ... vanish due to:

$$\log p(EM, \nabla) \;=\; \sum_{k>0} \frac{(-1)^{k-1}}{k} p_1(EM, \nabla)^k \;=\; \log(1 + p_1(EM, \nabla))$$

The corresponding argument for Chern forms is much simpler, because the trace is multiplicative for endomorphisms on a complex line, we get an identity of differential forms

$$c_1(LM, \nabla)^k \;=\; \Big(\mathrm{tr} \Big(-\frac{R^\nabla}{2\pi i} \Big) \Big)^k \;=\; \mathrm{tr} \Big(-\frac{R^\nabla}{2\pi i} \Big)^k \tag{7}$$

for every complex line bundle LM. For the time being we are more interested in multiplicative sequences of Pontryagin forms and so we will restrict our discussion to the case of real vector bundles from now on, mutatis mutandis all arguments and definitions presented immediately translate into statements about multiplicative sequences of Chern forms.

Definition 3.1 (Multiplicative Sequences of Pontryagin Forms)
The multiplicative sequence of Pontryagin forms associated to an even formal power series $F(z) = 1 + O(z^2)$ with formal logarithm $\log F(z) = \sum_{k>0} f_k z^{2k}$ is the differential form

$$F(VM, \nabla) \;=\; \det{}^{\frac{1}{2}} F\Big(\frac{R^\nabla}{2\pi i} \Big) \;:=\; \exp \Big(\frac{1}{2} \sum_{k>0} f_k \, \mathrm{tr} \Big(\frac{R^\nabla}{2\pi i} \Big)^{2k} \Big)$$

associated to a connection ∇ on a real vector bundle VM over a smooth manifold M.

The construction of multiplicative sequences is trivially multiplicative under the product of (admissable) even formal power series $(F \tilde{F})(VM, \nabla) = F(VM, \nabla) \tilde{F}(VM, \nabla)$. Historically however the name refers to the multiplicativity under the direct sum of vector bundles

$$F(VM \oplus \tilde{V}M, \nabla \oplus \tilde{\nabla}) \;=\; F(VM, \nabla) F(\tilde{V}M, \tilde{\nabla}) \tag{8}$$

which follows easily from the additivity of the trace of the powers of the curvature

$$\mathrm{tr} \Big(\frac{R^{\nabla \oplus \tilde{\nabla}}}{2\pi i} \Big)^{2k} \;=\; \mathrm{tr} \Big(\frac{R^\nabla}{2\pi i} \Big)^{2k} + \mathrm{tr} \Big(\frac{R^{\tilde{\nabla}}}{2\pi i} \Big)^{2k}$$

of the direct sum connection $\nabla \oplus \tilde{\nabla}$. Equation (6) immediately implies the characteristic property of the multiplicative sequence of Pontryagin forms parametrized by F, namely

$$F(EM, \nabla) \;=\; \exp \Big(\sum_{k>0} f_k \, p_1(EM, \nabla)^k \Big) \;=\; F(\sqrt{p_1(EM, \nabla)})$$

for every metric connection on a euclidian plane bundle EM. Last but not least we observe

$$F(VM, \nabla) \;=\; \exp \big(\Lambda^F \log p(VM, \nabla) \big) \tag{9}$$

due to equation (5), where $\Lambda^F : \Lambda^{4\bullet}T^*M \longrightarrow \Lambda^{4\bullet}T^*M$ multiplies homogeneous forms of degree $4k$ by $(-1)^{k-1}\,k\,f_k$. The right hand side of (9) expands into a formal power series in the Pontryagin forms $p_k(VM, \nabla)$ independent of the vector bundle VM. Sorting the summands of this formal power series according to homogeneity we get the "multiplicative sequence" of universal polynomials in the Pontryagin forms referred to in the name.

Among the more interesting uses of multiplicative sequences of Pontryagin forms is the construction of the quantized Pontryagin forms of a real vector bundle VM, which are closely related to the Chern characters of exterior powers of VM. More precisely the quantized Pontryagin forms $P_k(VM, \nabla), k \geq 1$, of are the coefficients of the multiplicative sequence

$$P(\,t,\,VM,\,\nabla\,) \;\;=\;\; 1 \;+\; t\,P_1(\,VM,\,\nabla\,) \;+\; t^2\,P_2(\,VM,\,\nabla\,) \;+\; \ldots$$

of Pontryagin forms associated to the formal power series $P(t,z) := 1 + 2\,t(\,\cosh\,z - 1\,)$. It is not too difficult to argue by the splitting principle or by the direct calculation sketched below that the quantized Pontryagin forms $P_k(VM, \nabla)$ for a metric connection ∇ on VM vanish for $k > \frac{1}{2}\dim VM$ similar to the original Pontryagin forms. Another similarity is

$$P_r(\,VM\,\oplus\,\tilde{V}M,\,\nabla\,\oplus\,\tilde{\nabla}\,) \;\;=\;\; \sum_{s=0}^{r} P_s(\,VM,\,\nabla\,)\,P_{r-s}(\,\tilde{V}M,\,\tilde{\nabla}\,)$$

which follows directly from the multiplicativity (8) of the sequence $P(t, VM, \nabla)$. Working out the expansion (9) for $P(t, VM, \nabla)$ explicitly we find moreover a universal formula

$$P_k(\,VM,\nabla\,) \;\;=\;\; p_k(\,VM,\nabla\,) + \frac{1}{12}\Big(\,p_1(\,VM,\nabla\,)\,p_k(\,VM,\nabla\,) - (k{+}1)\,p_{k+1}(\,VM,\nabla\,)\,\Big) + \ldots$$

for the quantized Pontryagin forms independent of the dimension of VM. The resulting congruence $P_k(VM, \nabla) \equiv p_k(VM, \nabla)$ modulo forms of higher degree is the main motivation for us to think of the forms $P_k(VM, \nabla) \in \Gamma(\Lambda^{\geq 4k}T^*M)$ as quantized Pontryagin forms.

Motivated by a similar construction in K–theory [A] we associate to every real vector bundle VM with connection the formal power series $\sum_{d\geq 0}\tau^d\,\mathrm{ch}\,\Lambda^d(VM, \nabla)$ of exterior powers of VM. This association is multiplicative in the vector bundle VM in the sense (8), but does not arise directly from a multiplicative sequence of Pontryagin forms, because in general the Chern character depends on the dimension of the vector bundle, too. Ignoring this point for the moment we find for a metric connection ∇ on a real plane bundle EM

$$\sum_{d\geq 0}\tau^d \mathrm{ch}\;\Lambda^d(\,EM,\nabla\,) \;\;=\;\; (1 + \tau e^z)\,(1 + \tau e^{-z}) \;\;=\;\; (1+\tau)^2\Big(1 + \frac{2\tau}{(1+\tau)^2}(\cosh\,z - 1)\Big)$$

where z is formally a root of $p_1(EM, \nabla)$. Applying the splitting principle we conclude

$$\sum_{d\geq 0}\tau^d\,\mathrm{ch}\;\Lambda^d(\,VM,\,\nabla\,) \;\;=\;\; (\,1\,+\,\tau\,)^{\dim\,VM}\,P(\,\frac{\tau}{(1+\tau)^2},\,VM,\,\nabla\,)$$

for every real vector bundle VM with metric connection ∇, or in terms of coefficients:

$$\mathrm{ch}\,\Lambda^d(\,VM,\,\nabla\,) \;=\; \sum_{k=0}^{d}\binom{\dim\,VM-2k}{d-k}P_k(\,VM,\,\nabla\,)$$

In passing we note that a very similar calculation proves the formula

$$\mathrm{ch}\,\mathrm{Sym}^d(\,VM,\,\nabla\,) \;=\; \sum_{k=0}^{d}\binom{\dim\,VM+d+k-1}{d-k}S_k(\,VM,\,\nabla\,)$$

for the Chern character of symmetric powers of a real vector bundle VM in terms of the coefficients $S_k(VM,\nabla)$, $k\geq 1$, of the multiplicative sequence $S(t,VM,\nabla)$ of Pontryagin forms associated to the formal power series $S(t,z)=(\,1-2t(\cosh\,z-1)\,)^{-1}$. The coefficients $S_k(VM,\nabla)\in\Gamma(\Lambda^{\geq 4k}T^*M)$ still vanish in degrees less than $4k$ and are independent of the dimension of VM, but the universal formula expressing them in terms of Pontryagin forms is considerably more complicated than the formula for quantized Pontryagin forms.

Eventually we want to push multiplicative sequences of Pontryagin forms, which result in closed differential forms $F(VM,\nabla)\in\Gamma(\Lambda^{4\bullet}T^*M)$ on M, to multiplicative sequences of Pontryagin classes in de Rham cohomology. The conditio sine qua non for doing so is that the de Rham cohomology class $F(VM)\in H^{4\bullet}_{\mathrm{dR}}(M)$ represented by the closed differential form $F(VM,\nabla)\in\Gamma(\Lambda^{4\bullet}T^*M)$ is independent of the connection ∇. En nuce this independence relies on the existence of transgression forms, explicit solutions $\mathrm{Trans}\,F(VM,\nabla^0,\nabla^1)$ to the transgression problem to make the difference between two representative closed forms exact:

$$F(\,VM,\,\nabla^1\,)\;-\;F(\,VM,\,\nabla^0\,)\;\;=\;\;d\,(\mathrm{Trans}\,F)(VM,\nabla^0,\nabla^1) \tag{10}$$

Needless to say these transgression forms are to live a life of their own, besides our use of them to calculate the values of multiplicative sequences on compact manifolds with boundary they lead to the definition of secondary characteristic classes of flat vector bundles. Although not completely straightforward we begin our discussion of transgression forms with:

Definition 3.2 (Logarithmic Transgression Form)
The logarithmic transgression form associated to a multiplicative sequence of Pontryagin classes parametrized by an even formal power series $F(z)=1+O(z^2)$ with logarithm $\log\,F(z)=\sum_{k>0}f_kz^{2k}$ at a tangent vector $(\nabla,\dot\nabla)$ to the space of connections on a (real) vector bundle VM over a differentiable manifold M is the following differential form on M:

$$\delta\,(\log\,F)(\,VM,\,\nabla,\,\dot\nabla\,)\;\;:=\;\;\sum_{k>0}k\,f_k\;\mathrm{tr}\Big(\,\frac{\dot\nabla}{2\pi i}\,\Big(\frac{R^\nabla}{2\pi i}\Big)^{2k-1}\,\Big)$$

The logarithmic transgression form describes the logarithmic derivative of the multiplicative sequence $\nabla\longmapsto F(\,VM,\,\nabla\,)$ along a curve curve $t\mapsto\nabla^t$ in the space of connections through:

$$\frac{d}{dt}\;\log\,F\,(\,VM,\,\nabla^t\,)\;\;=\;\;d\;\Big(\delta\,(\log\,F)(\,VM,\,\nabla^t,\,\frac{d}{dt}\nabla^t\,)\Big) \tag{11}$$

Upon integration over $[0, t]$ and exponentiation in the exterior algebra this equation becomes:

$$F(\,VM,\,\nabla^t\,) \;=\; F(\,VM,\,\nabla^0\,)\,\exp\Big(\,d\int_0^t \delta\,(\log\,F)(\,VM,\nabla^\tau,\,\frac{d}{d\tau}\nabla^\tau\,)\,d\tau\,\Big) \tag{12}$$

Taking the derivative of this equation in t and integrating once more over $[0, 1]$ we see that

$$(\,\mathrm{trans}\,F\,)(\,VM,\,\nabla^0,\,\nabla^1\,) \;:= \tag{13}$$
$$\int_0^1 \exp\Big(\,\int_0^t d\,\delta\,(\log\,F)(\,VM,\nabla^\tau,\,\frac{d}{d\tau}\nabla^\tau\,)\,d\tau\,\Big)\;\delta\,(\log\,F)(\,VM,\nabla^t,\,\frac{d}{dt}\nabla^t\,)\,dt$$

solves the following reformulation of the original transgression problem (10):

$$F(\,VM,\,\nabla^1\,)\;-\;F(\,VM,\,\nabla^0\,)\;=\;F(\,VM,\,\nabla^0\,)\,d\,(\mathrm{trans}\,F)(\,VM,\,\nabla^0,\,\nabla^1\,) \tag{14}$$

Given the rather ardeous formula (13) for the transgression form it may be hard to believe that it can be of any use besides showing that the de Rham cohomology class $F(VM)$ represented by $F(VM, \nabla)$ is well defined. The importance of formula (13) however lies in the fact that the solution to the modified transgression problem (14) is expressed solely in terms of the time dependence of the logarithmic transgression form $\delta(\log\,F)(VM, \nabla^t, \frac{d}{dt}\nabla^t)$. In the explicit calculation of transgression forms in Section 4 this becomes a crucial advantage.

Remark 3.3 (Multiplicative Sequences of Chern Forms)
Similarly to multiplicative sequences of Pontryagin forms (or classes) we can define multiplicative sequences of Chern forms for complex vector bundles VM over a manifold M. The differential form associated to a formal power series $F(z) = 1 + O(z)$ with formal logarithm $\log\,F(z) = \sum_{k>0} f_k z^k$ and a connection ∇ on a complex vector bundle VM over M reads

$$F(\,VM,\,\nabla\,)\;=\;\det F\Big(-\frac{R^\nabla}{2\pi i}\Big)\;:=\;\exp\Big(\sum_{k>0} f_k\,\mathrm{tr}\Big(-\frac{R^\nabla}{2\pi i}\Big)^k\Big)$$

while the corresponding logarithmic transgression form is defined as:

$$\delta\,(\log\,F)(\,VM,\,\nabla,\,\dot\nabla\,)\;:=\;\sum_{k>0} k\,f_k\,\mathrm{tr}\Big(\,-\frac{\dot\nabla}{2\pi i}\Big(-\frac{R^\nabla}{2\pi i}\Big)^{k-1}\Big)$$

Quantized Chern classes $C_k(VM, \nabla)$, $k \geq 1$, are defined as well as the coefficients of the multiplicative sequence $C(t, VM, \nabla)$ associated to the power series $C(t, z) := 1 + t(e^z - 1)$.

