Algebraic K-Theory*

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Abstract

This is an introductory course, with emphasis on concrete examples rather than general theory. The low dimensional K-groups K_0, K_1, and K_2 are defined explicitly and computed in examples related to number theory and arithmetic. There are also some discussion of the definition and properties of the higher algebraic K-groups.

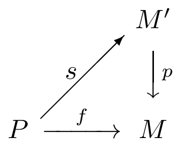
Contents

*Course given in the LMS-EPSRC INSTRUCTIONAL CONFERENCE ON K-THEORY. From 9^{th} to 15^{th} July, 1995 Lancaster University.
Based in lecture notes by J. L. Cisneros-Molina.

LECTURE 1

1 Projective Modules

Definition: Let A be a ring with 1 and let P, M and M' be A-modules. An A-module P is *projective* if for every homomorphism $f\colon P \to M$ and for every epimorphism $p\colon M' \to M$, there exists an homomorphism $s\colon P \to M'$ such that $p \circ s = f$.

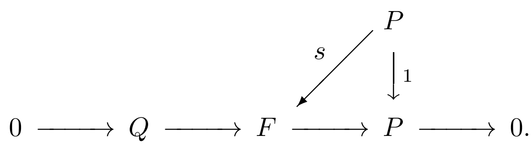


(1)

Example 1 Every free module is projective.

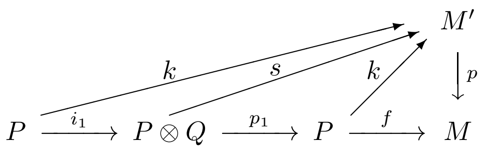
Proposition 1 *An A-module P is projective if and only if P is a summand of a free A-module.*

Proof: Let P be a projective module. On the commutative diagram (1) take $M' = F$, $M = P$ and $f = 1_P$ where F is a free A-module and $f = 1_P$ is the identity on P. Let $Q = \ker(P)$, then we have the following exact sequence



The existence of s implies that the sequence splits and therefore $F = P \oplus Q$.

Now let F be a free A-module such that $F = P \oplus Q$. Let $f\colon P \to M$ be a homomorphism and let $p\colon M' \to M$ be an epimorphism. Considere the following diagram



where $i_1\colon P \to F$ is the inclusion and $p_1\colon F \to P$ is the projection onto the first summand. Since F is projective, there exists s such that $p \circ s = f \circ p_1$. Let $k = s \circ i_1\colon P \to M'$, then

$$p \circ k = p \circ s \circ i_1 = f \circ p_1 \circ i_1 = f$$

since $p_1 \circ i_1 = 1_P$. $\square$

Proposition 2 *The following statements are equivalent:*

i) *P is a finitely generated projective A-module.*

ii) *There exists $n \in \mathbb{N}$ and an A-module Q such that $P \oplus Q = A^n$.*

iii) *There exists $n \in \mathbb{N}$ and $e \in M_n A$, the $n \times n$ matrices with entries in A, such that $e = e^2$ and $P = A^n e$.*

Let $\mathbb{P}_A$ be the category of finitely generated projective A-modules and let $\mathrm{Iso}(\mathbb{P}_A)$ be the isomorphism classes of $\mathbb{P}_A$, $\mathrm{Iso}(\mathbb{P}_A)$ is an abelian monoid under the operation

$$\langle P \rangle + \langle Q \rangle = \langle P \oplus Q \rangle$$

where $\langle P \rangle$ denotes the isomorphism class of the projective A-module P.

2 The Universal Group

Definition: Let I be an abelian monoid. Then there exist an abelian group $I^{\#}$(unique up to isomorphism) called *the universal group of I* or *the Grothendiek group of I* and a homomorphism $\phi \colon I \to I^{\#}$ which have the following *universal property:* Given a homomorphism $h \colon I \to G$ from I to an arbitrary group G there is a unique homomorphism of abelian groups $h' \colon I^{\#} \to G$ such that the following diagram is commutative

$$\begin{array}{ccc} I & \overset{\phi}{\to} & I^{\#} \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle h'} & \\ & G & \end{array}$$

2.1 Three constructions of $I^{\#}$

1. $I^{\#}$is the free abelian group with generators $[\alpha]$ with $\alpha \in I$ under the relations $[\alpha + \beta] = [\alpha] + [\beta]$.

2. Consider the relation $\sim$ in $I \times I$ given by

$$(\alpha, \beta) \sim (\alpha', \beta') \Leftrightarrow \exists \gamma \in I \text{ such that } \\ \alpha + \beta' + \gamma = \alpha' + \beta + \gamma.$$

   This is an equivalence relation. Then $I^{\#} = I \times I / \sim$ and the operation is defined by

$$(\alpha, \beta) + (\alpha', \beta') = (\alpha + \alpha', \beta + \beta')$$

   with identity $(0, 0)$ and inverse $-(\alpha, \beta) = (\beta, \alpha)$.

3. Assume that there exist $\alpha_0 \in I$ such that for every $\alpha \in I$ there exists $n \in \mathbb{N}$ and $\beta \in I$ such that $\alpha + \beta = n\alpha_0$. Then $I^{\#} = I \times \mathbb{N} / \sim$ where

$$(\alpha, n) \sim (\alpha', n') \Leftrightarrow \exists m \in \mathbb{N} \text{ such that } \\ \alpha + n'\alpha_0 + m\alpha_0 = \alpha' + n\alpha_0 + m\alpha_0.$$

This different abelian groups all have the required universal property. In the case of the construction 3 this is given by the following diagram

$$\begin{array}{ccc} I & \stackrel{\phi}{\to} & I\times\mathbb{N}/\sim \\ & {}_f\searrow \quad \swarrow_u & \\ & G & \end{array} \qquad \begin{array}{r} \phi(\alpha) = (\alpha, 0) \\ (\alpha, n) = \phi(\alpha) - n\phi(\alpha_0) \\ \text{Define } u(\alpha, n) = f(\alpha) - nf(\alpha_0). \end{array}$$

3 The Group K_0A

Definition: Let $K_0A = \mathrm{Iso}(\mathbb{P}_A)^{\#}$ i.e., the universal group of the abelian monoid $\mathrm{Iso}(\mathbb{P}_A)$.

According with the different constructions of the universal group, we can consider K_0A in three different ways

1. K_0A is the free abelian group with generators $[P]$ for every $P \in \mathbb{P}_A$ subject to the relations
$$[P \oplus Q] = [P] + [Q].$$
2. K_0A is the group of differences $[P] - [Q]$.
3. K_0A is the group of differences $[P] - [A^n]$.

Example 2 Let F be a field or a skew-field, then $\mathbb{P}_F$ is the set of vector spaces over F. The equivalence classes are characterized by the dimension of the vector spaces, therefore $\mathrm{Iso}(\mathbb{P}_F) \cong \mathbb{N}$ and $K_0F = \mathbb{Z}$.

Example 3 Let $A = \mathbb{Z}$. Then $\mathbb{P}_{\mathbb{Z}}$ comprises the finitely generated free abelian groups $\mathbb{Z}^n$ for $n \geq 0$. Therefore $\mathrm{Iso}(\mathbb{P}_{\mathbb{Z}}) \cong \mathbb{N}$ and $K_0\mathbb{Z} = \mathbb{Z}$. The same holds for principal ideal domains (P.I.D.).

