Dispersive Estimates for Schrödinger Equations and Applications

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We begin by looking at the autonomous linear first-order system

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0,$$

where A is a given n by n matrix.
Then a straightforward calculation shows that the solution is given by

$$x(t) = \exp(tA)x_0,$$

where the exponential function is defined by the usual power series

$$\exp(tA) = \sum_{j=0}^{\infty} \frac{t^j}{j!} A^j,$$

which converges by comparison with the real-valued exponential function since

$$\|\sum_{j=0}^{\infty} \frac{t^j}{j!} A^j\| \le \sum_{j=0}^{\infty} \frac{|t|^j}{j!} \|A\|^j = \exp(|t|\|A\|).$$

One of the basic questions concerning the solution is the long-time behavior: Does the solutions remain bounded for all times (stability) or does it even converge to zero (asymptotic stability)?
This question is usually answered by determining the spectrum (i.e. the eigenvalues) of A:

Theorem

A linear constant coefficient ODE is asymptotically stable iff all eigenvalues of A have negative real part. It is stable iff all eigenvalues have non-positive real part and for those with real part zero the algebraic and geometric multiplicities coincide.

This can easily been seen by resorting to the Jordan canonical form of A.

Proof.

If U maps A to Jordan canonical form $U^{-1}AU = J_1 \oplus \cdots \oplus J_m$, where $J = \alpha\mathbb{I} + N$ are the Jordan blocks, then

$$\exp(tA) = U \begin{pmatrix} \exp(tJ_1) & & \\ & \ddots & \\ & & \exp(tJ_m) \end{pmatrix} U^{-1},$$

with the exponential of one Jordan block given by

$$\exp(tJ) = \mathrm{e}^{\alpha t} \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \dots & \frac{t^{k-1}}{(k-1)!} \\ & 1 & t & \ddots & \vdots \\ & & 1 & \ddots & \frac{t^2}{2!} \\ & & & \ddots & t \\ & & & & 1 \end{pmatrix}.$$

In particular, in the stable case one has

$$\|\exp(tA)\| \le C, \qquad t \ge 0,$$

and in the asymptotically stable case

$$\|\exp(tA)\| \le C\mathrm{e}^{-\alpha t}, \qquad t \ge 0,$$

for some $C \ge 1$ and some $\alpha > 0$. Hence one has a quite good qualitative understanding of this case.

We remark that the first case is typical for physical systems which are e.g. of Hamiltonian type. For such systems the conservation of energy implies stability but clearly this is incompatible with asymptotic stability. If one adds some friction to the system, one has energy dissipation and hence gets asymptotic stability.

Trying to apply these ideas to partial differential equations things get more tricky. To understand this consider the heat equation which can formally be written as

$$\dot{x}(t) = Ax(t), \qquad A = \Delta,$$

where $\Delta = \sum_{j=1}^{n} \partial_{x_j}^2$ is the usual Laplacian. Now the underlying space X will be a suitable function space (e.g., some $L^p(\mathbb{R}^n)$) and Δ is considered as a linear operator on this space. If A were a bounded operator on X we could define $\exp(tA)$ as before by using power series. This gives a continuous group of operators

$$t \mapsto T(t) = \exp(tA).$$

However, our operator is not bounded (this is already reflected by the fact that we need to restrict its domain to functions which can be differentiated at least twice in some suitable sense) and hence this simple approach does not work here. Moreover, the fact that a Gaussian initial condition concentrates and eventually becomes unbounded for negative times, shows that we cannot expect $T(t)$ to be defined for all $t \in \mathbb{R}$.

Nevertheless, there is a well-developed theory under what conditions on A the abstract Cauchy problem

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0,$$

in a Banach space X has a solution given by

$$x(t) = T(t)x_0,$$

where $T(t)$, $t \geq 0$, forms a strongly continuous semigroup of (bounded) operators.

A strongly continuous operator semigroup (also C_0 semigoup) is a family of bounded operators $T(t) : X \to X$, $t \geq 0$, such that

1. $T(t)x \in C([0,\infty), X)$ for every $x \in X$ (strong continuity) and
2. $T(0) = \mathbb{I}$, $T(t+s) = T(t)T(s)$ for every $t, s \geq 0$ (semigroup property).

Note: Requiring uniform continuity instead of strong continuity is equivalent to A being bounded.