On compact manifolds M of dimension divisible by 4 we can integrate the top term of the multiplicative sequence $F(TM, \nabla)$ of Pontryagin forms over M. For closed manifolds the resulting value $\langle\,F(TM,\,\nabla), [M]\,\rangle$ of the multiplicative sequence on M only depends on the de Rham cohomology class $F(TM)$ of the differential form $F(TM, \nabla)$ and is thus independent of the connection ∇, for compact manifolds with boundary things are more difficult and the dependence of $\langle\, F(TM,\,\nabla), [M]\,\rangle$ on ∇ inevitably involves the values of transgression forms. In general it is quite difficult to calculate the values of multiplicative sequences or transgression forms, for complex projective spaces $\mathbb{C}P^n$ however the problem can be solved completely. The complex tangent bundle of the complex projective space is isomorphic to

$$T^{1,0}\mathbb{C}P^n \;=\; \mathrm{Hom}\,(\,\mathcal{O}(-1),\,\mathbb{C}^{n+1}/_{\mathcal{O}(-1)}\,) \;=\; (\,n+1\,)\,\mathcal{O}(1)\;-\;\mathbb{C}$$

where $\mathbb{C}$ and $\mathbb{C}^{n+1}$ denote the trivial complex vector bundles over $\mathbb{C}P^n$ of rank 1 and $n+1$ and $\mathcal{O}(-1)$ is the tautological complex line bundle with dual "hyperplane" bundle $\mathcal{O}(1)$. Forgetting the complex structure we get an isomorphism of real vector bundles

$$T\,\mathbb{C}P^n\,\oplus\,\mathbb{R}^2 \;=\; (\,n+1\,)\,\mathcal{O}(-1)^{\mathbb{R}} \tag{15}$$

which results in the equality $F(T\mathbb{C}P^n) = F(\mathcal{O}(-1)^{\mathbb{R}})^{n+1}$ for every multiplicative sequence of Pontryagin classes as trivial or more generally flat vector bundles VM only contribute a factor $F(VM) = 1$. In order to calculate the first Pontryagin class of the real plane bundle $\mathcal{O}(-1)^{\mathbb{R}}$ we consider its complexification $\mathcal{O}(-1)^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathcal{O}(-1) \oplus \mathcal{O}(1)$, which tells us

$$p_1(\,\mathcal{O}(-1)^{\mathbb{R}}\,) \;=\; -\,c_2(\,\mathcal{O}(-1)\oplus\mathcal{O}(1)\,) \;=\; -\,c_1(\,\mathcal{O}(-1)\,)\,c_1(\,\mathcal{O}(1)\,) \;=\; \Big(\frac{\omega^{\mathrm{FS}}}{\pi}\Big)^2$$

where ω^{FS} is the Kähler form of the Fubini–Study metric on $\mathbb{C}P^n$, because the first Chern form of the complex line bundles $\mathcal{O}(k)$ with respect to a Fubini–Study type connection reads:

$$c_1(\,\mathcal{O}(\,k\,),\,\nabla^{\mathrm{FS}}\,) \;=\; k\,\frac{\omega^{\mathrm{FS}}}{\pi}$$

All in all our consideration starting with the isomorphism (15) imply the following equality

$$F(\,T\mathbb{C}P^n,\,\nabla^{\mathrm{FS}}\,) \;=\; F(\,\frac{\omega^{\mathrm{FS}}}{\pi}\,)^{n+1}$$

in de Rham cohomology, which must be true already on the level of differential forms, because both sides are left invariant and thus parallel on the symmetric space $\mathbb{C}P^n$. Using the volume integral $\int_{\mathbb{C}P^n}\frac{1}{n!}(\omega^{\mathrm{FS}})^n = \frac{\pi^n}{n!}$ of the complex projective space we find $\langle\,(\frac{\omega^{\mathrm{FS}}}{\pi})^n,\,[\mathbb{C}P^n]\,\rangle = 1$ or:

$$\langle\;F(\,T\mathbb{C}P^n\,),\;[\,\mathbb{C}P^n\,]\,\rangle \;=\; \mathrm{res}_{z=0}\left[\frac{F(z)^{n+1}}{z^{n+1}}dz\right] \tag{16}$$

In this way we have reduced the problem of calculating the values $\langle\,F(T\mathbb{C}P^n),\,[\mathbb{C}P^n]\,\rangle$ of the multiplicative sequence F on complex projective spaces to the combinatorial problem of finding the residues on the right hand side of equation (16). Introducing the odd formal power series $\phi(z) = z + O(z^3)$ explicitly as the (composition) inverse of $\frac{z}{F(z)} = z + O(z^3)$ or implicitly by $F(\phi(z)) = \frac{\phi(z)}{z}$ we can apply the transformation rule for residues to get:

$$\mathrm{res}_{z=0}\left[\frac{F(z)^{n+1}}{z^{n+1}}\,dz\right] \;=\; \mathrm{res}_{z=0}\left[\frac{F(\phi(z))^{n+1}}{\phi(z)^{n+1}}\,\phi'(z)\,dz\right] \;=\; \mathrm{res}_{z=0}\left[\frac{\phi'(z)}{z^{n+1}}\,dz\right]$$

Theorem 3.4 (Hirzebruch's Proof of the Signature Theorem [Hir])
Consider a multiplicative sequence of Pontryagin classes associated to an even formal power series $F(z) = 1 + O(z^2)$*. The odd formal power series* $\phi(z) = z + O(z^3)$ *defined as the composition inverse of the odd power series* $\frac{z}{F(z)} = z + O(z^3)$ *or implicity by* $F(\phi(z)) = \frac{\phi(z)}{z}$ *is the generating power series for the values of the multiplicative sequence associated to* F

$$\phi'(z) \;\;=\;\; 1 \;+\; \sum_{k>0} \langle\, F(\; T\mathbb{C}P^{2k}\;),\; [\;\mathbb{C}P^{2k}\;]\,\rangle z^{2k}$$

on the complex projective spaces. Note that $\langle\, F(T\mathbb{C}P^k),\, [\mathbb{C}P^k]\,\rangle = 0$ *for odd* k *by definition.*

Using Theorem 3.4 we can construct multiplicative sequences of Pontryagin classes assuming arbitrarily prescribed values on the even and vanishing on the odd complex projective spaces. Collecting these values in the formal power series $\phi'(z) := 1 + \sum_{k>0} \phi'_k z^{2k}$ we simply choose the multiplicative sequence parametrized by the quotient $F(z) = \frac{z}{f(z)}$, where f is the unique formal power series solution to the differential equation $f' = \frac{1}{\phi'(f)}$ with initial value $f(0) = 0$. In particular there is a unique multiplicative sequence which takes the values 1 and 0 on even and odd complex projective spaces respectively, namely the sequence associated to

$$L(z) \;\;=\;\; \frac{z}{\tanh\, z}$$

because $l(z) = \tanh\, z$ is the unique solution to the differential equation $l' = 1 - l^2$ with initial value $l(0) = 0$. On the other hand the even and odd complex projective spaces have signature 1 and 0 respectively and generate the rational oriented cobordism ring due to a result of Thom. With signature being an oriented cobordism invariant Hirzebruch concluded

$$\mathrm{sign}\, M \;\;=\;\; \langle\, L(TM),\, [M]\,\rangle$$

for every closed manifold M. Instead of pursuing these ideas further and start a discussion of general rational oriented cobordism invariants or "genera" we want to use Theorem 3.4 to calculate the values of a particular genus, the $\widehat{A}$–genus, parametrized by

$$\widehat{A}(z) \;\;=\;\; \frac{\frac{z}{2}}{\sinh\, \frac{z}{2}}$$

on the complex projective spaces. With $\widehat{a}(z) = 2\sinh\frac{z}{2}$ being a solution to the differential equation $\widehat{a}'(z) = (1 + \frac{1}{4}\widehat{a}^2(z))^{\frac{1}{2}}$ we immediately find the generating formal power series

$$1 \;+\; \sum_{k>0} \langle\, \widehat{A}(T\mathbb{C}P^{2k}),\, [\mathbb{C}P^{2k}]\,\rangle\, z^{2k} \;\;=\;\; \Big(\, 1 + \frac{z^2}{4}\,\Big)^{-\frac{1}{2}} \;\;=\;\; \sum_{k\geq 0} \Big(-\frac{1}{16}\Big)^k \binom{2k}{k} z^{2k} \qquad (17)$$

by Newton's power series expansion $(1+z)^e = \sum_{k\geq 0} \binom{e}{k} z^k$ for $e = -\frac{1}{2}$ and $\binom{-\frac{1}{2}}{k} = (-\frac{1}{4})^k \binom{2k}{k}$.

In a similar vein it is possible to calculate the values of multiplicative sequences of Pontryagin classes on the quaternionic projective spaces $\mathbb{H}P^n$. Thinking of $\mathbb{H}P^n$ as the base of the quaternionic Hopf fibration we get a Fubini–Study type submersion metric g^{FS} from the round unit sphere metric on the total space S^{4n+3} with scalar curvature $\kappa^{\mathrm{FS}} = 16\,n\,(n+2)$. The role of the Kähler form ω^{FS} on $\mathbb{C}P^n$ is taken by the Kraines form Ω^{FS}, a parallel 4–form depending quadratically on the metric g^{FS}. By construction the volume of the quaternionic projective space $\mathbb{H}P^n$ with respect to g^{FS} is the quotient of two unit sphere volumes:

$$\mathrm{vol}(\,\mathbb{H}P^n,\ g^{\mathrm{FS}}\,) \quad=\quad \langle\ \frac{1}{(2n+1)!}(\,\Omega^{\mathrm{FS}}\,)^n,\ [\,\mathbb{H}P^n\,]\ \rangle \quad=\quad \frac{\mathrm{vol}(\,S^{4n+3}\,)}{\mathrm{vol}(\,S^3\,)} \quad=\quad \frac{\pi^{2n}}{(2n+1)!}$$

Moreover the quaternionic projective space $\mathbb{H}P^n$ comes endowed with a self–dual complex plane bundle $H(\mathbb{H}P^n)$, whose first Pontryagin class u satisfies the quaternionic volume identity $\langle\,u^n,\,[\mathbb{H}P^n]\,\rangle = 1$ according to $u = (\frac{\kappa^{\mathrm{FS}}}{16\,n\,\pi\,(n+2)})^2\,\Omega^{\mathrm{FS}} = \frac{1}{\pi^2}\,\Omega^{\mathrm{FS}}$, compare [W]. The tangent bundle of $\mathbb{H}P^n$ can be written in terms of $H(\mathbb{H}P^n)$ by means of the isomorphism

$$T\,\mathbb{H}P^n \otimes_{\mathbb{R}} \mathbb{C} \quad\cong\quad H(\mathbb{H}P^n)\otimes\mathbb{C}^{2n+2} \;-\; \mathrm{Sym}^2 H(\mathbb{H}P^n) \;-\; \mathbb{C}$$

of self–dual complex vector bundles, which gives rise to an identity in de Rham cohomology:

$$F(\,T\,\mathbb{H}P^n,\ \nabla^{FS}\,) \quad=\quad \frac{F(\sqrt{u})^{2n+2}}{F(2\sqrt{u})}$$

This identity must already be true on the level of differential forms as both sides are left invariant and thus parallel differential forms. Using the integral $\langle\,u^n,\,[\mathbb{H}P^n]\,\rangle = 1$ we find

$$\langle\ F(\,T\,\mathbb{H}P^n\,),\ [\mathbb{H}P^n]\ \rangle \quad=\quad \mathrm{res}_{z=0}\left[\frac{F(z)^{2n+2}}{F(2z)}\,\frac{dz}{z^{2n+1}}\right] \quad=\quad \frac{1}{2}\,\mathrm{res}_{z=0}\left[\frac{\phi^{-1}(2\phi(z))\,\phi'(z)}{z^{2n+2}}\,dz\right]$$

upon the substitution $z \rightsquigarrow \phi(z)$ under the residue, where $\phi(z) = z + O(z^3)$ is the composition inverse of $\frac{z}{F(z)} = z + O(z^3)$ as before. Interestingly we can use this formula for the "identity" multiplicative sequence with values in the rational oriented cobordism ring $\Omega^{\mathrm{SO}}\otimes\mathbb{Q}$:

$$\phi'(z) \quad:=\quad 1 \;+\; \sum_{n>0}\,[\,\mathbb{C}P^{2n}\,]\,z^{2n} \quad\in\quad (\Omega^{\mathrm{SO}}\otimes\mathbb{Q})[\![\,z\,]\!]$$

The formula for the value of a multiplicative sequence on $\mathbb{H}P^n$ then becomes the equality

$$z \;+\; \sum_{n>0}\,[\,\mathbb{H}P^n\,]\,z^{2n+1} \quad=\quad \frac{1}{2}\,\phi^{-1}(\,2\phi(z)\,)\,\phi'(z)$$

of formal power series with coefficients in $\Omega^{\mathrm{SO}}\otimes\mathbb{Q}$, or in terms of rational oriented cobordisms

$$\begin{aligned}
[\mathbb{H}P^1] &= 0\\
[\mathbb{H}P^2] &= -\,2\,[\mathbb{C}P^4] \;+\; 3\,[\mathbb{C}P^2]^2\\
[\mathbb{H}P^3] &= -\,8\,[\mathbb{C}P^6] \;+\; 24\,[\mathbb{C}P^4]\,[\mathbb{C}P^2] \;-\; 16\,[\mathbb{C}P^2]^3\\
[\mathbb{H}P^4] &= -\,\tfrac{82}{3}\,[\mathbb{C}P^8] \;+\; 90\,[\mathbb{C}P^6]\,[\mathbb{C}P^2] \;+\; 45\,[\mathbb{C}P^4]^2 \;-\; 200\,[\mathbb{C}P^4]\,[\mathbb{C}P^2]^2 \;+\; \tfrac{280}{3}\,[\mathbb{C}P^2]^4
\end{aligned}$$

etc. It is gratifying to see that the 4–sphere $\mathbb{H}P^1$ bounds over $\mathbb{Q}$ according to this calculation.

4 Solution of the Hypersurface Transgression Problem

On closed manifolds the evaluation of multiplicative sequences of Pontryagin forms factorizes over the Pontryagin classes in de Rham cohomology. Similar computations on manifolds with boundary involve additional boundary contributions in form of the integrals of associated transgression forms. In this section we want to study Definition 3.2 of the logarithmic transgression form associated to a multiplicative sequence in more detail for the standard transgression problem associated to a (boundary) hypersurface $M \subset M^{\mathrm{out}}$. In particular the formula obtained in Corollary 4.3 for the logarithmic transgression form of a permeable hypersurface $M \subset M^{\mathrm{out}}$ only depends on the 3–form $g(d^{\nabla}\mathrm{I\!I}, \mathrm{I\!I}) \in \Gamma(\Lambda^3 T^*M)$.

In order to describe the standard transgression problem associated to a hypersurface let us consider a compact subset $W \subset M^{\mathrm{out}}$ of an oriented manifold M^{out} with smooth boundary $M = \partial W$ and fix a collar neighborhood $]-\varepsilon,\varepsilon[\times M \supset M$ of M such that the coordinate r of $]-\varepsilon,\varepsilon[$ is zero on M and negative on the interior of W. A Riemannian metric g^{collar} on M^{out} is called a collar metric, if it restricts on the collar $]-\varepsilon,\varepsilon[\times M$ to the product

$$g^{\mathrm{collar}}\Big|_{]-\varepsilon,\varepsilon[\times M} \;=\; dr \otimes dr \;+\; g$$

of the standard metric on $]-\varepsilon,\varepsilon[$ with a Riemannian metric g on M, the vector field $N := \frac{\partial}{\partial r}$ is thus the outward pointing normal vector field to M. In the context of Theorem 1.1 we are interested in the value of a multiplicative sequence of Pontryagin forms

$$\langle\, F(\,TM^{\mathrm{out}},\ \nabla^{\mathrm{collar}}\,), [\,W\,]\,\rangle \;:=\; \int_W F(\,TM^{\mathrm{out}},\ \nabla^{\mathrm{collar}}\,) \tag{18}$$

for the Levi–Civita connection ∇^{collar} of a collar metric g^{collar}. An exact evaluation of the integral (18) may be impossible due to lack of precise information concerning the collar metric g^{collar}, however in favorable situations there might be another Riemannian metric g^{out} on M^{out} such that the calculation of the integral $\langle\, F(TM^{\mathrm{out}}, \nabla^{\mathrm{out}}), [\,W\,]\,\rangle$ for the Levi–Civita connection of g^{out} is feasible. In this case Stokes' Theorem allows us to convert the difference

$$\int_W F(TM^{\mathrm{out}}, \nabla^{\mathrm{out}}) \;-\; \int_W F(TM^{\mathrm{out}}, \nabla^{\mathrm{collar}}) \;=\; \int_M \mathrm{Trans}\, F(TM^{\mathrm{out}}, \nabla^{\mathrm{collar}}, \nabla^{\mathrm{out}}) \tag{19}$$

into a boundary term involving the transgression form $(\mathrm{Trans}\,F)(TM^{\mathrm{out}}, \nabla^{\mathrm{collar}}, \nabla^{\mathrm{out}})$. In turn the integration over M only depends on the restriction of the transgression form

$$(\,\mathrm{Trans}\,F\,)(\,TM^{\mathrm{out}}, \nabla^{\mathrm{collar}}, \nabla^{\mathrm{out}}\,)\Big|_M \;=\; F(TM^{\mathrm{out}}\big|_M, \nabla)\,(\,\mathrm{trans}\,F\,)(TM^{\mathrm{out}}\big|_M, \nabla, \nabla^{\mathrm{out}}\big|_M)$$

to the hypersurface $M \subset M^{\mathrm{out}}$, which agrees by naturality with the transgression form for the restricted connections $\nabla = \nabla^{\mathrm{collar}}\big|_M$ and $\nabla^{\mathrm{out}}|_M$ on $TM^{\mathrm{out}}|_M$. Of course the restricted connection ∇ is just the Levi–Civita connection for the metric g on M extended trivially to a metric connection on $TM^{\mathrm{out}}|_M \cong TM \oplus \mathrm{Norm}\,M$ so that $F(TM^{\mathrm{out}}|_M, \nabla) = F(TM, \nabla)$. According to Section 2 the homotopy $t \longmapsto t\nabla^{\mathrm{out}} + (1-t)\nabla^{\mathrm{collar}}$ between the connections ∇^{collar} and ∇^{out} on TM^{out} restricts to the linear interpolation $t \longmapsto \nabla + t\,\mathrm{I\!I} \wedge N$ between ∇ and $\nabla^{\mathrm{out}}|_M$ on $TM^{\mathrm{out}}|_M$, in consequence the associated transgression form depends crucially on the second fundamental form $\mathrm{I\!I}$ of $M \subset M^{\mathrm{out}}$. Summarizing these considerations we define:

Definition 4.1 (Logarithmic Transgression Form of a Hypersurface)
The logarithmic transgression form of a hypersurface $M \subset M^{\mathrm{out}}$ in a Riemannian manifold M^{out} with metric g^{out} is the logarithmic transgression form of (the derivative of) the linear interpolation $\nabla^t = \nabla + t\,\mathrm{I\!I} \wedge N$ between the metric connections ∇ and $\nabla + \mathrm{I\!I} \wedge N$ on $TM^{\mathrm{out}}|_M$:

$$\delta\,(\log\,F)(\,TM^{\mathrm{out}}\big|_M\,,\nabla^t,\mathrm{I\!I}\wedge N\,)\;\;=\;\;\sum_{k>0} k\,f_k\,\mathrm{tr}\left(\frac{\mathrm{I\!I}\wedge N}{2\pi i}\left(\frac{R^t}{2\pi i}\right)^{2k-1}\right)$$

In this formula the curvature R^t of ∇^t is given by the Gauß–Codazzi–Mainardi equation (4):

$$R^t \;\;=\;\; R \;+\; t\,d^\nabla\mathrm{I\!I}\,\wedge\,N \;+\; t^2\,\mathrm{I\!I}^\sharp\,\otimes\,\mathrm{I\!I}$$

Although this definition of the logarithmic transgression form is useful for theoretical considerations, it is not suitable for actual calculations, because its time dependence involves the time dependent curvature R^t of the connection ∇^t on $TM^{\mathrm{out}}|_M$. For the rest of this section we want to study the definition of the logarithmic transgression form in more detail in order to derive a more convenient expansion for $\delta(\log\,F)(\,TM^{\mathrm{out}}|_M\,,\nabla^t,\mathrm{I\!I}\wedge N\,)$. Let us begin this endeavour with the definition of the basic geometric differential forms

$$\xi \;:=\; \sum_{k\geq 0} g(\,d^\nabla\mathrm{I\!I},\,R^k\,\mathrm{I\!I}\,) \qquad \psi \;:=\; \sum_{k\geq 0} g(\,\mathrm{I\!I},\,R^k\,\mathrm{I\!I}\,) \qquad \Psi \;:=\; \sum_{k\geq 0} g(\,d^\nabla\mathrm{I\!I},\,R^k\,d^\nabla\mathrm{I\!I}\,) \tag{20}$$

encoding the geometry of the hypersurface $M \subset M^{\mathrm{out}}$. Evidently $\xi \in \Gamma(\Lambda^{\mathrm{odd}}T^*M)$ is an odd differential form with non–zero homogeneous components $\xi_3, \xi_5, \ldots$ living in degrees at least 3. Similarly ψ and Ψ are differential forms with non–zero homogeneous components $\psi_4, \psi_8, \ldots$ and $\Psi_4, \Psi_8, \ldots$ concentrated in positive degrees divisible by 4, because say

$$g(\,\mathrm{I\!I},\,R^k\,\mathrm{I\!I}\,) \;\;=\;\; (-1)^k\,g(\,R^k\,\mathrm{I\!I},\,\mathrm{I\!I}\,) \;\;=\;\; -\,(-1)^k\,g(\,\mathrm{I\!I},\,R^k\,\mathrm{I\!I}\,)$$

where the stray sign change in the last equality is caused by passing the odd vector valued form $\mathrm{I\!I}$ past the odd form $R^k\,\mathrm{I\!I}$. A similar argument implies that the last possible variant

$$i^{N+1}\xi \;\;:=\;\; \sum_{k\geq 0} g(\,\mathrm{I\!I},\,R^k\,d^\nabla\mathrm{I\!I}\,) \;\;=\;\; \sum_{k\geq 0}(-1)^k\,g(\,d^\nabla\mathrm{I\!I},\,R^k\,\mathrm{I\!I}\,)$$

in the definition (20) of the basic geometric differential forms results in a differential form $i^{N+1}\xi$ on M, which is the image of ξ under the operator $i^{N+1} : \Lambda^{\mathrm{odd}}T^*M \longrightarrow \Lambda^{\mathrm{odd}}T^*M$, which multiplies a homogeneous odd form of degree r by i^{r+1}. Using the second Bianchi identity $d^\nabla R = 0$ and the identity $d^\nabla(\,d^\nabla\mathrm{I\!I}\,) = R\,\mathrm{I\!I}$ for the metric connection ∇ on TM we readily find the exterior differential system satisfied by the differential forms ξ, ψ and Ψ

$$\begin{aligned}
d\xi &\;=\; \sum_{k\geq 0}\Big(\,g(\,R\,\mathrm{I\!I},\,R^k\,\mathrm{I\!I}\,) \;+\; g(\,d^\nabla\mathrm{I\!I},\,R^k\,d^\nabla\mathrm{I\!I}\,)\Big) &&=\;\; \Psi \;-\; \psi\\
d\psi &\;=\; \sum_{k\geq 0}\Big(\,g(\,d^\nabla\mathrm{I\!I},\,R^k\,\mathrm{I\!I}\,) \;-\; g(\,\mathrm{I\!I},\,R^k\,d^\nabla\mathrm{I\!I}\,)\Big) &&=\;\; \xi \;-\; i^{N+1}\xi\\
d\Psi &\;=\; \sum_{k\geq 0}\Big(\,g(\,R\,\mathrm{I\!I},\,R^k\,d^\nabla\mathrm{I\!I}\,) \;+\; g(\,d^\nabla\mathrm{I\!I},\,R^k\,R\,\mathrm{I\!I}\,)\Big) &&=\;\; \xi \;-\; i^{N+1}\xi
\end{aligned} \tag{21}$$

where the missing summands care for themselves. In terms of the homogeneous components of the differential forms ξ, ψ and Ψ these equations can be written schematically:

$$\mathbb{R}\,\xi_3 \xrightarrow{\;d\;} \mathbb{R}\,\Psi_4 \oplus \mathbb{R}\,\psi_4 \xrightarrow{\;d\;} \mathbb{R}\,\xi_5 \qquad \mathbb{R}\,\xi_7 \xrightarrow{\;d\;} \mathbb{R}\,\Psi_8 \oplus \mathbb{R}\,\psi_8 \xrightarrow{\;d\;} \mathbb{R}\,\xi_9 \qquad \ldots$$

Perhaps surprisingly the main problem in understanding the logarithmic transgression form of a hypersurface is to express the time dependent variants of the geometric differential forms

$$\xi^t \;:=\; \sum_{k\geq 0} g(\,d^\nabla \mathrm{I\!I}, (R^t)^k\,\mathrm{I\!I}\,) \qquad \psi^t \;:=\; \sum_{k\geq 0} g(\,\mathrm{I\!I}, (R^t)^k\,\mathrm{I\!I}\,) \qquad \Psi^t \;:=\; \sum_{k\geq 0} g(\,d^\nabla \mathrm{I\!I}, (R^t)^k\,d^\nabla \mathrm{I\!I}\,)$$

in terms of the basic geometric forms ξ, ψ and Ψ. In passing we note that the differential forms ξ^t, ψ^t and Ψ^t satisfy the same exterior differential system (21) as the basic geometric forms ξ, ψ and Ψ, in the argument given above we need only observe $d^\nabla \mathrm{I\!I} = d^{\nabla^t} \mathrm{I\!I}$ and replace the metric connection ∇ by the metric connection ∇^t. Moreover the possibly non–zero homogeneous components of the differential forms ψ^t and Ψ^t are concentrated in positive degrees divisible by 4 as before, while ξ^t is an odd differential form with vanishing component in degree 1. In difference to the basic geometric differential forms ξ, ψ and Ψ however the time dependence of ξ^t, ψ^t and Ψ^t allows us study the ordinary differential equation

$$\begin{aligned}
\frac{d}{dt}\,\xi^t &= -\,2\,t\,\xi^t \wedge (\,\psi^t + \Psi^t\,) \\
\frac{d}{dt}\,\psi^t &= -\,2\,t\,\psi^t \wedge \psi^t + 2\,t\,\xi^t \wedge (\,i^{N+1}\xi^t\,) \\
\frac{d}{dt}\,\Psi^t &= -\,2\,t\,\Psi^t \wedge \Psi^t - 2\,t\,\xi^t \wedge (\,i^{N+1}\xi^t\,)
\end{aligned}$$

satisfied by the triple $(\xi^t,\,\psi^t,\,\Psi^t)$, which is a direct consequence of the corresponding equation for the time dependent curvature $R^t = R + t\,d^\nabla \mathrm{I\!I} \wedge N + t^2\,\mathrm{I\!I}^\sharp \otimes \mathrm{I\!I}$. In fact we find

$$\begin{aligned}
\frac{d}{dt}\,\xi^t &= \sum_{k,\,l\geq 0} g\Big(\,d^\nabla \mathrm{I\!I},\,(R^t)^k \Big((d^\nabla \mathrm{I\!I})^\sharp \wedge N - N^\sharp \otimes (d^\nabla \mathrm{I\!I}) + 2\,t\,\mathrm{I\!I}^\sharp \otimes \mathrm{I\!I}\Big)\,(R^t)^l\,\mathrm{I\!I}\,\Big) \\
&= \sum_{k,\,l\geq 0} \Big(\,g(\,d^\nabla \mathrm{I\!I},\,(R^t)^k\,N\,) \wedge g(\,d^\nabla \mathrm{I\!I},\,(R^t)^l\,\mathrm{I\!I}\,) - g(\,d^\nabla \mathrm{I\!I},\,(R^t)^k\,d^\nabla \mathrm{I\!I}\,) \wedge g(\,N,\,(R^t)^l\,\mathrm{I\!I}\,)\Big) \\
&\qquad - 2\,t \sum_{k,\,l\geq 0} g(\,d^\nabla \mathrm{I\!I},\,(R^t)^k\,\mathrm{I\!I}\,) \wedge g(\,\mathrm{I\!I},\,(R^t)^l\,\mathrm{I\!I}\,) \\
&= -\,2\,t\,\Psi^t \wedge \xi^t - 2\,t\,\xi^t \wedge \psi^t
\end{aligned}$$

using $R^t\,N = -\,t\,d^\nabla \mathrm{I\!I}$, replacing the leftmost $d^\nabla \mathrm{I\!I}$ by $\mathrm{I\!I}$ and the rightmost $\mathrm{I\!I}$ by $d^\nabla \mathrm{I\!I}$ respectively in the second line provides the analoguous differential equations for ψ^t and Ψ^t.

Lemma 4.2 (Solution of Special Ordinary Differential Equation)
Consider the exterior algebra $\Lambda\,T^*$ of alternating forms on a vector space T and the subspace $\Lambda^{4\bullet}_\circ T^* \subset \Lambda\,T^*$ of forms concentrated in degrees divisible by 4 without constant term.

Moreover let $i^{N+1} : \Lambda^{\mathrm{odd}}T^* \longrightarrow \Lambda^{\mathrm{odd}}T^*$ be the operator on the subspace $\Lambda^{\mathrm{odd}}T^* \subset \Lambda T^*$ of odd forms on T multiplying a homogeneous odd form of degree r by i^{r+1}. For every triple of forms $(\xi,\, \psi,\, \Psi) \in \Lambda^{\mathrm{odd}}T^* \oplus \Lambda^{4\bullet}_{\circ}T^* \oplus \Lambda^{4\bullet}_{\circ}T^*$ the formal power series $(\xi^t,\, \psi^t,\, \Psi^t)$ in t with

$$\begin{aligned}
\xi^t &:= \xi \wedge (\, 1 \,+\, t^2\,\psi\,)^{-1} \wedge (\, 1 \,+\, t^2\,\Psi\,)^{-1} \\
\psi^t &:= \psi \wedge (\, 1 \,+\, t^2\,\psi\,)^{-1} \,+\, t^2\,\xi \wedge (\, i^{N+1}\xi\,) \wedge (\, 1 \,+\, t^2\,\psi\,)^{-2} \wedge (\, 1 \,+\, t^2\,\Psi\,)^{-1} \\
\Psi^t &:= \Psi \wedge (\, 1 \,+\, t^2\,\Psi\,)^{-1} \,-\, t^2\,\xi \wedge (\, i^{N+1}\xi\,) \wedge (\, 1 \,+\, t^2\,\psi\,)^{-1} \wedge (\, 1 \,+\, t^2\,\Psi\,)^{-2}
\end{aligned}$$

is the unique solution with initial value $(\xi,\, \psi,\, \Psi)$ in 0 to the ordinary differential equation:

$$\begin{aligned}
\frac{d}{dt}\,\xi^t &= -\,2\,t\,\xi^t \wedge (\, \psi^t \,+\, \Psi^t\,) \\
\frac{d}{dt}\,\psi^t &= -\,2\,t\,\psi^t \wedge \psi^t \,+\, 2\,t\,\xi^t \wedge (\, i^{N+1}\xi^t\,) \\
\frac{d}{dt}\,\Psi^t &= -\,2\,t\,\Psi^t \wedge \Psi^t \,-\, 2\,t\,\xi^t \wedge (\, i^{N+1}\xi^t\,)
\end{aligned}$$

Proof: Concerning the ordinary differential equation considered it may be interesting to know that the modified square $\xi \wedge (\, i^{N+1}\xi\,)$ of an odd form ξ is always concentrated in degrees divisible by 4 without constant term, because the number operator is additive on products:

$$i^N(\, \xi \wedge (\, i^{N+1}\xi\,)\,) \;=\; (\, i^N\xi\,) \wedge (\, i^N\, i^{N+1}\xi\,) \;=\; (\, i^N\xi\,) \wedge (\, -\,i\,\xi\,) \;=\; \xi \wedge (\, i^{N+1}\xi\,)$$

In consequence the standard theorems about the existence and uniqueness of solutions to ordinary differential equations guarantee the existence for all times of a unique solution staying in $\Lambda^{\mathrm{odd}}T^* \oplus \Lambda^{4\bullet}_{\circ}T^* \oplus \Lambda^{4\bullet}_{\circ}T^*$ for every initial value in $\Lambda^{\mathrm{odd}}T^* \oplus \Lambda^{4\bullet}_{\circ}T^* \oplus \Lambda^{4\bullet}_{\circ}T^*$. Of course for the ordinary differential equation considered a straightforward induction on degree shows that the homogeneous components of ξ^t, ψ^t and Ψ^t of degree r are polynomials in t and the homogeneous components of degree less than r of the initial values ξ, ψ and Ψ.