Example 4 Let A a Dedekind domain, (e.g., let F be a number field (finite $\mathbb{Q}$-extension), then A is the integral clousure of $\mathbb{Z}$ in F). A *fractional ideal* is a finitely generated A-submodule. Let

$$Pic(A) = \text{ideal class group of A} = \frac{\text{fractional ideals}}{\text{principal fractional ideals}}$$

then we have that $P \in \mathbb{P}_A$ if and only if P can be written as

$$P = a_1 \oplus \cdots \oplus a_n \qquad a_i \text{ fractional ideals.}$$

Then it turns out that

$$\begin{array}{c} K_0A = \mathbb{Z} \oplus Pic(A) \\ [A_1 \oplus \cdots \oplus A_n] \mapsto (n, \text{ideal class of } \{A_1 \cdots A_n\}) \end{array}$$

4 Serre-Swan Theorem

Definition: A *vector bundle* consists of a space E, a continuous map $\pi\colon E \to X$ and a structure of complex vector space on each fibre $E_x = \pi^{-1}\{x\}$ such that this situation is locally trivial, i.e., there exists a covering U_α and isomorphisms

$$E|_{U_\alpha} \cong \mathbb{C}^{n_\alpha}_{U_\alpha}$$

respecting the structure of vector space on the fibres. Here $\mathbb{C}^{n_\alpha}_{U_\alpha}$ denotes the trivial bundle over U_α

$$U_\alpha \times \mathbb{C}^{n_\alpha} \xrightarrow{\text{proj}} U_\alpha.$$

Definition: A *vector bundle map* between two vector bundles $\pi\colon E \to X$ and $\pi'\colon E' \to X$ is a map $\phi\colon E \to E'$ such that the following diagram commutes

$$\begin{array}{ccc} E & \overset{\phi}{\to} & E' \\ {}_{\pi}\searrow & & \swarrow_{\pi'} \\ & X & \end{array}$$

and restricted to the fibres is a linear transformation of vector spaces.

Theorem 1 (Serre-Swan) *Let X be a compact Hausdorff space and let $A = C(X)$ the continuous complex valued functions on X. Then $\mathbb{P}_A$ is equivalen to the category of complex vector bundles over X.*

To prove the theorem we need the following lemmas

Lemma 1 *If e is an idempotent endomorphism of a vector bundle E over X, then eE and $e^\perp E = (1-e)E$ are vector bundles.*

PROOF: We need to show that eE is locally trivial. Since this is a local question, we can assume that E is the trivial bundle $E = \mathbb{C}^n_X$. Then e is a continuous family $\{e_x\}$ of idempotent matrices $n \times n$ in $M_n(\mathbb{C})$. Fix a point x_0 in X and put

$$T_x = e_x e_{x_0} + e^\perp_x e^\perp_{x_0}\colon \mathbb{C} \to \mathbb{C}.$$

This is a continuous family of matrices such that

a) $e_x T_x = T_x e_{x_0}$.

b) $T_x = 1$ at $x = x_0$.

We have that b) implies that T_x^{-1} exists for x near x_0, then $T = \{T_x\}$ gives an automorphism of $\mathbb{C}^n$ such that $T^{-1}eT$ is constant for all values e_{x_0}. □

Lemma 2 *Given E a vector bundle over X there exists another vector bundle E' such that $E \oplus E'$ is isomorphic to $\mathbb{C}^n_X$ for some n.*

Proof: Since we are assuming X compact, there exists a finite open covering U_α and isomorphisms

$$g_\alpha\colon E|_{u_\alpha} \to \mathbb{C}^{n_\alpha}_{U_\alpha} \qquad \alpha = 1, \ldots, N.$$

Choose a partition of unity $\{\rho_\alpha\}$, i.e., a family of functions $\rho_\alpha\colon X \to \mathbb{C}$ such that supp $\rho_\alpha \subset U_\alpha$, $\rho_\alpha \geq 0$ and $\sum \rho_\alpha = 1$. Let

$$\chi_\alpha = \frac{\rho_\alpha}{\sum \rho_\alpha^2}$$

then we have that $\sum \chi_\alpha^2 = 1$. Now define the maps $i\colon E \to \bigoplus_{\alpha=1}^N \mathbb{C}^{n_\alpha}_{U_\alpha}$ and $p\colon \bigoplus_{\alpha=1}^N \mathbb{C}^{n_\alpha}_{U_\alpha} \to E$ by

$$E \xrightarrow{i=\begin{pmatrix}\chi_1 g_1 \\ \vdots \\ \chi_N g_N\end{pmatrix}} \bigoplus_{\alpha=1}^N \mathbb{C}^{n_\alpha}_{U_\alpha} \xrightarrow{p=(\chi_1 g_1^{-1},\ldots,\chi_N g_N^{-1})} E$$

then we have that

$$p \circ i = \sum \chi_\alpha g_\alpha^{-1} \chi_\alpha g_\alpha = \sum \chi_\alpha^2 = 1.$$

Therefore E is a retract of $\bigoplus_{\alpha=1}^N \mathbb{C}^{n_\alpha}_{U_\alpha}$. Since $p \circ i = 1$, then $(i \circ p)^2 = i \circ p$ and by Lemma 1 we have

$$\bigoplus_{\alpha=1}^N \mathbb{C}^{n_\alpha}_{U_\alpha} = (i \circ p)\Big(\bigoplus_{\alpha=1}^N \mathbb{C}^{n_\alpha}_{U_\alpha}\Big) \oplus (i \circ p)^\perp \Big(\bigoplus_{\alpha=1}^N \mathbb{C}^{n_\alpha}_{U_\alpha}\Big).$$

□

Proof of Serre-Swan Theorem: Recall that $A = C(X)$. Let *Vect*(X) be the category of vector bundles over X. Let $\pi\colon E \to X$ a vector bundle in *Vect*(X), we denote by $\Gamma(X, E)$ the set of sections of E, i.e., the set of maps $s\colon X \to E$ such that $\pi \circ s = 1_X$. We have a funtor from *Vect*(X) to the category of A-modules given by

$$\begin{aligned} \mathit{Vect}(X) &\xrightarrow{\Gamma} A\text{-modules} \\ E &\mapsto \Gamma(X, E). \end{aligned}$$

Actually, Γ is a funtor from *Vect*(X) to $\mathbb{P}_A$, the category of finitely generated projective A-modules, since by lemma 2 E is a direct summand of a trivial bundle and hence $\Gamma(E)$ is a direct summand of $\Gamma(X, \mathbb{C}^n_X) = A^n$ which is a finitely generated free A-module, therefore by proposition 1 $\Gamma(E)$ is projective.

To prove the equivalence of the categories *Vect*(X) and $\mathbb{P}_A$ we have to see that the funtor $\Gamma\colon \mathit{Vect}(X) \to \mathbb{P}_A$ is:

fully faithfull We need to show that

$$\mathrm{Hom}_{\mathit{Vect}}(E, E') \cong \mathrm{Hom}_A(\Gamma(E), \Gamma(E')) \tag{2}$$

But if this is true for the bundles $E = E_1$ and $E = E_2$ then it is true for the bundle $E = E_1 \oplus E_2$ and if it is true for E, then it is true for any summand of E (retract of an isomorphism is an isomorphism). Hence it reduces to E trivial but in this case is clear.

surjective Let $P \in \mathbb{P}_A$, by proposition 2 iii) there exists $n \in \mathbb{N}$ and an idempotent $e \in \mathrm{Hom}_A(P, P)$ such that $P = A^n e$. Since the funtor is fully faithfull there exists and idempotent $\hat{e} \in \mathrm{Hom}_{Vect}(\mathbb{C}^n_X, \mathbb{C}^n_X)$ corresponding to e under the isomorphism (2). By lemma 1 $\hat{e}\mathbb{C}^n_X \in Vect(X)$ and we have that $\Gamma(\hat{e}\mathbb{C}^n_X) = P$. □