In this setting A is called the infinitesimal generator of $T(t)$ and can be recovered via

$$Ax = \lim_{t\downarrow 0} \frac{1}{t}(T(t)x - x)$$

with domain precisely those $x \in X$ for which the above limit exists.
We are interested in bounded semigroups satisfying $\|T(t)\| \le 1$ which are known as semigroups of contraction.

Theorem (Hille–Yosida theorem)

A linear (possibly unbounded) operator A is the infinitesimal generator of a contraction semigroup if and only if

1. *A is densely defined, closed and*
2. *the resolvent set of A contains the positive half line and for every $\lambda > 0$ we have*
$$\|(A-\lambda)^{-1}\| \le \frac{1}{\lambda}.$$

Another version which is often easier to check is

Theorem (Lumer–Phillips theorem)

Let X be a complex Hilbert space with scalar product $\langle .,. \rangle$. A linear (possibly unbounded) operator A is the infinitesimal generator of a contraction semigroup if and only if

1. *A is densely defined, closed and*
2. *the resolvent set of A contains the positive half line and A is dissipative*

$$\mathrm{Re}\langle x, Ax \rangle \le 0$$

for every x in the domain of A.

Note: The theorem also holds in Banach spaces if the scalar product is replaced by a duality section.
Note: In the finite dimensional case dissipativity implies that all eigenvalues have non-positive real part.

In our case the underlying partial differential equation will be linear with constant coefficients and just as in the ODE case we will be able to obtain an explicit formula for $T(t)$ and hence we will not further pursue these ideas here.

So far we have only looked at linear equations while most more realistic problems are nonlinear. However, in many situations nonlinear problems can be considered as perturbations of linear problems via

$$\dot{x}(t) = Ax(t) + g(t, x(t)).$$

In such an situation we can use our knowledge about the linear problem ($g \equiv 0$) to show that asymptotic stability persists for the perturbation under suitable assumptions:

Theorem

Suppose A is asymptotically stable

$$\|\exp(tA)\| \le C\mathrm{e}^{-\alpha t}, \qquad t \ge 0,$$

for some constants $C, \alpha > 0$. Suppose that

$$|g(t,x)| \le g_0|x|, \qquad |x| \le \delta,\ t \ge 0,$$

for some constant $0 < \delta \le \infty$. Then, if $g_0 C < \alpha$, the solution $x(t)$ starting at $x(0) = x_0$ satisfies

$$\|x(t)\| \le C\mathrm{e}^{-(\alpha - g_0 C)t}|x_0|, \qquad |x_0| \le \frac{\delta}{C},\ t \ge 0.$$

Proof.

Our point of departure is the following integral equation

$$x(t) = \exp(tA)x_0 + \int_0^t \exp((t-s)A)g(s, x(s))ds.$$

which follows from Duhamel's formula for the in homogenous problem and is clearly equivalent to our original Cauchy problem. Inserting our assumptions we obtain

$$|x(t)| \le C\mathrm{e}^{-\alpha t}|x_0| + \int_0^t C\mathrm{e}^{-\alpha(t-s)} g_0 |x(s)| ds$$

for $t \ge 0$ as long as $|x(s)| \le \delta$ for $0 \le s \le t$.

Proof (Cont.).

Introducing $y(t) = |x(t)|\mathrm{e}^{\alpha t}$ we get

$$y(t) \le C|x_0| + \int_0^t C g_0 y(s) ds$$

and Gronwall's inequality implies $y(t) \le C|x_0|\mathrm{e}^{C g_0 t}$. This is the desired estimate

$$|x(t)| \le C|x_0|\mathrm{e}^{-(\alpha - C g_0)t}, \qquad 0 \le t \le T,$$

for solutions which satisfy $|x(t)| \le \delta$ for $0 \le t \le T$. If we start with $|x_0| \le \delta/C$ we have $|x(t)| \le \delta$ for sufficiently small T (note $C \ge 1$) by continuity and the above estimate shows that this remains true for all times. □

Discussion of the previous theorem

First of all note that in applications the linear part Ax will capture the linear behavior of the right hand side at 0 and the remainder $g(t,x)$ will vanish faster than linear near 0. Hence in our assumption we will be able to make g_0 as small as we want at the expense of choosing δ small. In particular, asymptotic stability persists under nonlinear perturbations.

Secondly, it is important to emphasize that the two main ingredients for this proof were

- Duhamel's formula and
- a good qualitative understanding of the linear part.

In particular, it generalizes to situations where a corresponding estimate for the linear part is available (e.g. periodic equations).