On the other hand the power series $(\xi^t,\, \psi^t,\, \Psi^t)$ in question certainly evaluates to the given intial value $(\xi^0,\, \psi^0,\, \Psi^0) \,=\, (\xi,\, \psi,\, \Psi)$ in $t \,=\, 0$. In order to verify it is actually a solution to the ordinary differential equation considered we observe that the components ψ^t and Ψ^t of the power series can be rewritten in terms of the component ξ^t to read:

$$\begin{aligned}
\psi^t &:= \psi \wedge (\, 1 \,+\, t^2\,\psi\,)^{-1} \,+\, t^2\,\xi^t \wedge (\, i^{N+1}\xi\,) \wedge (\, 1 \,+\, t^2\,\psi\,)^{-1} \\
\Psi^t &:= \Psi \wedge (\, 1 \,+\, t^2\,\Psi\,)^{-1} \,-\, t^2\,\xi^t \wedge (\, i^{N+1}\xi\,) \wedge (\, 1 \,+\, t^2\,\Psi\,)^{-1}
\end{aligned}$$

With ξ^t being an odd form we know of course $\xi^t \wedge \xi^t \,=\, 0$ and thus we find immediately:

$$\frac{d}{dt}\,\xi^t \;=\; \xi^t \wedge \Big(-\,2\,t\,\psi\,(\,1 \,+\, t^2\,\psi\,)^{-1} - 2\,t\,\Psi\,(\,1 \,+\, t^2\,\Psi\,)^{-1}\Big) \;=\; -\,2\,t\,\xi^t \wedge (\, \psi^t \,+\, \Psi^t\,)$$

Verifying the differential equation for ψ^t and Ψ^t is somewhat more involved. To begin with we rewrite the definition of the component ψ^t of the power series $(\xi^t,\, \psi^t,\, \Psi^t)$ a second time

$$\psi^t \;:=\; \Big(\, \psi \,+\, t^2\,\xi^t \wedge (\, i^{N+1}\xi\,)\,\Big) \wedge (\, 1 \,+\, t^2\,\psi\,)^{-1}$$

in order to use the product rule and the formula just derived for $\frac{d}{dt}\xi^t$ in the calculation:

$$\begin{aligned}\frac{d}{dt}\,\psi^t &= \Big(2\,t\,\xi^t + t^2\,\xi^t \wedge \Big(-\,2\,t\,\psi \wedge (1 + t^2\,\psi)^{-1} - 2\,t\,\Psi \wedge (1 + t^2\,\Psi)^{-1}\Big)\Big) \\ &\qquad \wedge (i^{N+1}\xi) \wedge (1 + t^2\,\psi)^{-1} + \psi^t \wedge (-\,2\,t\,\psi \wedge (1 + t^2\,\psi)^{-1}) \\ &= -\,2\,t\,\psi^t \wedge \psi^t + 2\,t\,\xi^t \wedge (i^{N+1}\xi) \wedge (1 + t^2\,\psi)^{-1} \\ &\qquad \wedge \Big(1 - t^2\,\psi \wedge (1 + t^2\,\psi)^{-1} - t^2\,\Psi \wedge (1 + t^2\,\Psi)^{-1} + t^2\,\psi^t \Big) \\ &= -\,2\,t\,\psi^t \wedge \psi^t + 2\,t\,\xi^t \wedge (i^{N+1}\xi) \wedge (1 + t^2\,\psi)^{-1} \wedge (1 + t^2\,\Psi)^{-1}\end{aligned}$$

In the last line we used the tautology $1 - t^2\,\Psi \wedge (1 + t^2\,\Psi)^{-1} = (1 + t^2\,\Psi)^{-1}$ and the fact that wedging with the odd form ξ^t kills the difference $\psi^t - \psi \wedge (1 + t^2\,\psi)^{-1}$. On the other hand the operator i^{N+1} commutes by its very definition with taking the product

$$i^{N+1}(\,\xi^t\,) \;=\; (\,i^{N+1}\xi\,) \wedge (1 + t^2\,\psi)^{-1} \wedge (1 + t^2\,\Psi)^{-1}$$

with a form concentrated in degrees divisible by 4. Mutatis mutandis the same arguments imply the validity of the ordinary differential equation claimed for the component Ψ^t. □

Applying the solution formula of Lemma 4.2 in every point $p \in M$ we conclude that the time dependent geometric differential forms ξ^t, ψ^t and Ψ^t can be expressed completely in terms of the basic geometric differential forms ξ, ψ and Ψ of a hypersurface $M \subset M^{\mathrm{out}}$, say

$$\begin{aligned}&\sum_{k\geq 0} g(\,d^\nabla \mathrm{I\!I},\,(R^t)^k\,\mathrm{I\!I}\,) \qquad (22) \\ &= \Big(\sum_{k\geq 0} g(\,d^\nabla \mathrm{I\!I},\,R^k\,\mathrm{I\!I}\,)\Big) \wedge \Big(1 + t^2 \sum_{\substack{k\geq 0\\ k\ \text{odd}}} g(\,\mathrm{I\!I},\,R^k\,\mathrm{I\!I}\,)\Big)^{-1} \wedge \Big(1 + t^2 \sum_{\substack{k\geq 0\\ k\ \text{even}}} g(\,d^\nabla \mathrm{I\!I},\,R^k\,d^\nabla \mathrm{I\!I}\,)\Big)^{-1}\end{aligned}$$

for the time dependent form ξ^t. What links this formula (22) to the logarithmic transgression form $\delta(\log F)(TM^{\mathrm{out}}|_M, \nabla^t, \mathrm{I\!I} \wedge N)$ of a hypersurface $M \subset M^{\mathrm{out}}$ associated to a multiplicative sequence of Pontryagin forms is that the traces occuring in Definition 4.1

$$\begin{aligned}\mathrm{tr}\left(\frac{\mathrm{I\!I} \wedge N}{2\pi i}\left(\frac{R^t}{2\pi i}\right)^{2k-1}\right) &= \frac{1}{(\,2\,\pi\,i\,)^{2k}}\,\mathrm{tr}\Big(\,\big(\,\mathrm{I\!I}^\sharp \otimes N - N^\sharp \otimes \mathrm{I\!I}\,\big)\,(\,R^t\,)^{2k-1}\Big) \\ &= \frac{(\,-1\,)^k}{(\,4\pi^2\,)^k}\,\Big(\,g(\,\mathrm{I\!I},\,(\,R^t\,)^{2k-1}\,N\,) - g(\,N,\,(\,R^t\,)^{2k-1}\,\mathrm{I\!I}\,)\Big) \\ &= 2\,t\,\frac{(\,-\,1\,)^{k-1}}{(\,4\,\pi^2\,)^k}\,g(\,d^\nabla \mathrm{I\!I},\,(\,R^t\,)^{2k-2}\,\mathrm{I\!I}\,) \qquad (23)\end{aligned}$$

of the logarithmic transgression form are essentially the homogeneous components ξ^t_{4k-1} of degree $4k-1$ of the time dependent geometric differential form ξ^t!

Corollary 4.3 (Logarithmic Transgression Form of Permeable Hypersurfaces)
Consider the linear interpolation $t \longmapsto \nabla^t := \nabla + t\,\mathrm{I\!I} \wedge N$ *between the two natural metric*

connections $\nabla^0 := \nabla$ and $\nabla^1 := \nabla^{\mathrm{out}}|_M$ on the restriction $TM^{\mathrm{out}}|_M$ of the outer tangent bundle to a permeable hypersurface $M \subset M^{\mathrm{out}}$. The logarithmic transgression form for this interpolating curve and a multiplicative sequence parametrized by $F(z) = 1 + O(z^2)$ reads

$$\delta\,(\log\,F)(\;TM^{\mathrm{out}}\big|_M\,,\,\nabla^t,\,\mathbb{I}\wedge N\;)\;\;=\;\;\frac{d}{dt}\left(\frac{t^2\,\xi}{4\pi^2}\,LF\Big(\sqrt{\frac{t^2\,d\xi}{4\pi^2}}\Big)\right)$$

in terms of the 3–form $\xi := g(d^\nabla\mathbb{I},\mathbb{I})$ and the power series $LF(z) := \frac{\log F(z)}{z^2}$, in consequence:

$$(\,\mathrm{trans}\,F\,)(\,TM^{\mathrm{out}}\big|_M\,,\nabla,\nabla+\mathbb{I}\wedge N\,)\;\;=\;\;\int_0^1 F\Big(\sqrt{\frac{t^2\,d\xi}{4\pi^2}}\Big)\,\frac{d}{dt}\left(\frac{t^2\,\xi}{4\pi^2}\,LF\Big(\sqrt{\frac{t^2\,d\xi}{4\pi^2}}\Big)\right)dt$$

With F and LF being even power series the ill–defined square root $\sqrt{d\xi}$ never materializes.

Proof: Recall that a permeable hypersurface is characterized by the vanishing of the 3–form $R\,\mathbb{I} = 0$. In consequence the basic geometric form $\psi = 0$ of a permeable hypersurface $M \subset M^{\mathrm{out}}$ vanishes identically, while $\xi = \xi_3 = g(\,d^\nabla\mathbb{I},\mathbb{I}\,)$ is a pure 3–form with exterior derivative $d\xi = \Psi = \Psi_4$. The expansion (22) of the time dependent differential form ξ^t thus simplifies to $\xi^t = \xi\wedge(\,1 + t^2\,d\xi\,)^{-1}$ so that the formula (23) for the traces occurring in the definition of the logarithmic transgression form turns into the formula

$$\begin{aligned}\delta\,(\,\log\,F\,)(\;TM^{\mathrm{out}}\big|_M\,,\,\nabla^t,\,\mathbb{I}\wedge N\;)\;\;&=\;\;\sum_{k>0}k\,f_k\,\mathrm{tr}\left(\frac{\mathbb{I}\wedge N}{2\pi i}\Big(\frac{R^t}{2\pi i}\Big)^{2k-1}\right)\\&=\;\;\frac{d}{dt}\left(\sum_{k>0}f_k\,\frac{t^2\,\xi}{4\pi^2}\wedge\Big(\frac{t^2\,d\xi}{4\pi^2}\Big)^{k-1}\right)\end{aligned}$$

where the f_k, $k > 0$, are the coefficients of the power series $(\log\,F)(z) =: \sum_{k>0} f_k z^{2k}$ as before. Applying the exterior derivative to the logarithmic transgression form we get

$$d\,\delta\,(\log\,F)(TM^{\mathrm{out}}\big|_M\,,\nabla^\tau,\mathbb{I}\wedge N)\;\;=\;\;\frac{d}{d\tau}\left(\sum_{k>0}f_k\Big(\frac{\tau^2\,d\xi}{4\pi^2}\Big)^k\right)\;\;=\;\;\frac{d}{d\tau}(\log\,F)\Big(\sqrt{\frac{\tau^2\,d\xi}{4\pi^2}}\Big)$$

the integrand of the interior integral in the definition (13) of the transgression form:

$$\begin{aligned}&(\,\mathrm{trans}\,F\,)(\,TM^{\mathrm{out}}\big|_M\,,\nabla,\nabla+\mathbb{I}\wedge N\,)\\&\quad=\;\int_0^1\exp\left(\int_0^t\frac{d}{d\tau}(\log\,F)\Big(\sqrt{\frac{\tau^2\,d\xi}{4\pi^2}}\Big)\,d\tau\right)\,\frac{d}{dt}\left(\frac{t^2\,\xi}{4\pi^2}\,LF\Big(\sqrt{\frac{t^2\,d\xi}{4\pi^2}}\Big)\right)dt\qquad\square\end{aligned}$$

5 Special Values of Multiplicative Sequences

In this section we will use the transgression forms of hypersurfaces discussed in Section 4 in order to calculate some special values of multiplicative sequences of Pontryagin forms introduced in Section 3. After a brief discussion of the Berger metrics g^ρ of parameter $\rho > -1$ on the unit sphere $S^{2n-1} \subset \mathfrak{p}$ of a hermitean vector space $\mathfrak{p}$ we calculate the integral of the differential form $F(TB^{2n}, \nabla^{\mathrm{collar},\rho})$ over the unit ball B^{2n} for the Levi–Civita connection $\nabla^{\mathrm{collar},\rho}$ of a suitable collar metric $g^{\mathrm{collar},\rho}$, which induces to a Berger metric g^ρ on the boundary $S^{2n-1} = \partial B^{2n}$. The collar metric $g^{\mathrm{collar},\rho}$ used in our calculations is actually the metric of the symmetric space $\mathbb{C}P^n$ and so we will rely heavily on the formulas detailed in Appendix B describing symmetric metrics in (conical) exponential coordinates.

Let us begin our calculations by a discussion of the natural CR–structure on odd dimensional spheres S^{2n-1}, $n \geq 1$. Consider a hermitean vector space $\mathfrak{p}$ of complex dimension n. The real part g of the hermitean form is a scalar product on $\mathfrak{p}$ making $\mathfrak{p}$ a euclidian vector space of dimension $2n$ endowed with an orthogonal complex structure $I \in \mathrm{End}\,\mathfrak{p}$ satisfying $I^2 = -1$ and $g(IX, IY) = g(X, Y)$ for all $X,\, Y \in \mathfrak{p}$. The identification $T_P\mathfrak{p} \cong \mathfrak{p}$ of the tangent spaces in points $P \in \mathfrak{p}$ with $\mathfrak{p}$ itself makes g a (flat) Kähler metric with integrable complex structure I on $\mathfrak{p}$ with associated Kähler form $\omega(X, Y) := g(IX, Y)$ and Riemannian volume form $\frac{1}{n!}\,\omega^n$ for the induced orientation. The Kähler metric g restricts to the standard round metric on the unit sphere $S^{2n-1} \subset \mathfrak{p}$, moreover the restrictions of the vector field and 1–form

$$C_P \;:=\; IP \qquad\qquad \gamma_P(\,X\,) \;:=\; C_P^\sharp(\,X\,) \;=\; g(\,IP,\,X\,)$$

on $\mathfrak{p}$ to S^{2n-1} are called the Reeb vector field and the contact form respectively. Evidently the trivial connection ∇^{triv} on $\mathfrak{p}$ is torsion free and thus the Levi–Civita connection for the Kähler metric g, in particular we find $d\gamma = 2\,\omega$. The Euler vector field $E_P := P$ on $\mathfrak{p}$ restricts to the outward pointing unit normal field on S^{2n-1} and satisfies $E \lrcorner\, \omega = \gamma$, hence the Riemannian volume form the standard round metric g on $S^{2n-1} \subset \mathfrak{p}$ reads:

$$\mathrm{vol}_g \;:=\; (\,E \lrcorner\, \frac{1}{n!}\,\omega^n\,)\Big|_{S^{2n-1}} \;=\; \frac{1}{2^{n-1}\,(n-1)!}\,\gamma \wedge (\,d\gamma\,)^{n-1}$$

In consequence the restriction of γ to S^{2n-1} is really a contact form on S^{2n-1}, moreover

$$\langle\, \frac{\gamma}{2\pi} \wedge (\,\frac{d\gamma}{2\pi}\,)^{n-1},\; [\,S^{2n-1}\,]\,\rangle \;=\; \frac{\Gamma(n)}{2\,\pi^n}\int_{S^{2n-1}} \frac{1}{2^{n-1}(n-1)!}\,\gamma \wedge (\,d\gamma\,)^{n-1} \;=\; 1 \qquad (24)$$

due to the standard volume $\frac{2\,\pi^n}{\Gamma(n)}$ of the unit sphere in dimension $2n-1$. In general the Reeb vector field on a contact manifold with contact form γ is the unique vector field C satisfying $\gamma(C) = 1$ and $C \lrcorner\, d\gamma = 0$, properties evidently satisfied by our Reeb field C.