LECTURE 2

5 The Group $K_1 A$

Let $GL_n(A)$ denote the invertible $n \times n$ matrices over A. We have an inclusion $GL_n(A) \subset GL_{n+1}(A)$ given by $\alpha \mapsto \left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and we can define

$$GL(A) = \bigcup_n GL_n(A).$$

Definition: Let e_{ij}, $(i \neq j)$ be the matrix with 1 in the i-th row, j-th column and zero elsewhere. Let $a \in A$, an *elementary matrix* e^a_{ij} is a matrix of the form

$$e^a_{ij} = 1 + e_{ij}.$$

It is easy to check the following relations:

$$e_{ij} e_{kl} = \delta_{jk} e_{il} \tag{3}$$
$$e^a_{ij} e^b_{ij} = e^{a+b}_{ij} \tag{4}$$

where δ_{ij} is the Kronecker delta defined by $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

Definition: The *commutator* $[x, y]$ of two elements x and y of a group is defined by

$$[x, y] = xyx^{-1}y^{-1}.$$

It is immediate that

$$[x, y]^{-1} = [y, x]. \tag{5}$$

Proposition 3 *The commutator of elementary matrices satisfies the following realtions:*

$$[e^a_{ij}, e^b_{kl}] = \begin{cases} 1 & j \neq k \text{ and } i \neq l \\ e^{ab}_{il} & \text{if } j = k \text{ and } i \neq l \\ e^{-ba}_{kj} & \text{if } j \neq k \text{ and } i = l. \end{cases}$$

PROOF: We will check the last two cases, the other is similar. Using the relations (3) and (4) in the definition of the commutator we have:

$$\begin{aligned} e_{ij}^a e_{jl}^b &= (1 + ae_{ij})(1 + be_{jl}) \\ &= 1 + ae_{ij} + be_{jl} + abe_{il} \end{aligned}$$

$$\begin{aligned} e_{ij}^a e_{jl}^b e_{ij}^{-a} &= (1 + ae_{ij} + be_{jl} + abe_{il})(1 - ae_{ij}) \\ &= 1 + ae_{ij} + be_{jl} + abe_{il} - ae_{ij} \\ &= 1 + be_{jl} + abe_{il} \end{aligned}$$

$$\begin{aligned} [e_{ij}^a, e_{jl}^b] &= (1 + be_{jl} + abe_{il})(1 - be_{jl}) \\ &= 1 + be_{jl} + abe_{il} - be_{jl} \\ &= e_{il}^{ab}. \end{aligned}$$

For the last case we use (5) and the previous case:

$$\begin{aligned} [e_{ij}^a, e_{ki}^b] &= [e_{ki}^b, e_{ij}^a]^{-1} \\ &= (e_{kj}^{ba})^{-1} \\ &= e_{kj}^{-ba} \end{aligned}$$

□

Definition: The *elementary group* $E_n(A)$ is the subgroup of $GL_n(A)$ generated by e_{ij}^a for $1 \leq i, j \leq n$, $i \neq j$ and $a \in A$. The inclusion $GL_n(A) \hookrightarrow GL_{n+1}(A)$ restricts to the inclusion $E_n(A) \hookrightarrow E_{n+1}(A)$ and we can define

$$E(A) = \bigcup_n E_n(A).$$

Definition: A group G is called *perfect* if it is equal to its *commutator* subgroup $[G, G]$, i.e., $[G, G]$ is the subgroup generated by $[g, g']$ for every g and g' in G. The group $G_{\mathrm{ab}} = G/[G, G]$ is the maximal abelian quotient group of G.

Proposition 4 *$E_n(A)$ is perfect for $n \geq 3$.*

PROOF: Given i and k choose j such that $j \neq i$ and $j \neq k$. Then by proposition 3 we have that

$$e_{ik}^a = [e_{ij}^a, e_{jk}^1] \in [E_n(A), E_n(A)],$$

this shows that all generators are commutators. □

Lemma 3 (Whitehead) $E(A) = [GL(A), GL(A)]$.

Proof: By proposition 4 we have that for $n \geq 3$, $[E(A), E(A)] = E(A) \subset GL(A)$. We only need to show that $[GL(A), GLA] \subset E(A)$. Let $\alpha \in GL_n(A)$ and let I be the $n \times n$ identity matrix. We have that

$$\begin{pmatrix} I & \alpha \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -\alpha^{-I} & I \end{pmatrix} \begin{pmatrix} I & \alpha \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ -\alpha^{-I} & 0 \end{pmatrix}$$

is in $E_{2n}(A)$ since $\left(\begin{smallmatrix} I & \alpha \\ 0 & I \end{smallmatrix}\right)$ can be expressed as product of elementary matrices as follows

$$\left(\begin{array}{c|c} I & \begin{matrix} \alpha_{1,n+1} & \cdots & \alpha_{1,2n} \\ \vdots & & \vdots \\ \alpha_{n,n+1} & \cdots & \alpha_{n,2n} \end{matrix} \\ \hline 0 & I \end{array}\right) = \prod_{\substack{1 \leq i \leq n \\ n+1 \leq j \leq 2n}} e_{ij}^{\alpha_{ij}} \tag{6}$$

and analogously for $\left(\begin{smallmatrix} I & 0 \\ -\alpha^{-1} & I \end{smallmatrix}\right)$. Now consider

$$\begin{pmatrix} 0 & \alpha \\ -\alpha^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$$

which is also in $E_{2n}(A)$ since $\left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right)$ can be reduced to I_{2n} using elementary operations by rows:

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \sim \begin{pmatrix} I & -I \\ I & 0 \end{pmatrix} \sim \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Hence

$$\begin{pmatrix} [\alpha, \beta] & & \\ & I & \\ & & I \end{pmatrix} =$$

$$\begin{pmatrix} \alpha & & \\ & \alpha^{-1} & \\ & & I \end{pmatrix} \begin{pmatrix} \beta & & \\ & I & \\ & & \beta^{-1} \end{pmatrix} \begin{pmatrix} \alpha & & \\ & \alpha^{-1} & \\ & & I \end{pmatrix}^{-1} \begin{pmatrix} \beta & & \\ & I & \\ & & \beta^{-1} \end{pmatrix}^{-1}$$

where all the matrices in the right hand side are in $E_{3n}(A)$. Therefore the image of $[GL_n(A), GL_n(A)]$ in $GL_{3n}(A)$ is contained in $E_{3n}(A)$ and taking the union over n we get

$$[GL(A), GL(A)] \subset E(A)$$

finishing the proof of the lemma. □

Definition: Let $K_1 A = GL(A)_{\mathrm{ab}} = GL(A)/E(A)$ i.e., the maximal abelian quotient group of $GL(A)$.

Example 5 Let F be a field. Left multiplication by e_{ij}^a add a times the j-th row to the i-th row. It is known that $E(F) = \ker\{GL_n(F) \xrightarrow{\det} F^\times\}$ where $F^\times$ comprises the non-zero elements under the multiplication. Hence $GL_n(F)/E_n(F) = F^\times$.