Generalizing the previous theorem

The generalization to strongly continuous semigroups is a bit more tricky. First of all any solution of the inhomogenous abstract Cauchy problem

$$\dot{x}(t) = Ax(t) + g(t), \qquad x(0) = x_0,$$

will still be given by Duhamel's formula

$$x(t) = T(t)x_0 + \int_0^t T(t-s)g(s)ds$$

provided g is integrable. However, the above formula might not be a (classical) solution of the abstract Cauchy problem and one speaks of a mild solution (it can be shown that one has uniqueness and hence there will be no classical solution in such a situation).
So solutions of our integral equation in the nonlinear situation might not be solutions of the original problem.

Concerning (uniform) exponential stability of semigroups

$$\|T(t)\| \le C\mathrm{e}^{-\alpha t}$$

for some $C, \alpha > 0$ we mention:

Theorem (Gearhart–Prüss–Greiner)

Let X be a Hilbert space. Then a strongly continuous semigroup $T(t)$ is uniformly exponentially stable if and only if the resolvent set of the generator A contains the half-plane $\{z \in \mathbb{C} | \mathrm{Re}(z) > 0\}$ and satisfies

$$\sup_{\mathrm{Re}(z)>0} \|(A-z)^{-1}\| < \infty.$$

Note: Since the resolvent $(A-z)^{-1}$ is analytic in z and $\|(A-z)^{-1}\| \to \infty$ as z approaches the spectrum, the condition implies that the spectrum (which is closed) must be contained in $\{z \in \mathbb{C} | \mathrm{Re}(z) < 0\}$. Since for bounded operators we have $\|(A-z)^{-1}\| \le (|z| - \|A\|)^{-1}$ for $|z| > \|A\|$ this condition reduces to our previous one for finite matrices.

Finally we want to look at the conservative case. Unfortunately this case is much less stable under perturbations as the trivial example

$$\dot{x} = x^2$$

shows. In fact, the solution is given by

$$x(t) = \frac{x_0}{1 - x_0 t}$$

which blows up at the finite positive time $t = \frac{1}{x_0}$ if $x_0 > 0$.

Theorem

Suppose A is stable

$$\|\exp(tA)\| \le C, \qquad t \ge 0,$$

for some constant $C \ge 1$. Suppose that

$$|g(t,x)| \le g_0(t)|x|, \qquad |x| \le \delta,\ t \ge 0,$$

for some constant $0 < \delta \le \infty$ and some function $g_0(t)$ with $G_0 = \int_0^\infty g_0(t) < \infty$. Then the solution $x(t)$ starting at $x(0) = x_0$ satisfies

$$|x(t)| \le C\exp(CG_0)|x_0|, \qquad |x_0| \le \frac{\delta}{C\exp(CG_0)},\ t \ge 0.$$

Proof.

As in the previous proof we start with

$$x(t) = \exp(tA)x_0 + \int_0^t \exp((t-s)A)g(s,x(s))ds$$

and using our estimates we obtain

$$|x(t)| \le C|x_0| + \int_0^t Cg_0(s)|x(s)|ds.$$

Hence an application of Gronwall's inequality

$$|x(t)| \le C|x_0| \exp\left(C \int_0^t g_0(s)ds \right) \le C|x_0| \exp\left(CG_0\right)$$

finishes the proof.

Discussion of the previous theorem

Observe that in this case the linear evolution does not provide any decay and hence we had to assume that our perturbation provides the necessary decay. In finite dimension there is nothing much we can do. However, in infinite dimensions there are different norms and while one might be conserved, another one might decay!

To understand this let as look again at the heat equation whose solution in $\mathbb{R}^n$ is given by

$$u(t,x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \mathrm{e}^{-\frac{|x-y|^2}{4t}} g(y) dy.$$

For positive initial condition g a simple application of Fubini shows $\|u(t,.)\|_1 = \|g\|_1$ that the L^1 norm is conserved. Physically this corresponds to energy conservation. Moreover, Young's inequality shows $\|u(t,.)\|_p \le \|g\|_p$ for every $1 \le p \le \infty$. However, if we use different norms on both sides we get decay

$$\|u(t,.)\|_\infty \le \frac{1}{(4\pi t)^{n/2}} \|g\|_1.$$

Of course this decay is no longer exponential. Since we have equality for the fundamental solution of the heat equation the above estimate is optimal.