Digressing for a second we recall that every Killing vector field K on a Riemannian manifold M with Levi–Civita connection ∇ and curvature tensor R determines a skew symmetric endomorphism field $\mathfrak{K} \in \Gamma(\Lambda^2 TM)$ such that $(\,K,\,\mathfrak{K}\,)$ satisfies the extended Killing equation:

$$\nabla_X K \;=\; \mathfrak{K}X \qquad\qquad \nabla_X \mathfrak{K} \;=\; R_{X,\,K}$$

In fact a trilinear form skew symmetric in two arguments is uniquely determined by its skew symmetrization in some other two arguments. Both trilinear forms $g(\,R_{X,K}Y,\,Z\,)$ and

$g(\,(\nabla_X\mathfrak{K})Y,\,Z\,)\;=\;g(\,\nabla^2_{X,Y}K,\,Z\,)$ are skew symmetric in Y and Z however and share the same skew symmetrization $g(\,R_{X,Y}K,\,Z\,)$ in X and Y by the first Bianchi identity.

The vector field C provides a nice example of the extended Killing equation. Evidently C is a Killing field for the flat Kähler metric g on $\mathfrak{p}$ with parallel skew symmetric covariant derivative $\nabla^{\mathrm{triv}}C\;=\;I$ consistent with $R^{\mathrm{triv}}\;=\;0$. After restriction to the unit sphere S^{2n-1} the same argument becomes more interesting, the Reeb field C is a Killing field on S^{2n-1} for the standard round metric g with curvature $R_{X,Y}\;=\;-\,(X\wedge Y)$, hence:

$$\nabla_XC\;=\;\mathfrak{C}X\qquad\qquad\nabla_X\mathfrak{C}\;=\;C\wedge X\tag{25}$$

where the skew symmetric endomorphism field $\mathfrak{C}$ evaluates in $P\in S^{2n-1}$ to the restriction

$$\mathfrak{C}_P\,X\;\;:=\;\;\mathrm{pr}_{\{\,P\,\}^\perp}(\,\nabla^{\mathrm{triv}}_XC\,)\;\;=\;\;\begin{cases}IX & \text{for } X\,\in\,\{\,C_P\,\}^\perp\\ 0 & \text{for } X\,\in\,\mathbb{R}C_P\end{cases}$$

of the complex structure I to $T_PS^{2n-1}\;=\;\{\,P\,\}^\perp$ defining the CR–structure on S^{2n-1}. Concluding this brief introduction to the CR–geometry of odd dimensional spheres we define the standard Berger metric g^ρ with parameter $\rho\;>\;-1$ by scaling the radius of Hopf circles, the intersections of $S^{2n-1}\;\subset\;\mathfrak{p}$ with complex lines in $\mathfrak{p}$, by the factor $\sqrt{1+\rho}$

$$g^\rho\;\;:=\;\;g\;+\;\rho\,\gamma\otimes\gamma\tag{26}$$

every Riemannian metric proportional to g^ρ will be called a Berger metric of parameter ρ. Unless otherwise stated however the musical isomorphisms $\sharp$ and $\flat$ will always refer to the standard metric g on S^{2n-1} regardless of which Berger metric we are studying. For the Levi–Civita connection of g we find say $\nabla_X\gamma\;=\;(\nabla_XC)^\sharp\;=\;g(\,\mathfrak{C}X,\cdot\,)$ using (25) and obtain

$$\begin{aligned}(\,\nabla_Xg^\rho\,)(\,Y,\,Z\,)\;&=\;\rho\,g(\,\mathfrak{C}X,\,Y\,)\gamma(\,Z\,)\;+\;\rho\,\gamma(\,Y\,)\,g(\,\mathfrak{C}X,\,Z\,)\\&=\;g^\rho(\,Y,\,\rho\,(\gamma\cdot\mathfrak{C})_XZ\,)\;+\;g^\rho(\,\rho\,(\gamma\cdot\mathfrak{C})_XY,\,Z\,)\end{aligned}$$

where the endomorphism–valued 1–form $\gamma\cdot\mathfrak{C}$ on S^{2n-1} is defined via:

$$(\,\gamma\,\cdot\,\mathfrak{C}\,)_X\;\;:=\;\;\gamma\otimes\mathfrak{C}X\;+\;\gamma(\,X\,)\,\mathfrak{C}$$

Since $(\gamma\cdot\mathfrak{C})_XY$ is symmetric in X and Y we conclude that $\nabla^\rho\;:=\;\nabla\;+\;\rho\,(\gamma\cdot\mathfrak{C})$ defines the Levi–Civita connection for every metric proportional to the standard Berger metric g^ρ.

Geometrically the Berger metrics g^ρ, $\rho\;>\;-1$, arise as the metrics on S^{2n-1} induced on the distance spheres in complex projective space. Consider the involutive automorphism of the group $\mathbf{SU}(n+1)$ of special unitary transformations of $\mathbb{C}^{n+1}$ given by conjugation $\theta\,:\,\mathbf{SU}(n+1)\longrightarrow\mathbf{SU}(n+1)$, $A\longmapsto SAS$, with the reflection S along the first standard basis vector. The automorphism θ fixes the subgroup $\mathbf{S}(\mathbf{U}(1)\times\mathbf{U}(n))\;\subset\;\mathbf{SU}(n+1)$ of special unitary transformations preserving the orthogonal decomposition $\mathbb{C}^{n+1}\;=\;\mathbb{C}\oplus\mathbb{C}^n$, hence the quotient $\mathbf{SU}(n+1)/\mathbf{S}(\mathbf{U}(1)\times\mathbf{U}(n))$ can be interpreted as the set $\mathbb{C}P^n$ of all such decompositions of $\mathbb{C}^{n+1}$ into a line and its orthogonal complement. On the Lie algebra of trace–free, skew hermitean matrices $\mathfrak{su}(n+1)$ of $\mathbf{SU}(n+1)$ the differential automorphism is still given by conjugation with S, its eigenspace decomposition $\mathfrak{su}(n+1) = \mathfrak{k} \oplus \mathfrak{p}$ reads:

$$\mathfrak{k} \ := \ \{ \begin{pmatrix} a & 0 \\ 0 & A \end{pmatrix} \mid A = -A^H, \ a + \operatorname{tr} A = 0 \} \qquad \mathfrak{p} \ := \ \{ \begin{pmatrix} 0 & -X^H \\ X & 0 \end{pmatrix} \mid X \in \mathbb{C}^n \}$$

The characterizing property of the Fubini–Study metric g^{FS} on $\mathbb{C}P^n$ is that the corresponding scalar product on $\mathfrak{p}$ is the standard scalar product $g(X,Y) := \mathrm{Re}\ X^H Y$ arising from the isomorphism $\mathfrak{p} \cong \mathbb{C}^n$ indicated above, evidently g is the restriction of the $\mathbf{SU}(n+1)$–invariant trace form $B(X,Y) = -\frac{1}{2}\,\mathrm{Re}\,\mathrm{tr}\,XY$ on $\mathfrak{su}(n+1)$ to $\mathfrak{p}$. Last but not least we will need the triple product on $\mathfrak{p} \cong \mathbb{C}^n$ describing the geometry of the symmetric space $\mathbb{C}P^n$:

$$\begin{aligned} &\Big[\frac{1}{2} \begin{pmatrix} 0 & -X^H \\ X & 0 \end{pmatrix}^2, \begin{pmatrix} 0 & -Y^H \\ Y & 0 \end{pmatrix} \Big] \\ &\qquad = \begin{pmatrix} 0 & +X^H X Y^H - 2\,X^H Y X^H + Y^H X X^H \\ -\,X X^H Y + 2\,X Y^H X - Y X^H X & 0 \end{pmatrix} \end{aligned}$$

In terms of the scalar product g and the complex structure I on $\mathfrak{p}$ the triple product reads:

$$[\ \tfrac{1}{2} X^2, \, Y\ ] \ = \ g(\, X, \, Y\,)\, X \ - \ 3\, g(\, IX, \, IY\,)\, IX \ - \ g(\, X, \, X\,)\, Y \tag{27}$$

In accordance with the general theory of symmetric spaces discussed in Appendix B the endomorphisms $(\operatorname{ad} P)^2 : \ X \longmapsto [\ \frac{1}{2} P^2, \, X\,]$ of $\mathfrak{p}$ are symmetric for all $P \in \mathfrak{p}$ with eigenvalues 0, -4 and -1 for unit vectors $P \in S^{2n-1}$ on the pairwise orthogonal eigenspaces $\mathbb{R}\,P$, $\mathbb{R}\,IP$ and $\{P, IP\}^\perp$. In order to make use of Corollary B.1 and other formulas of Appendix B we introduce conical exponential coordinates $\widetilde{\exp} : \ \mathbb{R}^+ \times S^{2n-1} \longrightarrow \mathbb{C}P^n$, $(r, P) \longmapsto \exp(rP)$, and conclude that the pull back of the Fubini–Study metric on $\mathbb{C}P^n$ becomes

$$\widetilde{\exp}^*\, g^{\mathrm{FS}} \ = \ dr \otimes dr \ + \ \sin^2 r\, (\ g \ - \ \sin^2 r\, (\, \gamma \otimes \gamma\,)\ ) \tag{28}$$

due to $(r\frac{\sin 2r}{2r})^2 = \sin^2 r\,(\,1 - \sin^2 r\,)$. In particular the Fubini–Study metric g^{FS} restricts on the distance spheres S^{2n-1}_r of radius $0 < r < \frac{\pi}{2}$ smaller than the injectivity radius $\frac{\pi}{2}$ of $\mathbb{C}P^n$ to the Berger metric $-\rho\, g^\rho$ with parameter $\rho := -\sin^2 r \in \,]-1, \, 0\,[$.

Recall that the argument in Appendix B leading to the description of the Riemannian metric on a symmetric space depends only on the description of the forward parallel transport and may thus be applied with due modifications to any parallel tensor. In particular we can mimick the calculations leading to Corollary B.1 for the parallel Kähler form ω^{FS} on $\mathbb{C}P^n$

$$\begin{aligned} (\,\widetilde{\exp}^*\, \omega^{\mathrm{FS}}\,)_{(r,P)}(\,\frac{\partial}{\partial r}, \ X\ ) \ &= \ g(\, I\, \frac{\sinh\ \operatorname{ad} rP}{\operatorname{ad} rP}\, P\, , \ \frac{\sinh\ \operatorname{ad} rP}{\operatorname{ad} rP}\, rX\ ) \ = \ \frac{\sin\, 2r}{2}\, \gamma(\, X\,) \\ (\,\widetilde{\exp}^*\, \omega^{\mathrm{FS}}\,)_{(r,P)}(\ X\, , \ Y\ ) \ &= \ g(\, I\, \frac{\sinh\ \operatorname{ad} rP}{\operatorname{ad} rP}\, rX, \ \frac{\sinh\ \operatorname{ad} rP}{\operatorname{ad} rP}\, rY\ ) \end{aligned}$$

and conclude in light of the identity $d\gamma(X,Y) = 2\, g(\, IX, \, Y\,)$ for all $X, \, Y \in T_P S^{2n-1}$:

$$\widetilde{\exp}^*\, \omega^{\mathrm{FS}} \ = \ \sin r\ \cos r\ dr \wedge \gamma \ + \ \frac{\sin^2 r}{2}\, d\gamma \tag{29}$$

Please note that γ and $d\gamma$ in this and subsequent formulas denote the pull back to $\mathbb{R}^+ \times S^{2n-1}$ of the differential forms of the same name on the unit sphere $S^{2n-1} \subset \mathfrak{p}$ so that in particular $\frac{\partial}{\partial r} \lrcorner\, d\gamma \;=\; 0$. In view of Corollary B.2 the addition theorem $\frac{2}{\tan 2z} \;=\; \frac{1}{\tan z} \;-\; \tan z$ implies that the second fundamental form of the distance sphere $S^{2n-1}_r \subset \mathbb{R}^+ \times S^{2n-1}$ reads

$$\mathrm{I\!I} \;\;=\;\; -\frac{1}{\tan r}\,\mathrm{id} \;+\; \tan r\,\gamma\,C \;\;=\;\; -\frac{1}{\tan r}\Big(\,\mathrm{id} \;+\; \frac{\rho}{\rho\,+\,1}\,\gamma\,C\,\Big) \tag{30}$$

with $\rho \;:=\; -\sin^2 r$ as before. Being torsion–free the Levi–Civita connection ∇^ρ for the Berger metric $-\rho g^\rho$ on S^{2n-1}_r kills the soldering form $d^{\nabla^\rho}\mathrm{id} \;=\; 0$ while $\nabla^\rho C \;=\; (\rho+1)\mathfrak{C}$, so

$$d^{\nabla^\rho}\mathrm{I\!I} \;\;=\;\; \tan r\,\Big(\,d\gamma\,C \;-\; (\,\rho\,+\,1\,)\,\gamma\,\mathfrak{C}\,\Big)$$

where $\mathfrak{C}$ has to be interpreted as a vector valued 1–form of course. Recalling the definition $\xi \;:=\; -\rho g^\rho(d^{\nabla^\rho}\mathrm{I\!I}, \mathrm{I\!I})$ of of the basic differential 3–form ξ of the distance sphere S^{2n-1}_r we find

$$\xi \;\;=\;\; \rho\,g^\rho\Big(\,d\gamma\,C \;-\; (\,\rho\,+\,1\,)\,\gamma\,\mathfrak{C},\,\mathrm{id} + \frac{\rho}{\rho\,+\,1}\,\gamma\,C\,\Big) \;\;=\;\; \rho^2\,\gamma\,\wedge\,d\gamma \tag{31}$$

omitting the auxiliary calculations:

$$g^\rho(\,C,\mathrm{id}\,) \;=\; (\,\rho\,+\,1\,)\,\gamma \qquad g^\rho(\,C,C\,) \;=\; \rho+1 \qquad g^\rho(\,\mathfrak{C},\mathrm{id}\,) \;=\; d\gamma \qquad g^\rho(\,\mathfrak{C},C\,) \;=\; 0$$

Strictly speaking ξ is only the homogeneous part of degree three of the form ξ defined in Section 4, however all the higher degree parts vanish, because $S^{2n-1}_r \;\subset\; \mathbb{R}^+ \times S^{2n-1}$ is a permeable hypersurface! According to Definition 2.2 a permeable hypersurface is characterized by a covariantly closed $d^{\nabla^\rho}(d^{\nabla^\rho}\mathrm{I\!I}) \;=\; 0$ second fundamental form. Due to the extended Killing equation (25) for $\nabla_X\mathfrak{C} \;=\; C \wedge X$ the covariant derivative of $\mathfrak{C}$ with respect to ∇^ρ

$$\begin{aligned}(\,\nabla^\rho_X\mathfrak{C}\,)\,Y \;\;&=\;\; (\,\nabla_X\mathfrak{C}\,)\,Y \;+\; \rho\,[\,\gamma(X)\,\mathfrak{C} \;+\; \gamma \otimes \mathfrak{C}X,\; \mathfrak{C}\,]\,Y\\ &=\;\; \gamma(Y)\,X \;-\; g(\,X,\,Y\,)\,C \;-\; \rho\,\gamma(Y)\,\mathfrak{C}^2 X\end{aligned}$$

becomes $d^{\nabla^\rho}\mathfrak{C} \;=\; -\gamma(\mathrm{id} - \rho\mathfrak{C}^2)$ after skew symmetrization in X, Y so that $d^{\nabla^\rho}\mathrm{id} \;=\; 0$ implies:

$$d^{\nabla^\rho}(\,d^{\nabla^\rho}\mathrm{I\!I}\,) \;\;=\;\; \tan r\,\gamma\,d^{\nabla^\rho}(\,\nabla^\rho C\,) \;\;=\;\; -\,\tan r\,(\,\rho\,+\,1\,)\,\gamma^2\,(\,\mathrm{id}\,-\,\rho\,\mathfrak{C}^2\,) \;\;=\;\; 0$$