6 The Group K_2A

Definition: Let $n \geq 2$. The *Steinberg group* $St_n(A)$ (also $St(A)$) is the group with generators x_{ij}^a with $i \neq j$ and $a \in A$ subjet to the relations

$$x_{ij}^a x_{ij}^b = x_{ij}^{a+b} \tag{7}$$

$$[x_{ij}^a, x_{kl}^b] = \begin{cases} 1 & j \neq k \text{ and } i \neq l \\ x_{il}^{ab} & \text{if } j = k \text{ and } i \neq l \\ x_{kj}^{-ba} & \text{if } j \neq k \text{ and } i = l. \end{cases} \tag{8}$$

There is a canonical surjection

$$St(A) \xrightarrow{\phi} E(A)$$

given by

$$\phi(x_{ij}^a) = e_{ij}^a \tag{9}$$

Definition: The group K_2A is defined as the kernel of the canonical surjection (9) i.e.

$$K_2A = \ker\{\phi\colon St(A) \to E(A)\}.$$

Definition: The *center* of a group G is defined by

$$Z(G) = \{x \in G | xg = gx \ \forall g \in G\}$$

Lemma 4 $Z(E(A)) = 1$

Proof: Let α be in the center of $E(A)$ an n suficiently big such that $\alpha \in E(A)$ Hence in $E_{2n}(A)$ we have that

$$\begin{pmatrix} \alpha & \alpha \\ 0 & I \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & I \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} \alpha & I \\ 0 & I \end{pmatrix}$$

and therefore $\alpha = I$ and $Z(E(A)) = 1$. □

Proposition 5 $\ker \phi = Z(St(A))$.

Proof: Firstly let show that $Z(St(A)) \subset \ker \phi$. Let β be in the center of $St(A)$ and $\gamma = \phi(\beta) \in E(A)$, where ϕ is the canonical surjection (9). Since ϕ is surjective, $\gamma \in Z(E(A))$. Hence by lemma 4 $\gamma = 1$ and $\beta \in \ker \phi$.

Now let show that $\ker \phi \subset Z(St(A))$. Let C_n the subgroup of $St(A)$ generated by x_{in}^a with $i \neq n$, $a \in A$ and fixed n.

Claim *The restriction of ϕ to C_n*

$$\phi|_{C_n} \colon C_n \to \phi(C_n)$$

is injective.

Proof of claim: Since $[x_{in}^a, x_{jn}^b] = 1$ C_n is abelian and any element γ of C_n can be written as a finite product $\gamma = \prod_{i \neq n} x_{in}^{a_i}$. Hence

$$\phi(\gamma) = \prod_{i \neq n} e_{in}^{a_i} = \begin{pmatrix} 1 & & & & a_1 & \\ & 1 & & & a_2 & \\ & & \ddots & & \vdots & \\ & & & 1 & a_{n-1} & \\ & & & & 1 & \\ & & & & a_{n+1} & 1 \end{pmatrix}$$

and therefore $\phi(C_n) \cong \bigoplus_{i \neq n} A$.

Consider now the following surjection

$$\bigoplus_{i \neq n} A \xrightarrow{\psi} C_n$$
$$(a_i)_{i \neq n} \mapsto \prod_{i \neq n} x_{in}^{a_i}.$$

Since the $x_{in}^{a_i}$ commute this product is independent of ordering if ψ is an homomorphism, but by (7)

$$(a_i + a_i') \mapsto \prod_{i \neq n} x_{in}^{a_i + a_i'} = \prod_{i \neq n} x_{in}^{a_i} x_{in}^{a_i'} = \prod_{i \neq n} x_{in}^{a_i} \prod_{i \neq n} x_{in}^{a_i'}$$

and we get the following commutative diagram

$$\begin{array}{ccc} \bigoplus_{i \neq n} A & \xrightarrow{\psi} & C_n \\ & {\scriptstyle\cong}\searrow \quad \swarrow{\scriptstyle\phi} & \\ & \phi(C_n) & \end{array}$$

which clearly implies the claim. □

Proof of the proposition: Take $\alpha \in \ker \phi$ and write it as a finite product of x_{ij}^a's. Choose n different from any i, j ocurring in the representation of α. Then α normalizes C_n, i.e. $\alpha C_n \alpha^{-1} \subset C_n$ since

$$x_{ij}^a x_{kn}^b x_{ij}^{-a} = \begin{cases} x_{kn}^b & \text{if } k \neq i, j \\ x_{in}^{ab} x_{jn}^{-b} & k = j. \end{cases} \tag{10}$$

Let $\gamma \in C_n$. Then $\alpha\gamma\alpha^{-1} \in C_n$ and $\phi(\alpha\gamma\alpha^{-1}) = \phi(\gamma)$ because $\alpha \in \ker \phi$. By the claim $\phi|_{C_n}$ is injective and $\alpha\gamma\alpha^{-1} = \gamma$. Therefore α centralizes C_n. Similarly, let R_n be the subgroup of $St(A)$ generated by x_{nj}^a with $j \neq n$, $a \in A$ and n fixed as before. By a similar argument α centralizes R_n, but $C_n \cup R_n$ generates $St(A)$ since if $i \neq j$

$$x_{ij}^a \in C_n \text{ if } j = n$$
$$x_{ij}^a \in R_n \text{ if } i = n$$

and $x_{ij}^a = [x_{in}^a, x_{nj}^1] \in [C_n, R_n]$ if $i \neq n$ and $j \neq n$. Therefore α centralizes $St(A)$, i.e. $\ker \phi \subset Z(St(A))$. □

LECTURE 3

7 Motivation

In section 5 we defined K_1A as

$$K_1A = GL(A)/E(A).$$

From topological K-Theory we have that

$$K^{-1}(X) = [X, GL(\mathbb{C})]$$

but

$$\mathrm{Hom}(X, GL(\mathbb{C})) = GL_n(\mathbb{C}).$$

and $\left(\begin{smallmatrix} 1 & ta \\ 0 & 1 \end{smallmatrix}\right)$ is a homotopy from $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$, so elementary matrices are homotopic to the identity . Therefore $K_1(C(X))$ is the algebraic analogue of $K^{-1}(X)$.

For $K^{-2}(X)$ we have

$$\begin{aligned} K^{-2}(X) &= K^{-1}(SX) = [SX, GL(\mathbb{C})] \\ &= [S^1, \mathrm{Hom}(X, GL(\mathbb{C}))] = [S^1, GL(C(X))]. \end{aligned}$$

So think on "loops in $GL(C(X))$". A chain of elementary matrices gives relations between them. On the other hand $K_2A = \ker\phi$ also gives relations between elementary matrices.

8 Central Extensions

Definition: A *central extension* of a group G is an exact sequence of groups

$$1 \to K \to E \to G \to 1$$

such that $K \subset Z(E)$, where $Z(E)$ denotes the center of E.

Definition: Two central extensions

$$1 \to K \to E \to G \to 1$$

and

$$1 \to K \to E' \to G \to 1$$

are *equivalent* if there exists an isomorphism $\psi\colon E \to E'$ such that the following diagram commutes

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle\psi} & & \| & & \\ 1 & \longrightarrow & K & \longrightarrow & E' & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

Definition: An *universal central extension* of a group G is a central extension

$$1 \to N \to U \to G \to 1$$

such that, given any central extension

$$1 \to K \to E \to G \to 1$$

there exists a unique homomorphism $h\colon U \to E$ such that the following diagram commutes

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & U & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow & & \downarrow h & & \| & & \\ 1 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

Note that if there exists a universal central extension this is unique up to isomorphism.

The following theorem is a well known characterization of universal central extensions.

Theorem 2 *A central extension*

$$1 \to N \to U \to G \to 1$$

is universal if and only if U is perfect and every central extension of U splits.

An immediate consequence of the definition of $St(A)$ is that it is perfect. By proposition 5 $\ker \phi = Z(St(A))$ and we have the canonical central extension

$$1 \to K_2A \to St(A) \to E(A) \to 1.$$

In fact, this is the universal central extension of $E(A)$ and to see this by theorem 2 it is enough to prove:

Theorem 3 *Any central extension*

$$1 \to C \to Y \xrightarrow{\psi} St(A) \to 1$$

splits, i.e. there exists an homomorphism s such that $\psi s =$ identity.