Conclusion

What you should have remembered so far:

- Nonlinear equations can be treated as perturbations of linear problems by virtue of Duhamel's formula!
- This requires a good qualitative understanding of the underlying linear problem.
- For dissipative systems we have exponential decay in time for the underlying semigroup. For conservative systems we can still get polynomial decay of the semigroup if we chose the right norms.

To keep the technical problems at a minimum and to be able to focus on some of the main ideas we will use the one-dimensional discrete Schrödinger equation as our model:

$$\mathrm{i}\dot{u}(t) = H_0 u(t), \quad t \in \mathbb{R},$$

where

$$(H_0 u)_n = -u_{n+1} + 2u_n - u_{n-1}, \quad n \in \mathbb{Z}.$$

We will investigate this operator and its associated group in the weighted spaces $\ell^p_\sigma = \ell^p_\sigma(\mathbb{Z})$, $\sigma \in \mathbb{R}$, associated with the norm

$$\|u\|_{\ell^p_\sigma} = \begin{cases} \left(\sum_{n\in\mathbb{Z}} (1+|n|)^{p\sigma} |u(n)|^p\right)^{1/p}, & p \in [1,\infty), \\ \sup_{n\in\mathbb{Z}} (1+|n|)^{\sigma} |u(n)|, & p = \infty. \end{cases}$$

Of course, the case $\sigma = 0$ corresponds to the usual $\ell^p_0 = \ell^p$ spaces without weight.

Making the plane wave ansatz

$$u_n(t) = \mathrm{e}^{-\mathrm{i}(\theta n - \omega t)}$$

we obtain a solution if and only if the dispersion relation

$$\omega = 2(\cos(\theta) - 1)$$

holds. Hence different plane waves travel at different speeds which is known as dispersion. According to physical intuition the superposition of different plane waves will lead to destructive interference and thus to decay of wave packets.

Mathematically a wave packet is a (continuous superposition) of plane waves given by

$$u_n(t) = \int_{-\pi}^{\pi} c(\theta) \mathrm{e}^{-\mathrm{i}(\theta n - \omega(\theta) t)} d\theta,$$

where $c(\theta)$ is the amplitude of the plane wave with frequency θ. Evaluating this formula at $t = 0$ shows that the initial condition

$$g_n = u_n(0) = \int_{-\pi}^{\pi} c(\theta) \mathrm{e}^{-\mathrm{i}\theta n} d\theta$$

is given by 2π the Fourier coefficients of $c(\theta)$:

$$c(\theta) = \sum_{k \in \mathbb{Z}} \frac{g_k}{2\pi} \mathrm{e}^{\mathrm{i}\theta k}.$$

Putting everything together we obtain

$$u_n(t) = (\exp(-\mathrm{i}tH_0)g)_n = \sum_{m\in\mathbb{Z}} K_{n,k}(t) g_k,$$

where

$$K_{n,k}(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}t(2-2\cos\theta)}\mathrm{e}^{-\mathrm{i}\theta|n-k|} d\theta.$$

Note: The last integral is Bessel's integral implying

$$K_{n,k}(t) = \mathrm{e}^{\mathrm{i}(-2t+\frac{\pi}{2}|n-k|)} J_{|n-k|}(2t),$$

where $J_\nu(z)$ denotes the Bessel function of order ν. But for our purpose the integral will be more suitable.

As $t \to \infty$ our integrand will oscillate faster and faster and hence is of oscillatory type. By the method of stationary phase it is well-known that the main contribution of such oscillatory integrals will come from the stationary phase points. By this method one can get precise asymptotes of the kernel along rays where $v = \frac{|n-k|}{t} = const$. However, we would rather like an estimate which is uniform in n and k. Such a quantitative estimate is provided by the van der Corput lemma. To this end we look at the oscillatory integral

$$\int_a^b A(\theta) \mathrm{e}^{\mathrm{i}t\phi(\theta)} d\theta,$$

where $A(\theta)$ is the amplitude and $\phi(\theta)$ is the phase (which is assumed to be real).

Lemma

Suppose that ϕ is real-valued and l times differentiable with $|\phi^{(l)}(\theta)| \geq 1$ for $\theta \in [a, b]$ with $\phi'(\theta)$ monotonic if $l = 1$. Then there is a universal constant c_l (independent of a, b, and ϕ) such that

$$\left| \int_a^b \mathrm{e}^{\mathrm{i}t\phi(\theta)} d\theta \right| \leq \frac{c_l}{t^{1/l}}$$

for $t > 0$.