By Corollary 4.3 the logarithmic transgression form of the distance sphere S^{2n-1}_r reads

$$\delta\,(\log\,F)(\,T(\mathbb{R}^+ \times S^{2n-1})\big|_{S^{2n-1}_r},\,(\nabla^\rho)^t,\,\mathrm{I\!I} \wedge \frac{\partial}{\partial r}\,) \;\;=\;\; \frac{d}{dt}\,\Big(\,\frac{t\,\rho\,\gamma}{2\pi}\,\frac{t\rho\,d\gamma}{2\pi}\,LF\Big(\,\frac{t\,\rho\,d\gamma}{2\pi}\,\Big)\,\Big) \tag{32}$$

for every multiplicative sequence of Pontryagin forms parametrized by an even power series $F(z) \;=\; 1 + O(z^2)$ with logarithm $LF(z) \;=\; \frac{\log F(z)}{z^2}$. With the decisive equations (16), (29) and (32) in place we can eventually embark on the proof of the main theorem of this article:

Theorem 5.1 (Special Values of Multiplicative Sequences)
Consider a smooth metric $g^{\mathrm{collar},\rho}$ on the closed unit ball $B^{2n} \subset \mathfrak{p}$, which is a product of the usual metric on $]-\varepsilon, 0]$ and the Berger metric of parameter $\rho \in]-1, 0[$ in a collar neighborhood $]-\varepsilon, 0] \times S^{2n-1}$ of the boundary S^{2n-1} of B^{2n}. The multiplicative sequence of Pontryagin forms parametrized by an even formal power series $F(z) = 1 + O(z^2)$ with associated composition inverse $\phi(z)$ of $\frac{z}{F(z)} = z + O(z^3)$ takes the following value on B^{2n}:

$$\int_{B^{2n}} F(\, TB^{2n},\, \nabla^{\mathrm{collar},\rho}\,) \;=\; \rho^n\, \mathrm{res}_{z=0}\left[\,\frac{F(\,z\,)^n}{z^{n+1}}\, dz\,\right] \;=\; \rho^n\, \mathrm{res}_{z=0}\left[\,\frac{(\log\phi)'(z)}{z^n}\, dz\,\right]$$

Proof: The special Berger metric $-\rho\, g^\rho$ with parameter $\rho \in]-1, 0[$ is realized by the geodesic distance sphere $S^{2n-1}_r \cong S^{2n-1}$ in $\mathbb{C}P^n$ of radius $r \in]0, \frac{\pi}{2}[$ with $\rho = -\sin^2 r$. The principal idea of the proof is to transgress from the Levi–Civita connection $\nabla^{\mathrm{collar},\rho}$ for the collar metric $g^{\mathrm{collar},\rho}$ to the Levi–Civita connection ∇^{FS} of the Fubini–Study metric g^{FS} of $\mathbb{C}P^n$ restricted to the ball $B^{2n}_r \cong B^{2n}_r \subset \mathfrak{p}$ of radius r. For the collar metric $g^{\mathrm{collar},\rho}$ the boundary $S^{2n-1} = \partial B^{2n}_r$ is totally geodesic, hence the transgression restricts on the boundary to the standard hypersurface transgression problem from ∇^ρ to ∇^{FS}. For the Fubini–Study metric g^{FS} on the other hand the boundary S^{2n-1}_r is permeable with the known logarithmic transgression form (32).

In a first step we employ the naturality of the multiplicative sequence associated to the even formal power series F under scaling $\mathrm{sc} :\; B^{2n}_r \longrightarrow B^{2n}$ and the transgression formula (14) to convert the integral $\langle\, F(TB^{2n},\, \nabla^{\mathrm{collar},\rho}), [B^{2n}]\,\rangle = \langle\, F(TB^{2n}_r,\, \mathrm{sc}^*\nabla^{\mathrm{collar},\rho}), [B^{2n}_r]\,\rangle$ into:

$$\begin{aligned}&\langle\; F(\,TB^{2n}_r,\;\nabla^{\mathrm{FS}}\,),\,[\,B^{2n}_r\,]\;\rangle \;-\; \langle\; F(\,TB^{2n},\;\nabla^{\mathrm{collar},\rho}\,),\,[\,B^{2n}\,]\;\rangle \qquad (33)\\ &\quad=\;\langle\; F(\,TS^{2n-1}_r,\;\nabla^\rho\,)\wedge(\mathrm{trans}\,F)(\,TB^{2n}_r\big|_{S^{2n-1}_r},\,\nabla^\rho,\,\nabla^\rho + \mathrm{I\!I}\wedge\frac{\partial}{\partial r}\,),\,[\,S^{2n-1}_r\,]\;\rangle\end{aligned}$$

Recall from the discussion of complex projective spaces in Section 3 that $F(T\mathbb{C}P^n, \nabla^{\mathrm{FS}})$ is parallel so that its top term is a constant multiple of the volume form of $\mathbb{C}P^n$. Using the formula (28) for the Fubini–Study metric g^{FS} in conical exponential coordinates we find that

$$\begin{aligned}\mathrm{Vol}(\,B^{2n}_r,\,g^{\mathrm{FS}}\,) \;&=\; \int_0^r \mathrm{Vol}(\,S^{2n-1},\,\sin^2 s\,(\,g\,-\,\sin^2 s\,\gamma\otimes\gamma\,)\,)\,ds\\ &=\; \mathrm{Vol}(\,S^{2n-1},\,g\,)\int_0^r\sqrt{1\,-\,\sin^2 s}\;\sin^{2n-1}s\;ds \;=\; \frac{\pi^n}{n!}\,\sin^{2n} r\end{aligned}$$

extended to the injectivity radius $r = \frac{\pi}{2}$ of $\mathbb{C}P^n$ becomes its volume $\frac{\pi^n}{n!}$ and conclude

$$\langle\; F(\,TB^{2n}_r,\;\nabla^{\mathrm{FS}}\,),\,[\,B^{2n}_r\,]\;\rangle \;=\; \rho^n\,\mathrm{res}_{z=0}\Big[\,\frac{F(z)^{n+1}}{z^{n+1}}\,dz\,\Big] \qquad (34)$$

from equation (16), the additional sign in $\rho = -\sin^2 r$ turns out to be immaterial, because both sides of the equation vanish if n is odd.

Moreover the explicit formula (29) for the Kähler form ω^{FS} in conical exponential coordinates implies $\omega^{\mathrm{FS}}\big|_{S^{2n-1}_r} = -\frac{1}{2}\,\rho\,d\gamma$. Using $F(T\mathbb{C}P^n,\nabla^{\mathrm{FS}}) = F(\frac{\omega}{\pi})^{n+1}$ we can rewrite the basic transgression formula (12) for the logarithmic transgression form (32) as

$$\begin{aligned}
F(\,TB^{2n}_r,\,\nabla^{\mathrm{FS}}\,)\big|_{S^{2n-1}_r} &= F(\,\frac{\omega^{\mathrm{FS}}}{\pi}\Big|_{S^{2n-1}_r}\,)^{n+1} \;=\; F(\,\frac{\rho\,d\gamma}{2\pi}\,)^{n+1}\\
&= F(\,TS^{2n-1}_r,\,\nabla^\rho\,)\wedge\exp\Big(d\int_0^t\frac{d}{d\tau}\Big(\,\frac{\tau\rho\gamma}{2\pi}\,\frac{\tau\rho\,d\gamma}{2\pi}\,LF(\frac{\tau\rho\,d\gamma}{2\pi})\Big)d\tau\Big)\Big|_{t=1}\\
&= F(\,TS^{2n-1}_r,\,\nabla^\rho\,)\wedge\exp\Big(\,(\,\log\,F\,)(\,\frac{t\rho\,d\gamma}{2\pi}\,)\,\Big)\Big|_{t=1}
\end{aligned}$$

recall that F is even with $LF(z) := \frac{\log\,F(z)}{z^2}$. With the differential form $F(\frac{\rho d\gamma}{2\pi})$ being invertible in the algebra of differential forms we conclude $F(TS^{2n-1}_r,\nabla^\rho) = F(\frac{\rho d\gamma}{2\pi})^n$. Reinserting this result into the preceding equation we obtain for the transgression form in equation (33)

$$\begin{aligned}
&F(\,TS^{2n-1}_r,\,\nabla^\rho\,)\;\wedge\;(\mathrm{trans}\,F\,)(\,TB^{2n}\big|_{S^{2n-1}_r}\,,\,\nabla^\rho,\,\nabla^\rho\,+\,\mathrm{II}^\rho\wedge N\,)\\
&\qquad= F(\,\frac{\rho\,d\gamma}{2\pi}\,)^n\int_0^1 F(\,\frac{t\rho\,d\gamma}{2\pi}\,)\,\frac{d}{dt}\Big(\,\frac{t\rho\gamma}{2\pi}\,\frac{t\rho\,d\gamma}{2\pi}\,LF(\,\frac{t\rho\,d\gamma}{2\pi}\,)\,\Big)\,dt
\end{aligned}$$

which integrates via equation (24) in the form $\langle\,\frac{\gamma}{2\pi}\wedge(\frac{d\gamma}{2\pi})^{n-1},\,[\,S^{2n-1}\,]\,\rangle = 1$ to the value

$$\begin{aligned}
&\mathrm{res}_{z=0}\Big[\,F(\rho z)^n\,\Big(\int_0^1 F(t\rho z)\,\frac{d}{dt}\Big(\,t^2\rho^2 z\,\frac{\log\,F(t\rho z)}{(t\rho z)^2}\,\Big)\,dt\Big)\,\frac{dz}{z^n}\,\Big]\\
&\quad= \mathrm{res}_{z=0}\Big[\,F(\rho z)^n\,\Big(\int_0^1 F(t\rho z)\,\frac{F'(t\rho z)}{F(t\rho z)}\,\rho\,dt\Big)\,\frac{dz}{z^n}\,\Big] \;=\; \rho^n\,\mathrm{res}_{z=0}\Big[\,\frac{F(z)^{n+1}-F(z)^n}{z^{n+1}}\,dz\,\Big]
\end{aligned}$$

with the formal variable $z := \frac{d\gamma}{2\pi}$. Combined with equations (33) and (34) this result implies

$$\langle\;F(\,TB^{2n},\;\nabla^{\mathrm{collar},\rho}\,),\,[\,B^{2n}\,]\;\rangle \;=\; \mathrm{res}_{z=0}\Big[\,\frac{F(z)^n}{z^{n+1}}\,\Big] \;=\; \mathrm{res}_{z=0}\Big[\,\frac{\phi(z)^n\,\phi'(z)}{z^n\,\phi(z)^{n+1}}\,dz\,\Big]$$

where the odd power series ϕ is the composition inverse of $\frac{z}{F(z)} = z + O(z^3)$ as before. □

In order to complete the proof of Theorem 1.1 stated in the introduction we still have to argue that Theorem 5.1 is not only true for the values $\rho\in\,]-1,\,0\,[$ of the Berger parameter ρ realized by the distance spheres in complex projective space. Evidently it is sufficient to show that the value $\langle\,F(\,TB^{2n},\;\nabla^{\mathrm{collar},\rho}\,),\,[\,B_{2n}\,]\,\rangle$ of the multiplicative sequence parametrized by F on the closed unit ball B^{2n} with a collar metric $g^{\mathrm{collar},\rho}$ for the Berger metric g^ρ on $S^{2n-1} = \partial B^{2n}$ is an analytic function in ρ. For this purpose we consider the auxiliary metric

$$g^{\mathrm{out},\rho} \;:=\; dr\,\otimes\,dr\;+\;r^2\,(\,g\;+\;\rho\,r^2\,\gamma\,\otimes\,\gamma\,)$$

on $\mathbb{R}^+\times S^{2n-1}$, which is smooth in 0 and positive definite on a neighborhood of the unit ball $B^{2n}\subset\mathfrak{p}$. Clearly this auxiliary Riemannian metric induces the Berger metric g^ρ on the boundary unit sphere $S^{2n-1} \subset B^{2n}$, nevertheless the boundary is not totally geodesic. Using the Lie derivative with respect to the unit normal field $\frac{\partial}{\partial r}$ as in Corollary B.2 we get

$$\mathbb{I} \;=\; -\frac{1}{2}\,\mathfrak{Lie}_{\frac{\partial}{\partial r}}\Big(\,dr \otimes dr \,+\, r^2 g \,+\, r^4 \rho\gamma \otimes \gamma\,\Big)\Big|_{r=1} \;=\; -g \,-\, 2\rho\gamma \otimes \gamma$$

or $\mathbb{I} \,=\, -(\,\mathrm{id} \,+\, \frac{\rho}{\rho+1}\gamma C\,)$ expressed as a vector valued 1–form. It is straightforward, but somewhat more work to calculate the Levi–Civita connection $\nabla^{\mathrm{out},\rho}$ for the metric $g^{\mathrm{out},\rho}$. Leaving the details of this calculation to the reader we only note that it is sufficient to do the calculations for $\frac{\partial}{\partial r}$ and vector fields $X,\, Y \,\in\, \Gamma(\,TS^{2n-1}\,)$ on S^{2n-1} extended constantly in r direction to $\mathbb{R}^+ \times S^{2n-1}$. Eventually we find the formulas

$$\nabla^{\mathrm{out},\rho}_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r} \;=\; 0 \qquad\qquad \nabla^{\mathrm{out},\rho}_{X}\frac{\partial}{\partial r} \;=\; \nabla^{\mathrm{out},\rho}_{\frac{\partial}{\partial r}}X \;=\; \frac{1}{r}X \,+\, \frac{\rho\, r}{\rho\, r^2 \,+\, 1}\,\gamma(X)\,C$$

and:

$$\nabla^{\mathrm{out},\rho}_{X}Y \;=\; \nabla_X Y \,-\, r\, g(X,Y)\,\frac{\partial}{\partial r} \,+\, \rho\,\Big(\,r^2\,\gamma(X)\,\mathfrak{C}Y \,+\, r^2\,\gamma(Y)\,\mathfrak{C}X \,-\, 2\,r\,\gamma(X)\,\gamma(Y)\,\frac{\partial}{\partial r}\,\Big)$$

For $\rho \,=\, 0$ these formulas evidently reduce to the well–known formulas for the trivial connection ∇^{triv} on $\mathfrak{p}$ expressed in polar coordinates. More important is that $\nabla^{\mathrm{out},\rho}$ is a rational function of ρ with a pole on the sphere of radius $\frac{1}{\sqrt{-\rho}}$. Hence the curvature tensor and the basic geometric forms on the boundary hypersurface $S^{2n-1} \,=\, \partial B^{2n}$ are all rational functions of ρ. The integrand differential forms occurring on the right hand side of the reformulation

$$\int_{B^{2n}} F(\,TB^{2n}, \nabla^{\mathrm{collar},\rho}\,) \;=\; \int_{B^{2n}} F(\,TB^{2n}, \nabla^{\mathrm{out},\rho}\,) \,-\, \int_{S^{2n-1}} \mathrm{Trans}\, F(\,TB^{2n}, \nabla^{\mathrm{collar},\rho}, \nabla^{\mathrm{out},\rho}\,)$$

of the transgression formula (19) are thus rational functions of ρ as well so that the left hand side $\langle\; F(\,TB^{2n},\; \nabla^{\mathrm{collar},\rho}\,),\, [\,B^{2n}\,]\;\rangle$ is certainly an analytic function of ρ.