Corolary 1 *If $Y = [Y, Y]$, then $Y \xrightarrow{\sim} St(A)$.*

Basic idea Suppose $y_1, y_2 \in Y$ are such that $\psi(y_1) = \psi(y_2)$ i.e. $y_1 = cy_2$ with $c \in C$. Then

$$[y, y'] = [cy_2, y'] = cy_2y'(cy_2)^{-1}{y'}^{-1} = cy_2y'y_2^{-1}c^{-1}{y'}^{-1} = [y_2, y'].$$

hence for $x \in St(A)$ $[\psi^{-1}(x), y']$ is a well defined element of Y, similarly $[\psi^{-1}(x), \psi^{-1}(x')]$ is a well defined element of Y. Since by the relations (8) $x_{ij}^a = [x_{in}^a, x_{nj}^1]$ for $n \neq i, j$, we can define s by

$$s(x_{ij}^a) = [\psi^{-1}(x_{in}^a), \psi^{-1}(x_{nj}^1)].$$

The hard point is to prove that this is independent of the choose of n.

Lemma 5 *$[\psi^{-1}(x^a_{ij}), \psi^{-1}(x^b_{kl})] = 1$ if $j \neq k$ and $i \neq l$.*

PROOF: Choose n different from k,l,i and j, then

$$[\psi^{-1}(x^b_{kn}), \psi^{-1}(x^1_{nl})] \subset \psi^{-1}[x^b_{kn}, x^1_{nl}] = \psi^{-1}(x^b_{kl})$$

and therefore

$$[\psi^{-1}(x^a_{ij}), \psi^{-1}(x^b_{kl})] = [\psi^{-1}(x^a_{ij}), [\psi^{-1}(x^b_{kn}), \psi^{-1}(x^1_{nl})]].$$

Choose $v \in \psi^{-1}(x^b_{kn})$, $w \in \psi^{-1}(x^1_{nl})$ and $u \in \psi^{-1}(x^a_{ij})$. We need to prove that

$$[u, [v, w]] = 1$$

but we have that

$$\begin{aligned} [u, [v, w]] &= u[v, w]u^{-1}[v, w]^{-1} \\ &= [uvu^{-1}, uwu^{-1}][v, w]^{-1} \\ &= [v, w][v, w]^{-1} \\ &= 1 \end{aligned}$$

if uvu^{-1} and v are congruent mod C and if wvw^{-1} and w are congruent mod C, but it is the case since $\psi(u)$ and $\psi(v)$ commute and also $\psi(u)$ and $\psi(w)$ commute by (10). □

Proposition 6 (Identities in any group) *Let X, Y and Z elements of a group. Then we have the following identities.*

1) $[X, [Y, Z]] = [XY, Z][Z, X][Z, Y]$.

2) $[XY, Z] = [X, [Y, Z]][Y, Z][X, Z]$.

PROOF:

$$[XY, Z][Z, X] = (xy)z(y^{-1}x^{-1})z^{-1}zxz^{-1}x^{-1} \tag{11}$$

$$[X, [Y, Z]] = x(yzy^{-1}z^{-1})x^{-1}(zyz^{-1}y^{-1}). \square \tag{12}$$

Lemma 6 *Let h, i, j, k be distinct and let $a, b, c \in A$. Then*

$$[\psi^{-1}(x^a_{hi}), [\psi^{-1}(x^b_{ij}), \psi^{-1}(x^c_{jk})]] = [[\psi^{-1}(x^a_{hi}), \psi^{-1}(x^b_{ij})], \psi^{-1}(x^c_{jk})].$$

PROOF: Pick $u \in \psi^{-1}(x^a_{hi})$, $v \in \psi^{-1}(x^b_{ij})$ and $v \in \psi^{-1}(x^c_{jk})$. Then by lemma 5 $[u, w] = 1$ and we have

$[u, v] \subset \psi^{-1}[x^a_{hi}, x^b_{ij}] = \psi^{-1}(x^{ab}_{hj})$ commutes with u and v.

$[v, w] \subset \psi^{-1}(x^{ba}_{ik})$ commutes with v and w.

$\left.\begin{array}{l} [u, [v, w]] \subset [\psi^{-1}(x^a_{hi}), \psi^{-1}(x^{ba}_{ik})] \\ [[u, v], w] \end{array}\right\} \subset \psi^{-1}(x^{abc}_{ik})$ commutes with u, v and w.

Using the identity *1)* for u, v and w and the fact that $[u, w] = 1$ we have

$$\begin{aligned}
[u, [v, w]] &= [uv, w][w, v] \\
&= [[u, v]vu, w][w, v] \\
&= [vu[u, v], w][w, v] \\
&= [vu, [[u, v], w]][[u, v], w][vu, w][w, v] \\
&= [[u, v], w]vuwu^{-1}v^{-1}w^{-1}wvw^{-1}v^{-1} \\
&= [[u, v], w]
\end{aligned}$$

□

PROOF OF THEOREM 3: Recall that we defined

$$s(x^a_{hk}) = [\psi^{-1}(x^a_{hn}), \psi^{-1}(x^1_{nk})].$$

Rewrite lemma 6 as

$$[\psi^{-1}(x^a_{hi}), \psi^{-1}(x^{bc}_{ik})] = [\psi^{-1}(x^{ab}_{hj}), \psi^{-1}(x^c_{jk})]$$

and putting $b = c = 1$ we get

$$[\psi^{-1}(x^a_{hi}), \psi^{-1}(x^1_{ik})] = [\psi^{-1}(x^a_{hj}), \psi^{-1}(x^1_{jk})].$$

This proves that the definition of s is independent of n □

Another well-known result is that every perfect group G has a universal central extension

$$\begin{array}{ccccccccc}
1 & \longrightarrow & H_2(G, \mathbb{Z}) & \longrightarrow & \hat{G} & \longrightarrow & G & \longrightarrow & 1 \\
 & & \exists! \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & C & \longrightarrow & Y & \longrightarrow & G & \longrightarrow & 1
\end{array}$$

This result combined with the fact that $St(A)$ is the universal central extension of $E(A)$ gives us the following relation between $H_2(E(A))$ and K_2A

$$H_2(E(A)) = \ker\{St(A) \to E(A)\} = K_2.$$

Other relations between group homology and algebraic K-Theory are:

$$\begin{aligned}
&K_1A = GL(A)/[GL(A), GL(A)] = H_1(GL(A), \mathbb{Z}) \\
&K_2A = H_2(E(A), \mathbb{Z}) \\
&K_3A = H_3(St(A), \mathbb{Z}).
\end{aligned}$$

LECTURE 4

9 Group Cohomology

Let G be a group and let M be a G-module. There are groups $H_i(G, M)$ and $H^i(G, M)$ called respectively the homology and cohomology groups of G with coefficients on M.

If C is an abelian group with trivial G-action we have the following facts:

a) $H^0(G, C) = C$.

b) $H^1(G, C) = \mathrm{Hom}(G, C) = \mathrm{Hom}(G_{\mathrm{ab}}, C)$.

c) $H^2(G, C) =$ set of isomorphism clases of central extensions of G by C.

10 Topological Interpretation

Any group G has a *classifying space* BG which is a pointed (nice) space unique up to homotopy tipe equivalence such that:

i) $\pi(BG) = G$.

ii) The universal covering of BG is contractible.