Note: If $|\phi^{(l)}(\theta)| \geq \delta > 0$ we can scale t and ϕ and the estimate reads

$$\left| \int_a^b \mathrm{e}^{\mathrm{i}t\phi(\theta)} d\theta \right| \leq \frac{c_l}{(\delta t)^{1/l}}.$$

Proof.

Denote the integral by $I(t)$. We will use induction and begin with $l = 1$. Integration by parts shows

$$
\begin{aligned}
I(t) &= \int_a^b \frac{1}{\mathrm{i}t\phi'(\theta)} \left(\frac{d}{d\theta} \mathrm{e}^{\mathrm{i}t\phi(\theta)} \right) d\theta \\
&= \frac{1}{\mathrm{i}t\phi'(\theta)} \mathrm{e}^{\mathrm{i}t\phi(\theta)} \Big|_a^b - \frac{1}{\mathrm{i}t} \int_a^b \left(\frac{d}{d\theta} \frac{1}{\phi'(\theta)} \right) \mathrm{e}^{\mathrm{i}t\phi(\theta)} d\theta
\end{aligned}
$$

and estimating the absolute value gives

$$
\begin{aligned}
t|I(t)| &\le \frac{1}{|\phi'(b)|} + \frac{1}{|\phi'(a)|} + \int_a^b \left| \frac{d}{d\theta} \frac{1}{\phi'(\theta)} \right| d\theta \\
&= |\phi'(b)|^{-1} + |\phi'(a)|^{-1} + |\phi'(b)^{-1} - \phi'(a)^{-1}| \le 2.
\end{aligned}
$$

This shows the case $l = 1$ with $c_1 = 2$.

Proof (Cont.).

Now suppose $|\phi^{(l+1)}(\theta)| \geq 1$. Then there can be at most one point $\theta_0 \in [a,b]$ with $\phi^{(l)}(\theta_0) = 0$ and we have $|\phi^{(l)}(\theta)| \geq \delta$ for $|\theta - \theta_0| \geq \delta$. If there is no such point choose θ_0 such that $|\phi^{(l)}|$ is minimal. Now split $I(t) = I_1(t) + I_2(t)$ where I_1 is the integral over $(a, \theta_0 - \delta) \cup (\theta_0 - \delta, b)$ and I_2 the integral over $(\theta_0 - \delta, \theta_0 - \delta)$. Then

$$|I_1(t)| \leq \frac{2c_l}{(\delta t)^{1/l}} \quad \text{and} \quad |I_2(t)| \leq 2\delta,$$

where we have used the induction hypothesis for both parts of the first integral. Choosing $\delta = t^{-1/(l+1)}$, to balance both contributions, finishes the proof with $c_{l+1} = 2(1 + c_l) = 2(2^l - 1))$. □

The $\ell^1 \to \ell^\infty$ decay

Theorem

For the time evolution of the free discrete Schrödinger equation the following dispersive decay estimate holds:

$$\|\mathrm{e}^{-\mathrm{i}tH_0}\|_{\ell^1 \to \ell^\infty} = \mathcal{O}(t^{-1/3}), \quad t \to \infty.$$

Proof.

We set $v := |n-k|/t \geq 0$. Then $K_{n,k}(t)$ is a oscillatory integral with the phase function

$$\phi_v(\theta) = 2 - 2\cos\theta + v\theta.$$

Then, since $\phi_v''(\theta) = 2\cos(\theta)$ and $\phi_v'''(\theta) = -2\sin(\theta)$, we can split our domain of integration into four intervals where either $|\phi_v''(\theta)| \geq \sqrt{2}$ or $|\phi_v'''(\theta)| \geq \sqrt{2}$. Applying the van der Corput lemma on each interval gives the desired claim. $\square$

The above decay rate is "sharp" as can be seen from the following well-known asymptotics of the Bessel function

$$J_t(t) \sim t^{-1/3}, \quad t \to \infty.$$

In fact, the slowest decay happens precisely along the ray $v = 2$, where we have a degenerate phase point with $\theta_2'(-\pi/2) = \theta_2''(-\pi/2) = 0$ and where the van der Corput lemma with $l = 3$ is the best we can do. Away from this ray we can apply the van der Corput lemma with $l = 2$ or $l = 1$ and hence improve the decay rate. Moreover, this is the case when n and k are restricted to a finite region or, as we will show next, are sufficiently localized.