A The Atiyah–Patodi–Singer Index Theorem

Our main motivation for studying multiplicative sequences on Berger spheres is the relationship between η–invariants and indices of twisted Dirac operators as formulated in the Atiyah–Patodi–Singer Index Theorem [APS] for manifolds with boundary. In this appendix we want to sketch the general setup of the Atiyah–Patodi–Singer Theorem and work out the details for the untwisted Dirac operator and the signature operator, whose index densities are multiplicative sequences of Pontryagin forms.

Recall that a $\mathbb{Z}_2$–graded Clifford module bundle EM^{out} on an even dimensional, oriented Riemannian manifold M^{out} with metric g^{out} is a hermitean vector bundle EM^{out} over M^{out} endowed with a hermitean connection $\nabla^{EM^{\mathrm{out}}}$, a parallel hermitian idempotent Γ^{out} with $(\Gamma^{\mathrm{out}})^2 \,=\, 1$ and a Clifford multiplication $\bullet : \; TM^{\mathrm{out}} \times EM^{\mathrm{out}} \longrightarrow EM^{\mathrm{out}}$, $(X, e) \longmapsto X\bullet e$, satisfying the Clifford relation $X\bullet Y\bullet + Y\bullet X\bullet = -2\,g^{\mathrm{out}}(X,Y)$ for all $X,\,Y\,\in\,\Gamma(\,TM^{\mathrm{out}}\,)$ such that the endomorphisms $Y\bullet$ are skew hermitean, anticommute with Γ^{out} and satisfy the following Leibniz rule for all $X,\,Y\,\in\,\Gamma(\,TM^{\mathrm{out}}\,)$ and $e\,\in\,\Gamma(\,EM^{\mathrm{out}}\,)$:

$$\nabla^{EM^{\mathrm{out}}}_X(\,Y\bullet e\,) \;=\; (\,\nabla^{\mathrm{out}}_X Y\,)\bullet e \;+\; Y\bullet\nabla^{EM^{\mathrm{out}}}_X e$$

Every $\mathbb{Z}_2$–graded Clifford bundle $EM^{\mathrm{out}} = E^+M^{\mathrm{out}} \oplus E^-M^{\mathrm{out}}$ splits into the parallel eigenspace subbundles for the eigenvalues ±1 of the grading operator Γ^{out} of equal rank $\dim\,E^+M^{\mathrm{out}} = \dim\,E^-M^{\mathrm{out}}$. Clifford bundles are introduced in order to axiomatize the definition of (twisted) Dirac operators, the Dirac operator associated to EM^{out} reads

$$D^{\mathrm{out}}:\quad \Gamma(\,EM^{\mathrm{out}}\,)\;\longrightarrow\;\Gamma(\,EM^{\mathrm{out}}\,),\qquad e\;\longmapsto\;\sum E_\mu\bullet\nabla^{EM^{\mathrm{out}}}_{E_\mu}e$$

where as usual $\{E_\mu\}$ denotes some local orthonormal basis in an open subset of M^{out}. The Dirac operator is a self–adjoint, elliptic differential operator of first order [LM], [BGV] anticommuting with Γ^{out}, in particular it is the sum of the two "chiral" Dirac operators $D^{\mathrm{out}+} : \Gamma(\,E^+M^{\mathrm{out}}\,) \longrightarrow \Gamma(\,E^-M^{\mathrm{out}}\,)$ and $D^{\mathrm{out}-} : \Gamma(\,E^-M^{\mathrm{out}}\,) \longrightarrow \Gamma(\,E^+M^{\mathrm{out}}\,)$.

In their paper [APS] Atiyah, Patodi and Singer undertook a detailed analysis of the behaviour of the Dirac operator D^{out} close to a smooth boundary hypersurface $M := \partial W$ of a compact subset $W \subset M^{\mathrm{out}}$ with non–trivial interior under some rather severe restrictions on the geometry of M^{out} and EM^{out} in a collar neighborhood of M. For the moment let us only assume that M is a totally geodesic hypersurface in M^{out} in the sense that the outward pointing unit normal vector field N on M is parallel for the Levi–Civita connection ∇^{out}. Under this rather weak assumption the restriction $EM := EM^{\mathrm{out}}|_M$ of EM^{out} to M comes along with three parallel, mutually anticommuting endomorphisms $N\bullet$, Γ^{out} and $\Gamma := -N\bullet\Gamma^{\mathrm{out}}$ with $(\Gamma^{\mathrm{out}})^2 = 1 = \Gamma^2$ and thus splits in two different ways into parallel eigensubbundles for the eigenvalues ±1 of Γ^{out} and Γ respectively:

$$EM \;=\; E^+M \,\oplus\, E^-M \;=\; E_+M \,\oplus\, E_-M$$

In contrast to the subbundles $E^+M = E^+M^{\mathrm{out}}|_M$ and $E^-M = E^-M^{\mathrm{out}}|_M$ the subbundles E_+M and E_-M are actually Clifford subbundles of EM of the same rank $\frac{1}{2}\dim EM$, because Γ commutes with Clifford multiplication $X\bullet$ by vectors $X \in TM$ tangent to M. In addition the projections $\mathrm{pr}_\pm := \frac{1}{2}(1\pm\Gamma)$ from EM to the eigenspaces $E_\pm M$ of Γ induce isomorphisms $E^+M \longrightarrow E_\pm M$ and $E^-M \longrightarrow E_\pm M$ with explicit left inverses

$$\iota_\pm:\quad E_\pm M \;\stackrel{\cong}{\longrightarrow}\; E^+M,\qquad e\;\longmapsto\; e\,\pm\,N\bullet e$$

the analoguous $e \longmapsto e\mp N\bullet e$ provide left inverses for the restrictions of $\mathrm{pr}_\pm$ to E^-M. In order to relate the η–invariant of the Dirac operator D associated to the Clifford bundle E_+M on M to the index of the Dirac operator D^{out} on W under suitable boundary conditions Atiyah, Patodi and Singer require that in some collar neighborhood $M^{\mathrm{out}} \supset]-\varepsilon,\varepsilon[\times M \supset M$ of M with collar coordinate $r = 0$ on M and $r \leq 0$ on W the geometric data g^{out} and

$\nabla^{EM^{\rm out}}$ is compatible with the product structure. More precisely the outer Riemannian metric $g^{\rm out} = dr \otimes dr + g$ is required to be a product on the collar and the Clifford bundle $EM^{\rm out}$ is required to be trivializable over the collar $EM^{\rm out} \supset]-\varepsilon, \varepsilon[\times EM \supset EM$ in such a way that the Clifford multiplication, the grading $\Gamma^{\rm out}$ and the hermitean metric do not dependent on r, while the connection reduces to $\nabla^{EM^{\rm out}} = dr \otimes \frac{\partial}{\partial r} + \nabla^{EM}$. For geometric Clifford bundles like the spinor bundle and the bundle of differential forms discussed below the latter assumption on $EM^{\rm out}$ is a straightforward consequence of the former on $g^{\rm out}$.

In case all these additional assumptions are met the hypersurface $M \subset]-\varepsilon, \varepsilon[\times M$ is totally geodesic with outward pointing normal field $N := \frac{\partial}{\partial r}$ and the specific form the Clifford multiplication $\bullet$ and the connection $\nabla^{EM^{\rm out}}$ take over the collar ensures that

$$D^{\rm out} \;=\; N\bullet \frac{\partial}{\partial r} \,+\, D \;=\; N\bullet\Big(\,\frac{\partial}{\partial r} \,-\, N\bullet D\,\Big)$$

where the operator D can be identified with the Dirac operator of the Clifford bundle EM over M, the apparent sign change when compared to [APS] is due to the fact that $\frac{\partial}{\partial r}$ is outward pointing. Besides being well–defined on sections of EM over M the operator $N\bullet D$ commutes with the grading operator $\Gamma^{\rm out}$. Conjugating its positive chirality part $N\bullet D^+$

$$\Gamma(\,E_\pm M\,) \stackrel{\iota_\pm}{\longrightarrow} \Gamma(\,E^+M\,) \stackrel{N\bullet D^+}{\longrightarrow} \Gamma(\,E^+M\,) \stackrel{{\rm pr}_\pm}{\longrightarrow} \Gamma(\,E_\pm M\,) \tag{35}$$

with the mutually inverse isomorphisms $\iota_\pm$ and ${\rm pr}_\pm$ we simply get the restriction of the operator $\pm D$ to $E_\pm M$, because $(N\bullet D^+)(\,e \pm N\bullet e\,) = \pm De + N\bullet De$ and the section $N\bullet De$ of $E_\mp M$ projects to 0. Note that this result is consistent with the fact that the linear isomorphism $N\bullet : E_+M \longrightarrow E_-M$ conjugates $+D$ on E_+M with $-D$ on E_-M. A similar analysis can be done for the negative chiral part $N\bullet D^-$ on sections of E^-M, which becomes $\mp D$ on sections of $E_\pm M$ under conjugation.

On its natural domain $W^1(E^+M^{\rm out})$ of square integrable sections of $E^+M^{\rm out}$ with square integrable distributional derivatives the positive chiral part $D^{\rm out+}$ of the Dirac operator on $EM^{\rm out}$ is in itself not a Fredholm operator due to the presence of the boundary. In order to obtain a well–defined index Atiyah, Patodi and Singer introduced a global boundary condition depending on the spectral decomposition $W^{\frac{1}{2}}(E^+M) = W^{\frac{1}{2}}_{<0}(E^+M) \oplus W^{\frac{1}{2}}_{\geq 0}(E^+M)$ under the selfadjoint elliptic differential operator $N\bullet D^+$ on the compact manifold M into the closed subspaces spanned by eigensections with negative and non–negative eigenvalues respectively. Now the restriction of $N\bullet D^+$ to the preimage $W^1_{<0}(E^+M^{\rm out})$ of the closed subspace $W^{\frac{1}{2}}_{<0}(E^+M)$ under the continuous trace map $W^1(E^+M^{\rm out}) \longrightarrow W^{\frac{1}{2}}(E^+M)$ is a Fredholm operator and its index is given by the Atiyah–Patodi–Singer Index Theorem

$$\begin{aligned} &{\rm index}\Big(\,D^{\rm out+} :\;\; W^1_{<0}(\,E^+M^{\rm out}\,) \;\longrightarrow\; W^0(\,E^-M^{\rm out}\,)\,\Big) \qquad (36)\\ &\quad = \int_W \widehat{A}(\,TM^{\rm out}, \nabla^{\rm out}\,)\,{\rm ch}\,(\,EM^{\rm out} \,:\, \$M^{\rm out}\,) \;-\; \frac{1}{2}\Big(\,\eta_{N\bullet D^+} \,+\, \dim\ker N\bullet D^+\,\Big) \end{aligned}$$

where $\eta_{N\bullet D^+}$ is the η–invariant of the differential operator $N\bullet D^+$ on sections of E^+M over M and ${\rm ch}(EM^{\rm out} : \$M^{\rm out})$ is the Chern character form of a locally defined "twisting" vector bundle on $M^{\rm out}$ canonically associated to the Clifford bundle $EM^{\rm out}$ with respect to a distinguished "twisting" connection arising from $\nabla^{EM^{\rm out}}$ and $\nabla^{\rm out}$. In calculations it is convenient to replace the differential operator $N\bullet D^+$ on sections of E^+M with the conjugated restriction $D|_{E_+M}$ of the Dirac operator D to sections of E_+M according to equation (35).

For the purpose of this article we need to discuss two special geometric $\mathbb{Z}_2$–graded Clifford bundles in more detail, the spinor bundle $EM^{\rm out} = \$M^{\rm out}$ on an even–dimensional manifold $M^{\rm out}$ with its half spinor grading and the differential forms bundle $\Lambda T^*M^{\rm out}$ on an oriented manifold $M^{\rm out}$ of dimension divisible by 4 graded by the Hodge $*$–isomorphism. To begin with the half spinor grading $\$M^{\rm out} = \$^+M^{\rm out} \oplus \$^-M^{\rm out}$ on the spinor bundle on a manifold $M^{\rm out}$ of even dimension $m+1 = \dim M^{\rm out}$ is induced by the outer grading operator $\Gamma^{\rm out}$ given by Clifford multiplication with the complex volume element of $M^{\rm out}$

$$\Gamma^{\rm out} \;:=\; i^{\lfloor\frac{m+2}{2}\rfloor}\, E_0\bullet E_1\bullet\,\ldots\, E_m\bullet \tag{37}$$

where $E_0,\,\ldots,\,E_m$ is a local oriented orthonormal basis for $M^{\rm out}$. Along the boundary hypersurface M we can choose $E_0 = N$ with the outward pointing unit vector field N and conclude that the grading operator $\Gamma := -N\bullet\Gamma^{\rm out}$ is again the Clifford multiplication with the complex volume form of M, because m is odd. In consequence the Clifford eigensubbundle E_+M of $EM = \$M^{\rm out}|_M$ for the eigenvalue $+1$ under Γ is exactly *the* spinor bundle of M and so the restriction of D to E_+M is *the* Dirac operator on M.

Turning from the spinor bundle to the bundle of differential forms things become significantly more involved. Consider the differential forms bundle $EM^{\rm out} = \Lambda T^*M^{\rm out}$ on $M^{\rm out}$ graded again by the Clifford multiplication $\Gamma^{\rm out}$ with the complex volume element (37), which agrees with the Hodge $*$–isomorphism up to constants, in order to be precise

$$\Gamma^{\rm out}\,\psi \;=\; i^{\lfloor\frac{m+2}{2}\rfloor}\,(-1)^{(m+1)|\psi|+\lfloor\frac{|\psi|}{2}\rfloor}\;*\,\psi$$

on homogeneous forms ψ on a manifold $M^{\rm out}$ of dimension $m+1$. Note in particular $\Gamma^{\rm out} = *$ in forms ψ of middle degree $|\psi| = 2k$ on manifolds of dimension $m+1 = 4k$ divisible by 4. In consequence the grading operator $\Gamma = -N\bullet\Gamma^{\rm out}$ is Clifford multiplication with the complex volume element of M for the induced orientation and thus commutes with Clifford multiplication with vectors tangent to M. In terms of the standard isomorphism $EM := \Lambda T^*M^{\rm out}|_M \cong \Lambda T^*M \oplus N\bullet\Lambda T^*M$ the operators $N\bullet$ and Γ read

$$N\bullet(\,\psi\oplus\tilde\psi\,) \;=\; (\,-\tilde\psi\,)\oplus\psi \qquad \Gamma\,(\,\psi\oplus\tilde\psi\,) \;=\; \Gamma\psi\oplus(\,-\Gamma\tilde\psi\,)$$

while the Dirac operator D becomes $(d+d^*)\oplus-(d+d^*)$. The twisted isomorphism

$$\Phi:\quad \Lambda T^*M \stackrel{\cong}{\longrightarrow} [\,\Lambda T^*M^{\rm out}\big|_M\,]_+, \qquad \psi \longmapsto (\,\psi^{\rm ev}+\Gamma\psi^{\rm ev}\,)\oplus(\,\psi^{\rm odd}-\Gamma\psi^{\rm odd}\,)$$

thus conjugates the Dirac operator D restricted to the eigensubbundle E_+M of EM for the eigenvalue $+1$ under Γ to $\Phi^{-1}\circ D\circ\Phi = \Gamma\,(d+d^*)$. The latter operator preserves the parity of forms and commutes with the grading operator Γ, which changes parity as M is odd dimensional. In consequence the η–invariant of $D|_{E_+M} \cong \Gamma\,(d+d^*)$ is twice the η–invariant of the restriction $\Gamma\,(d+d^*)^{\rm ev}$ of $\Gamma\,(d+d^*)$ to even or equivalently to odd forms.