Example 6

$$G = \mathbb{Z} \qquad BG = S^1.$$

It is a known fact that

$$H_i(G, C) = H_i(BG, C)$$

and

$$H^i(G.C) = H^i(BG, C)$$

By the universal coefficient theorem we have

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{i-1}(BG, \mathbb{Z}), C) \to H^i(BG, C) \to \mathrm{Hom}(H_i(BG, \mathbb{Z}), C) \to 0$$

and

$$H^1(G, C) \xrightarrow{\cong} \mathrm{Hom}(H_1(G, \mathbb{Z}), C)$$

therefore by b)

$$H_1(G, \mathbb{Z}) = G_{\mathrm{ab}}.$$

For $i = 2$ we also have

$$0 \to \mathrm{Ext}^1(H_1(G, \mathbb{Z}), C) \to H^2(G, C) \to \mathrm{hom}(H_2(G, \mathbb{Z}), C) \to 0.$$

In particular if G is perfect i.e. $G_{\mathrm{ab}} = 0$ then

$$H^2(G, C) = \mathrm{Hom}(H_2(G, \mathbb{Z}), C)$$

but by c) corresponding to the identity in $\mathrm{Hom}(H_2G, H_2G)$ there is a central extension

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H_2G & \longrightarrow & \tilde{G} & \longrightarrow & G & \longrightarrow & 1 \\ & & \Big\downarrow u & & \Big\downarrow & & \Big\| & & \\ 1 & \longrightarrow & C & \longrightarrow & u_*\tilde{G} & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

and given any homomorphism $u \in \mathrm{Hom}(H_2G, C)$ push-out gives a central extension of G by C

$$u_*\tilde{G} = C \times \tilde{G}/\{(-u(x), i(x))|x \in H_2G\}$$

so any central extension of G by C is induced by a unique homomorphism $u\colon H_2G \to C$.

Proposition 7 *$\tilde{G}$ is perfect.*

PROOF: Consider the following diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H_2G & \longrightarrow & \tilde{G} & \longrightarrow & G & \longrightarrow & 1 \\ & & \Big\downarrow \exists!\, u & & \Big\downarrow \exists & & \Big\| & & \\ 1 & \longrightarrow & B & \longrightarrow & [\tilde{G}, \tilde{G}] & \longrightarrow & G & \longrightarrow & 1 \\ & & \Big\downarrow i & & \Big\downarrow & & \Big\| & & \\ 1 & \longrightarrow & H_2G & \longrightarrow & \tilde{G} & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

where $B = H_2G \cap [\tilde{G}, \tilde{G}]$. Then $i \circ u =$ identity on H_2G because

$$i_*u_*\tilde{G} = i_*[\tilde{G}, \tilde{G}] = \tilde{G}$$

and then $B = H_2G$ and therefore $[\tilde{G}, \tilde{G}] = \tilde{G}$. □

Proposition 8 *$H_2\tilde{G} = 0$, i.e. every central extension of $\tilde{G}$ splits.*

PROOF: Given

$$E \xrightarrow{q} \tilde{G} \xrightarrow{p} G$$

where E is a perfect central extension of $\tilde{G}$. E acts in the following exact sequence of abelian groups

$$1 \to \ker q \to \ker pq \to \ker p \to 1$$

with the trivial action on $\ker q$ and $\ker p$, so we get a homomorphism

$$E \to \mathrm{Hom}(\ker p, \ker q)$$

therefore the action of E on $\ker pq$ is trivial.

So if $E = \tilde{\tilde{G}}$, then is a perfect central extension of $\tilde{G}$

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H_2\tilde{G} & \longrightarrow & \tilde{\tilde{G}} & \longrightarrow & \tilde{G} & \longrightarrow & 1 \\ & & & & \uparrow & & \downarrow & & \\ & & & & \tilde{G} & \longrightarrow & G & & \end{array}$$

Then the universal property of $\tilde{G}$ implies that $\tilde{G}$ lifts into $\tilde{\tilde{G}}$ which says

$$\tilde{\tilde{G}} = \tilde{G} \times H_2\tilde{G} \xrightarrow{\text{proj}} H_2\tilde{G}$$

and since it is perfect then $H_2\tilde{G} = 0$. □

Example 7 Let $G = A_5$ the group of rotations of the icosahedron. It is a simple non-abelian group of order 60 and it is also perfect i.e. $[G, G] = G$.

What is $\tilde{G}$ and $H_2(A_5)$?

Consider the following exact sequence

$$1 \to \{\pm 1\} \to SU(2) \to SO(3) \to 1.$$

We have that $SU(2) = S^3$ is simply connected and that $G = A_5$ is a subgroup of $SO(3)$ and pull-back gives us the following commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & S^3 & \longrightarrow & SO(3) \\ & & \| & & \uparrow & & \uparrow \\ 1 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & E & \longrightarrow & G \end{array}$$

The group E acts freely on S^3 and preservs orientation, so S^3/E is an oriented 3-manifold and it is called the Poincaré homology 3-sphere.

Because $\pi_i S^3 = 0$ for $i \leq 2$, S^3/E is close to BE. For $i = 2$

$$\pi_i(S^3/E) = \pi_i(S^3) = 0$$

and therefore

$$H_i(S^3/E) = H_i(E) \qquad \text{for } i = 1, 2.$$

We have that S^3/E is an oriented 3-manifold, $H_1(S^3/E) = H_1(E) = 0$ and by Poincaré duality $H_2(S^3/E) = 0$ then $H_2(E) = 0$. Therefore $H_1(E) = H_2(E) = 0$ implies that $E = \tilde{A}_5$ and $H_2(A_5) = \mathbb{Z}_2$.

Analogy:

connected	$\longleftrightarrow$	perfect
universal covering	$\longleftrightarrow$	universal central extension

$$O(3) \overset{i}{\supset} \underset{\substack{\text{connected}\\ \text{component}\\ \text{of } 1}}{SO(3)} \overset{\varphi}{\twoheadleftarrow} \underset{\substack{\text{universal}\\ \text{covering}\\ \text{of identity}\\ \text{component}}}{S^3}$$

The cokernel of i is $\pi_0(O(3))$ and the kernel of φ is $\pi_1(O(3))$.

$$GL(A) \overset{i}{\supset} \underset{\substack{\text{largest}\\ \text{perfect}\\ \text{subgroup}}}{E(A)} \overset{\phi}{\twoheadleftarrow} \underset{\substack{\text{universal}\\ \text{central}\\ \text{extension}}}{St(A)}$$

The cokernel of i is K_0A and the kernel of ϕ is K_2A.

Or think on

$$O \supset SO \leftarrow Spin.$$

Here $\pi_0 O = KO^{-1}(pt)$ and $\pi_1 O = KO^{-2}(pt)$.

LECTURE 5

11 The + Construction

Notation: In this section by a space we will mean a connected CW complex with basepoint and also we will denote by $[X, Y]$ the homotopy classes of basepoint preserving maps from X to Y.

Let X be a space such that $\pi_1(X)$ is perfect i.e. $\pi_1(X)_{\text{ab}} = H_1(X, \mathbb{Z})$ vanishes.

Problem: Construct a space X^+ such that $\pi_1(X^+) = 0$ and a map $i\colon X \to X^+$ such that

$$i_*\colon H_*(X) \xrightarrow{\cong} H_*(X^+).$$

Idea: Choose elements $\gamma_i \in \pi_1(X)$, $i \in I$ such that the normal subgroup they generate is the whole group $\pi_1(X)$.

Special case: one γ. Choose a loop $S^1 \xrightarrow{u} X$. Let $Y = X \cup_u e^2$, by the van Kampen theorem

$$\pi_1 Y = \pi_1 X \times_{\pi_1(S^1)} \pi_1(e^2) = \pi_1 X / \{{}^{\text{normal subgroup}}_{\text{generated by } \gamma}\} = 0.$$

From the homology exact sequence of the pair

$$\begin{array}{ccccc}
 & & & & H_3(Y, X) \\
 & & & \swarrow & \\
H_2(X) & \longrightarrow & H_2(Y) & \longrightarrow & H_2(X, Y) \\
 & & & \swarrow & \\
H_1(X) & \longrightarrow & H_1(Y) & \longrightarrow & H_1(X, Y)
\end{array}$$

we have that

$$H_n(X) \xrightarrow{\cong} H_n(Y) \qquad n \geq 3$$

and since $H_1(X) = 0$ because $\pi_1 X$ is perfect and $H_1(Y) = 0$ because Y is simply-connected we also have

$$0 \to H_2(X) \to H_2(Y) \to (Y) \to \mathbb{Z} \to 0.$$

By Hurewicz theorem $\pi_1(Y) = 0$ implies $\pi_2(Y) = H_2(Y)$ and there is a map $S^2 \xrightarrow{\nu} Y$ such that

$$H_2(S^2) \to H_2(Y) \to \mathbb{Z}. \quad (\text{composite } H_2(S^2) \xrightarrow{\cong} \mathbb{Z})$$

Put $X^+ = Y \cup_\nu e^3$. Using the homology exact sequence the pair (X^+, Y)

$$H_3(X^+) \to H_3(X^+, Y) \to H_2(Y) \to H_2(X^+) \to H_2(X^+, Y)$$

and the fact that $H_3(X^+, Y) = H_3(X^+/Y) = \mathbb{Z}$ since $X^+/Y = e^3/\partial e^3 = S^3$ we have that the image of ∂ is the class of ν and

$$H_2 X \cong H_2 X^+$$
$$H_n X \cong H_n Y \cong H_n X^+ \qquad n \geq 3.$$

Proposition 9 *Let X be a space with $\pi_1 X$ perfect. Let $i \colon X \to X^+$ be such that $\pi_1 X^+ = 0$ and $i_* \colon H_*(X) \xrightarrow{\cong} H_*(X^+)$. Then for all Y such that $\pi_1 Y = 0$ we have*

$$i^* \colon [X^+, Y] \xrightarrow{\cong} [X, Y] \qquad \begin{array}{ccc} X & \xrightarrow{i} & X^+ \\ & \searrow & \swarrow \exists! \\ & Y & \end{array}$$

Proof: *Surjectivity of i^**

Let $u \in [X, Y]$. Consider the following diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & X^+ \\ u\downarrow & & \downarrow u' \\ Y & \overset{i'}{\underset{r}{\rightleftarrows}} & Z \end{array}$$

where $Z = Y \cup_X X^+$. We have that i is a homology isomorphism and applying Mayer-Vietoris i' is also homology isomorphism. By van Kampen's theorem we have that $\pi_1(Z) = 0$ and by Whitehead theorem i' is a homotopy equivalence. Let $r \colon Z \to Y$ be a homotopy inverse for i'. Then

$$(ru')i = r(i'u) = (ri')u \sim (id)u = u$$

Injectivity of i^*

Let $g_0, g_1 \colon X^+ \to Y$ be such that $g_0 i$ and $g_1 i$ are both homotopic to u.

$$\begin{array}{ccc} X & \xrightarrow{i} & X^+ \\ u\downarrow & {}^{g_0}\swarrow & \downarrow u' \\ Y & \xrightarrow{i'} & Z \end{array}$$

where Z is as before. By the homotopy extension theorem we can deform g_0 and g_1 so that $g_0 i = g_1 i = u$.

Define

$$r_0 \colon Y \cup_X X^+ \xrightarrow{(id_Y, g_0)} Y$$

$$r_1 \colon Y \cup_X X^+ \xrightarrow{(id_Y, g_1)} Y.$$

Then $r_0 i' = r_1 i' = id_Y$. But i' is a homotopy equivalence and therefore $r_0 \sim r_1$. Since $r_0 u' = g_0$ and $r_1 u' = g_1$ then $g_0 \sim g_1$. □

Corolary 2 (X^+, i) *are determined up to homotopy equivalence.*

Lemma 7 *Let G be perfect and let*

$$1 \to H_2 G \to \tilde{G} \to G \to 1$$

its universal central extension. Let $C = H_2 G$. Then one has a map of fibrations

$$\begin{array}{ccccc} BC & \longrightarrow & B\tilde{G} & \longrightarrow & BG \\ \| & & \downarrow & & \downarrow \\ BC & \longrightarrow & B\tilde{G}^+ & \longrightarrow & BG \end{array}$$

Recall that $H_1(G) = H_2(\tilde{G}) = 0$ and that $\pi_1(BG) = G$. We have that

$$H^2(G, C) = H^2(BG, C) = [BG, K^{EM}(C, 2)]$$

where $K^{EM}(C, 2)$ is an Eilenberg-McLane space. Consider the following diagram

$$\begin{array}{ccccc} BC & = & BC & = & K^{EM}(C,1) \\ \downarrow & & \downarrow & & \downarrow \\ B\tilde{G} & \xrightarrow{\beta} & P & \longrightarrow & K^{EM}(C,2) \\ \downarrow & & \downarrow & & \downarrow \\ BG & \longrightarrow & BG^+ & \xrightarrow{\alpha} & K^{EM}(C,2) \end{array}$$

we have

$$\pi_2(BG^+) = H_2(BG^+) = H_2(BG) = C$$

and α induces an isomorphism in π_2

$$\pi_2(BG^+) \to \pi_1(BC) \to \pi_1(P) \to \pi_1(BG^+) = 0$$

and therefore $\pi_1(P) = 0$. The map β is a homology isomorphism and since $\pi_1(P) = 0$ then $BG^+ \simeq P$.

From $BC \to B\tilde{G}^+ \to BG^+$ we have

$$\pi_3(BG^+) = \pi_3(B\tilde{G}^+) = H_3(B\tilde{G}^+)$$

by Hurewicz theorem, since $H_1(\tilde{G}) = H_2(\tilde{G}) = 0$.

Consider now $G = E(A)$. Since in this case G is perfect we have that $\tilde{G} = St(A)$ and $H_2(G) = K_2A$.

Claim

$$\begin{aligned} \pi_2(BE(A)^+) &= H_2(E(A)) = K_2A \\ \pi_3(BE(A)^+) &= H_3(St(A)) \end{aligned}$$

Proof:

$$\begin{aligned} \pi_2(BE(A)^+) = H_2(BE(A)^+) = H_2(BE(A)) = H_2(E(A)) = K_2A \\ \pi_3(BE(A)^+) = \pi_3(BSt(A)^+) = H_3(BSt(A)) = H_3(St(A)) \end{aligned}$$

□

Theorem 4 *Let $N \subset \pi_1(X)$ be a perfect normal subgroup. Then there is a space X^+ (depending on N) and a map $i\colon X \to X^+$ such that*

a) Induces an isomorphism

$$\pi_1(X)/N \to \pi_1(X^+).$$

b) For any $\pi_1(X^+)$-module L one has

$$i_*\colon H_*(X, i^*L) \xrightarrow{\cong} H_*(X, L).$$

The pair (X^+, i) is determined up to homotopy equivalence.

Construction: Let $\tilde{X}$ be the covering space corresponding to $N \subset \pi_1 X$. Then $\pi_1\tilde{X} = N$ and since N is perfect we can apply to $\tilde{X}$ the + construction described on page 19 and get a simply-connected space $\tilde{X}^+$ and a map $\tilde{i}\colon \tilde{X} \to \tilde{X}^+$ wich induces an isomorphism on homology. By push-out in the diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow[\text{isomorphism}]{\text{homology}} & \tilde{X}^+ \\ \downarrow & & \downarrow \\ X & \longrightarrow & X^+ \end{array}$$

we get the desired space X^+ such that $\pi_1(X^+) = \pi_1(X)/N$.