B Riemannian Symmetric Spaces

Symmetric spaces are a classical topic of differential geometry, their rich algebraic structure makes various calculations feasible, which are difficult or even impossible for other homogeneous spaces, not to speak of general differentiable manifolds. Needless to say the book of Helgason [Hel] is a definite account about the canon of formulas for calculations in symmetric spaces. In this appendix we focus on the triple product formulation of symmetric spaces, which allows us to avoid studying Lie algebras in detail in our calculations for the Fubini–Study metric on the complex projective spaces $\mathbb{C}P^n$, $n \geq 1$.

Consider a Lie group G together with an involutive automorphism $\theta : G \longrightarrow G$ satisfying $\theta^2 = \mathrm{id}$. The differential of the automorphism θ in the identity element $e = \theta(e)$ of G defines an involutive automorphism still denoted by θ of the Lie algebra $\mathfrak{g} := T_eG$ of G. Being involutive the differential $\theta : \mathfrak{g} \longrightarrow \mathfrak{g}$ is diagonalizable with eigenspace decomposition

$$\mathfrak{g} \;=\; \mathfrak{k} \oplus \mathfrak{p} \tag{38}$$

where $\mathfrak{k}$ and $\mathfrak{p}$ are the eigenspaces for θ for the eigenvalues $+1$ and -1 respectively. The characteristic property $\theta\,[\,X,\,Y\,] = [\,\theta\,X,\,\theta\,Y\,]$ of automorphisms of Lie algebras like θ readily implies $[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$, after all involutive automorphisms are in one–to–one correspondence to $\mathbb{Z}_2$–graduations on Lie algebras.

In consequence the eigenspace $\mathfrak{k} \subset \mathfrak{g}$ for the eigenvalue $+1$ is the subalgebra of the subgroup $K_\theta \subset G$ of elements fixed by the original automorphism θ. Every subgroup $K_\theta \supset K \supset K_\theta^\circ$ of G in between K_θ and its connected component K_θ° of the identity $e \in K_\theta$ shares the same Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$ and is necessarily closed. The homogeneous space G/K is thus a smooth manifold endowed with diffeomorphisms called point reflections

$$\theta_{gK} : \quad G/_K \;\longrightarrow\; G/_K, \qquad \tilde{g}\,K \;\longmapsto\; g\,\theta(\,g^{-1}\,\tilde{g}\,)\,K$$

parametrized by points $gK \in G/K$ such that the differential of θ_{gK} in its fix point gK equals $(\theta_{gK})_{*gK} = -\mathrm{id}$. Choosing different subgroups K in between K_θ° and K_θ amounts to taking different covering symmetric spaces, in particular G/K_θ° is the universal covering space of G/K, while G/K_θ is its smallest symmetric quotient.

Whereas general symmetric spaces abound there exist significantly fewer Riemannian symmetric spaces, symmetric spaces G/K endowed with a Riemannian metric $g^{G/K}$ for which all point reflections θ_{gK}, $gK \in G/K$, are isometries. In case the description of a given symmetric space G/K is effective in the sense that no normal subgroup of K besides $\{1\}$ is normal in all of G the symmetric Riemannian metrics $g^{G/K}$ on G/K correspond to G–invariant, bilinear forms B on $\mathfrak{g}$, which make the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ orthogonal and restrict to a scalar product g on $\mathfrak{p}$. In order to avoid the question of effectiveness one can alternatively capture the local geometry of a symmetric space G/K by the triple product

$$\mathrm{Sym}^2\mathfrak{p} \otimes \mathfrak{p} \;\longrightarrow\; \mathfrak{p}, \qquad X \cdot Y \otimes Z \;\longmapsto\; [\,X \cdot Y,\,Z\,]$$

defined on $\mathfrak{p}$ by $[X \cdot Y, Z] := [X,[Y,Z]] + [Y,[X,Z]]$, note that the Jacobi identity implies:

$$[\,[\,X,\,Y\,],\,Z\,] \;=\; \frac{1}{3}\Big(\,[\,Z \cdot Y,\,X\,] \;-\; [\,Z \cdot X,\,Y\,]\,\Big) \tag{39}$$

An axiomatic triple product on a vector space $\mathfrak{p}$ is a trilinear map $(X, Y, Z) \longmapsto [\, X \cdot Y,\, Z\,]$ on $\mathfrak{p}$ with values in $\mathfrak{p}$, which is symmetric in the first two arguments as indicated and satisfies the linear variant $[\, X \cdot X,\, X\,] = 0$ of the Jacobi identity as well as the quadratic equation

$$\begin{aligned} &[\,[\,U \cdot U,\, V\,] \cdot X,\, Y\,] \;-\; [\,[\,U \cdot U,\, V\,] \cdot Y,\, X\,] \\ &\qquad = \; 2\,[\,(\,[\,U \cdot X,\, Y\,] - [\,U \cdot Y,\, X\,]\,) \cdot U,\, V\,] \;+\; [\,U \cdot U,\, [\,V \cdot X,\, Y\,] - [\,V \cdot Y,\, X\,]\,] \end{aligned}$$

for all $X, Y, U, V \in \mathfrak{p}$, which essentially means that the endomorphisms $[X, Y] \star : \mathfrak{p} \longrightarrow \mathfrak{p}$ *defined* by equation (39) act as derivations for the triple product for all $X, Y \in \mathfrak{p}$. Isometry classes of symmetric spaces up to covering are then classified by isomorphism classes of axiomatic triple products [Hel], for example a Riemannian symmetric space corresponds to a triple product on a euclidian vector space $\mathfrak{p}$ with scalar product g such that

$$\mathrm{Sec}(\, X,\, Y,\, U,\, V\,) \;:=\; -\,2\,g(\,[\,X \cdot Y,\, U\,],\, V\,) \tag{40}$$

is symmetric in U, V for all $X, Y \in \mathfrak{p}$. Under this additional assumption the Jacobi identity becomes the first Bianchi identity $\mathrm{Sec}(X, X, X, V) = 0$ for an algebraic sectional curvature tensor $\mathrm{Sec} \in \mathrm{Sym}^2\mathfrak{p}^* \otimes \mathrm{Sym}^2\mathfrak{p}^*$ with associated algebraic Riemannian curvature tensor:

$$R(\, X,\, Y,\, U,\, V\,) \;:=\; \frac{1}{3}\, g(\,[\,U \cdot X,\, Y\,] \;-\; [\,U \cdot Y,\, X\,],\, V\,) \tag{41}$$

In turn the quadratic equation imposed on the triple product ensures that the overdetermined differential equation $\nabla R = 0$ with initial value R is formally integrable. Its unique solution describes geometrically the germ of the corresponding Riemannian symmetric space.

The preceding discussion of triple products vindicates the idea that all geometric properties of a symmetric space G/K eventually come from its triple products or equivalently from the endomorphisms $(\mathrm{ad}\ P)^2 := [\,\frac{1}{2}\,P^2,\, \cdot\,]$. Presumably the single most important formula in this respect is the classical formula (42) for the backward parallel transport Φ^{-1} along radial geodesics in exponential coordinates. Recall that every symmetric space G/K carries a canonical, complete, torsion–free connection called the Loos connection on its tangent bundle. Identifying the tangent space $T_{eK}(\,G/K\,) \cong \mathfrak{g}/\mathfrak{k}$ of a symmetric space G/K in the base point $eK \in G/K$ with the (-1)–eigenspace $\mathfrak{p}$ of θ we can use the exponential map

$$\exp : \quad \mathfrak{p} \;\cong\; T_{eK}(\,G/K\,) \;\longrightarrow\; G/K, \qquad P \;\longmapsto\; \exp(\,P\,)\,K$$

for the Loos connection to pull it back to a torsion free connection $\nabla^{G/K}$ on $\mathfrak{p}$. The backward parallel transport Φ^{-1} for the latter connection along radial geodesics $t \longmapsto tP$ reads [Hel]:

$$\Phi_P^{-1} : \quad \mathfrak{p} \;\cong\; T_P\,\mathfrak{p} \;\longrightarrow\; T_0\,\mathfrak{p} \;\cong\; \mathfrak{p}, \qquad X \;\longmapsto\; \frac{\sinh\ \mathrm{ad}\,P}{\mathrm{ad}\,P}\, X \tag{42}$$

Evidently every even formal power series like $\frac{\sinh z}{z}$ in $\mathrm{ad}\,P$ is determined completely by the triple product on $\mathfrak{p}$. Incidentally the inverse formal power series $\frac{z}{\sinh z}$ has radius of convergence 2π and so the forward parallel transport Φ_P becomes singular in the cut locus of points $P \in \mathfrak{p}$, where $\mathrm{ad}\,P$ has an eigenvalue $\pm\, 2\pi i$.

The importance of formula (42) derives from the fact that the pull back of covariant parallel tensors on G/K to $\mathfrak{p}$ via the exponential map exp is governed by the backward parallel transport Φ^{-1}. On a Riemannian symmetric space G/K for example the Riemannian metric $g^{G/K}$ is parallel and pulls back to the following smooth metric on $\mathfrak{p}$

$$g^{G/K}_P(\,X,\,Y\,) \;=\; g(\,\Phi_P^{-1}X,\,\Phi_P^{-1}Y\,) \;=\; g(\,\frac{\sinh\,\operatorname{ad}P}{\operatorname{ad}P}X,\,\frac{\sinh\,\operatorname{ad}P}{\operatorname{ad}P}Y\,)$$

where g is the scalar product on $\mathfrak{p}$ defining $g^{G/K}$ in the first place. In order to make this formula suit our considerations in Section 5 we consider conical exponential coordinates

$$\widetilde{\exp}:\quad \mathbb{R}^+\times S^{m-1}\longrightarrow\mathfrak{p}\overset{\exp}{\longrightarrow}G/K,\qquad(\,r,\,P\,)\longmapsto\exp(\,rP\,)\,K$$

where $m = \dim\mathfrak{p}$ and S^{m-1} is the unit sphere in $\mathfrak{p}$ with respect to the scalar product g. The differential of the polar coordinates map cone : $\mathbb{R}^+\times S^{m-1}\longrightarrow\mathfrak{p}$, $(\,r,P\,)\longmapsto rP$, reads

$$\mathrm{cone}_{*(r,P)}(\,\frac{\partial}{\partial r}\,) \;=\; P\qquad\qquad \mathrm{cone}_{*(r,P)}(\,X\,) \;=\; r\,X$$

for all $X\in T_PS^{m-1}\subset T_{(r,P)}(\mathbb{R}^+\times S^{m-1})$. The factorization $\widetilde{\exp} = \exp\circ\,\mathrm{cone}$ thus implies

$$\begin{aligned}
(\,\widetilde{\exp}^*g^{G/K}\,)_{(r,P)}(\,\frac{\partial}{\partial r},\,\frac{\partial}{\partial r}\,) &= g(\,\frac{\sinh\,\operatorname{ad}rP}{\operatorname{ad}rP}\,P\,,\,\frac{\sinh\,\operatorname{ad}rP}{\operatorname{ad}rP}\,P\,\,) \;=\; 1\\
(\,\widetilde{\exp}^*g^{G/K}\,)_{(r,P)}(\,\frac{\partial}{\partial r},\,X\,) &= g(\,\frac{\sinh\,\operatorname{ad}rP}{\operatorname{ad}rP}\,P\,,\,\frac{\sinh\,\operatorname{ad}rP}{\operatorname{ad}rP}\,rX\,) \;=\; 0\\
(\,\widetilde{\exp}^*g^{G/K}\,)_{(r,P)}(\,X\,,\,Y\,) &= g(\,\frac{\sinh\,\operatorname{ad}rP}{\operatorname{ad}rP}\,rX,\,\frac{\sinh\,\operatorname{ad}rP}{\operatorname{ad}rP}\,rY\,)
\end{aligned}\tag{43}$$

for $X,Y\in T_PS^{m-1} = \{P\}^\perp$, note that $(\operatorname{ad}P)^2 := [\frac{1}{2}P^2,\cdot\,]$ is symmetric with respect to g.

Corollary B.1 (Riemannian Metric in Conical Exponential Coordinates)
Consider the isotropy representation $\mathfrak{p}$ of a Riemannian symmetric space G/K as a euclidian vector space of dimension m with scalar product g, unit sphere S^{m-1} and a triple product $[\,,]$. In consistency with the Lemma of Gauß the Riemannian metric on G/K pulls back under conical exponential coordinates $\mathbb{R}^+\times S^{m-1}\longrightarrow G/K$, $(r,P)\longmapsto\exp(rP)$, to the metric

$$\widetilde{\exp}^*g^{G/K} \;=\; dr\otimes dr \;+\; g^{G/K}_r$$

where $g^{G/K}_r$ for sufficiently small $r>0$ is the Riemannian metric on $S^{m-1}_r := \{r\}\times S^{m-1}$ defined in $P\in S^{m-1}_r$ in terms of the endomorphism $(\operatorname{ad}rP)^2 := r^2\,[\frac{1}{2}\,P^2,\cdot]$ by the formula:

$$(\,g^{G/K}_r\,)_P(\,X,\,Y\,) \;=\; r^2\,g(\,\frac{\sinh\,\operatorname{ad}rP}{\operatorname{ad}rP}X,\,\frac{\sinh\,\operatorname{ad}rP}{\operatorname{ad}rP}Y\,)$$

In particular the vector field $\frac{\partial}{\partial r}$ on $\mathbb{R}^+\times S^{m-1}$ is the unit normal field to the spheres $S^{m-1}_r\subset\mathbb{R}^+\times S^{m-1}$, which correspond to the geodesic distance spheres of radius $r>0$

in the symmetric space G/K. On the other hand the second fundamental form of a hypersurface can be calculated as the Lie derivative $\mathrm{II} = -\frac{1}{2}\,\mathfrak{Lie}_N g$ of the metric along the unit normal field N used to define II in the first place, in our case this consideration implies:

$$\begin{aligned}\mathrm{II}_P(\,X,\,Y\,) &= -\frac{1}{2}\,\mathfrak{Lie}_{\frac{\partial}{\partial r}}(\,dr\otimes dr\;+\;g_r^{G/K}\,)\\ &= -\frac{1}{2}\Big(\,g_r^{G/K}(\,\frac{1}{r}\,\frac{\mathrm{ad}\,rP}{\tanh\;\mathrm{ad}\,rP}X,\,Y\,)\;+\;g_r^{G/K}(\,X,\,\frac{1}{r}\,\frac{\mathrm{ad}\,rP}{\tanh\;\mathrm{ad}\,rP}Y\,)\Big)\end{aligned}$$

by $\frac{d}{dr}(r\frac{\sinh rz}{rz}) = \frac{\sinh rz}{rz}\,\frac{rz}{\tanh rz}$. Now $(\mathrm{ad}\,rP)^2$ is symmetric for g and a fortiori for $g_r^{G/K}$, thus:

Corollary B.2 (Second Fundamental Form of Distance Spheres)
The second fundamental form of the distance sphere S_r^{m-1} of radius $r>0$ in a Riemannian symmetric space G/K considered as an endomorphism on $T_P S_r^{m-1} = \{P\}^\perp$ is given by:

$$\mathrm{II}_P(\,X\,) = -\,\frac{1}{r}\,\frac{\mathrm{ad}\,rP}{\tanh\;\mathrm{ad}\,rP}X \qquad\qquad X\;\in\;\{\,P\,\}^\perp$$

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