Let $X = BGL(A)$. Then $\pi_1 X = GL(A)$ and $N = E(A) \subset GL(A)$ is perfect. By theorem 4 we can get $BGL(A)^+$ with

$$\pi_1(BGL(A)^+) = GL(A)/E(A) = K_1 A$$

and also

$$H_*(BGL(A), L) \to H_*(BGL(A)^+, L)$$

is an isomorphism for all modules over $K_1 A$

Taking the pull-back by i of the universal covering of $BGL(A)^+$ we get

$$\begin{array}{ccc} BE(A) & \xrightarrow[\text{isomorphism}]{\text{homology}} & B\widetilde{GL(A)}^+ \\ \downarrow & & \downarrow \\ BGL(A) & \longrightarrow & BGL(A)^+ \end{array}$$

therefore $BE(A)^+$ is the universal covering of $BGL(A)^+$. Then we have

$$\begin{aligned} \pi_n BGL(A)^+ &= \pi_n BE(A)^+ \qquad n \geq 2 \\ \pi_2 BGL(A)^+ &= K_2 A \\ \pi_3 BGL(A)^+ &= H_3(St(A)). \end{aligned}$$

Now we can define the groups $K_n A$ for every N.

Definition: $K_n A = \pi_n(BGL(A)^+)$ for $n \geq 1$.

LECTURE 6

12 Acyclic Maps

Proposition 10 *Let X and Y be CW complexes. For a map $f\colon X \to Y$ the following statements are equivalent:*

1. *The homotopy-fiber F of f is acyclic i.e. $\tilde{H}_*(F) = 0$. Here homotopy-fiber means to replace f by a Serre fibration and take the actual fiber.*

2. *$\pi_1 f\colon \pi_1 X \to \pi_1 Y$ is surjective and for any $\pi_1 Y$-module L we have*

$$f_*\colon H_*(X, f^* L) \xrightarrow{\sim} H_*(Y, L).$$

3. *Let $\tilde{Y}$ the universal covering of Y and take the pull-back by f*

$$\begin{array}{ccc} X \times_Y \tilde{Y} & \xrightarrow{f'} & \tilde{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

Then f' is a homology isomorphism.

Definition: Call f *acyclic* when any of the conditions on proposition 10 hold.

Corolary 3 *Acyclic maps are closed under composition, homotopy pull-back and homotopy push-outs.*

Theorem 5 *Given a perfect normal subgroup $N \subset \pi_1 X$, there is a unique (up to homotopy equivalence) acyclic map $f\colon X \to Y$ where $\pi_1 Y = \pi_1 X/N$ such that $N = \ker \pi_1(f)$.*

Moreover for any T

$$f^*\colon [Y,T] \xrightarrow{\sim} \{\alpha \in [X,T] | \pi_1(\alpha)\colon \pi_1 X \to \pi_1 T \text{ kills } N\}$$

Construction of Y: Let $\tilde{X}$ be the covering space corresponding to $N \subset \pi_1 X$. We get Y as the push-out inthe following diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\text{acyclic}} & (\tilde{X})^+ \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Let $X = BGL(A)$. We have that $E(A)$ is a perfect normal subgroup of $GL(A) = \pi_1(BGL(A))$. Applying theorem 5 we get

$$\begin{array}{ccc} BE(A) & \xrightarrow{\tilde{f}} & B\widetilde{GL(A)}^+ \\ \downarrow & & \downarrow \\ BGL(A) & \xrightarrow{f} & BGL(A)^+ \end{array}$$

where f is the unique acyclic map such that $\ker \pi_1(f) = E(A)$. Since f is acyclic, $\tilde{f}$ is also acyclic and we conclude that

$$\pi_2 BGL(A)^+ = \pi_2 BE(A)^+ = H_2(BE(A)^+) = H_2(E(A)) = K_2 A$$
$$\pi_3 BGL(A)^+ = \pi_3 BE(A)^+ = H_3(BSt(A))$$
$$K_n = \pi_n BGL(A)^+ \qquad \text{for } n \geq 1.$$

Let $G_\infty = \varinjlim BGL_k(\mathbb{C}^*) = BGL(\mathbb{C})$ with its natural topology. Corresponding to direct sum of vector bundles there is a h-space structure on G_∞.

The space $BGL(A)$ is not an h-space but $BGL(A)^+$ is an h-space.

We have that the maps

$$GL_n(\mathbb{C}) \to GL_{2n}(\mathbb{C})$$
$$\alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$$
$$\alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$$

are homotopic in the natural topology but the maps

$$\alpha \mapsto \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\alpha \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are conjugates via an element of $E(A)$.

Conjugate by elements of $E(A)$ on $BGL(A)$ is non trivial (provided base-point preserving maps are considered)

$$\pi_1(BGL(A)) = GL(A).$$

But conjugation of elements of $E(A)$ on $BGL(A)^+$ is trivial up to homotopy. Hence $BGL(A)^+$ is an h-space.

Example 8 Let F be a field. We have that $F[x_1, x_2, \ldots, x_n] \sim F$. Hence

$$K_0 F[x_1, x_2, \ldots, x_n] = K_0 F.$$

There is no periodicity.

Theorem 6 *Let F_q a finite fiel with order q. Then*

$$K_n F_q = \begin{cases} \mathbb{Z} & n = 0 \\ F_q^\times \backsimeq \mathbb{Z}_{q-1} & n = 1 \\ 0 & n = 2 \\ \mathbb{Z}_{q^2-1} & n = 3 \\ 0 & \\ \mathbb{Z}_{q^3} & \\ \vdots & \end{cases}$$

Ideas to proof this: Representations in characteristic p can be lifted to virtual representations over $\mathbb{C}$

$$BGL(F_q) \xrightarrow[\text{lifting}]{\text{Brower}} BU \xrightarrow{\psi^q - 1} BU \qquad BU = G_\infty.$$

Take the homotopy fibre of $BU \xrightarrow{\psi^q-1} BU$ and get a map

$$BGL(F_q) \to \text{h-fibre of } (BU \xrightarrow{\psi^q-1} BU).$$

So by the universal property of the + construction we get

$$BGL(F_q)^+ \to \text{h-fibre of } (BU \xrightarrow{\psi^q-1} BU).$$

Theorem 7 *The map* $GLF_q{}^+ \to$ *h-fibre of* $(BU \xrightarrow{\psi^q - 1} BU)$ *is a homotopy equivalence.*

Because $BGL(A)^+$ is an h-space one knows that $H_*(BGL(A)^+, \mathbb{Q})$ is a Hopf algebra and also by Milnor-Moore theorem

$$\begin{aligned} \pi_*(BGL(A)^+ \otimes \mathbb{Q}) &= PrimH_*(BGL(A)^+, \mathbb{Q}) \\ &= PrimH_*(BGL(A), \mathbb{Q}) \\ &= PrimH_*(GL(A), \mathbb{Q}). \end{aligned}$$

So rational K-Theory comes from calculating group cohomology.

Example 9 Let $A = \mathbb{Z}$. Then

$$\dim_{\mathbb{Q}}(K_i\mathbb{Z} \otimes \mathbb{Q}) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 1 \\ 0 & i = 2 \\ 0 & i = 3 \\ 0 & i = 4 \\ \mathbb{Z} & i = 5 \\ \vdots & \\ \mathbb{Z} & i = 9 \end{cases} \quad \text{non-periodic.}$$

Example 10 Let A a number field. Then $A \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$.

$$\dim_{\mathbb{Q}}(K_n A \otimes \mathbb{Q}) = \begin{cases} 1 & n = 0 \\ r_1 + r_2 & n = 1 \\ 0 & n = 2 \\ r_2 & n = 3 \\ 0 & n = 4 \\ r_1 + r_2 & n = 5 \\ 0 & n = 6 \\ r_2 & n = 7. \end{cases}$$

There is a periodicity phenomenon high-up but problems at bottom.