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Theorem

For the time evolution of the free discrete Schrödinger equation the following dispersive decay estimate holds:

$$\|\mathrm{e}^{-\mathrm{i}tH_0}\|_{\ell^2_\sigma \to \ell^2_{-\sigma}} = \mathcal{O}(t^{-1/2}), \quad t \to \infty, \quad \sigma > 2/3.$$

Note: The claim holds for $\sigma > 1/2$ but the proof is significantly more complicated.

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Proof.

The problematic direction which causes the slowest decay is $v = 2$, as pointed out before. Hence we distinguish the two cases $|n-k| \le t$ and $|n-k| \ge t$. In the region $|n-k| \le t$ have $v = |n-k|/t \le 1$ and avoid this degenerate phase point. Hence we can split the domain of integration into the region $[-\pi, -\frac{2\pi}{3}] \cup [-\frac{\pi}{3}, \frac{\pi}{3}] \cup [\frac{3\pi}{2}, \pi]$ where $|\phi_v''(\theta)| \ge 1$ and $[-\frac{2\pi}{3}, -\frac{\pi}{3}] \cup [\frac{\pi}{3}, \frac{3\pi}{2}]$ where $|\phi_v'(\theta)| \ge \sqrt{3} - 1 > 0$. Applying the van der Corput lemma on each of these intervals we obtain the bound

$$\sup_{|n-k| \le t} \left| K_{n,k}(t) \right| \le C t^{-1/2}.$$

On the other hand, our previous proof implies

$$\sup_{|n-k| \ge t} \left| K_{n,k}(t) \right| \le C t^{-1/3} = C t^{-1/2} t^{1/6} \le C t^{-1/2} |n-k|^{1/6}.$$

Proof (Cont.).

From this the desired estimate follows since $K_{n,k} = |n-k|^\alpha$ is bounded in $\ell^2_\sigma \to \ell^2_{-\sigma}$ iff $\tilde{K}_{n,k} = (1+|n|)^{-\sigma}|n-k|^\alpha(1+|k|)^{-\sigma}$ is bounded in ℓ^2. Finally,

$$\sum_{n,k} |\tilde{K}_{n,k}|^2 \le \sum_{n,k} (1+|n|)^{-2(\sigma-\alpha)}(1+|k|)^{-2(\sigma-\alpha)} = \left(\sum_n (1+|n|)^{-2(\sigma-\alpha)} \right)^2$$

shows that $\tilde{K}$ is Hilbert–Schmid (and hence bounded) if $2(\sigma-\alpha) > 1$. For $\alpha = \frac{1}{6}$ we obtain $\sigma > \frac{3}{2}$. $\square$

Now we would like to consider the perturbed Schrödinger equation

$$\mathrm{i}\dot{u}(t) = Hu(t), \quad t \in \mathbb{R},$$

where

$$(Hu)_n = (H_0 u)_n + q_n u_n, \quad n \in \mathbb{Z}.$$

We will assume that q_n is real-valued and decays at $|n| \to \infty$. In particular, $q \in \ell^\infty(\mathbb{Z})$ and thus H is a bounded self-adjoint operator in $\ell^2(\mathbb{Z})$. The main difference now is that the underlying difference equation $Hu = zu$ can no longer be solved explicitly!
Despite this lack of explicit solvability we will still be able to show qualitative features of the solutions. But first we will need a good formula for the time evolution.

We begin with some general remarks which are valid for an arbitrary self-adjoint operator H in a Hilbert space $\mathfrak{H}$.
The most important observation in this case that in order to understand the spectral features of H one needs to understands its resolvent

$$z \mapsto \mathcal{R}_H(z) = (H - z)^{-1}$$

which is an analytic map from the resolvent set $\rho(H) \subset \mathbb{C}$ to the set of bounded operators from $\mathfrak{H}$ to itself. Since the spectrum of a self-adjoint operator is confined to the real line, the resolvent is analytic in the upper and lower half planes. Because of

$$\mathcal{R}_H(\overline{z}) = \mathcal{R}_H(z)^*$$

we can restrict our attention to the upper half plane. Here we use the bar for complex conjugation and the star for the adjoint operator.

More specific, what needs to be understood is the limit of the resolvent as z approaches the real line. To see this recall Stone's formula.

Theorem

Let H be self-adjoint. Then

$$\frac{1}{2\pi\mathrm{i}}\int_{\lambda_1}^{\lambda_2}\big(\mathcal{R}_H(\lambda+\mathrm{i}\varepsilon)-\mathcal{R}_H(\lambda-\mathrm{i}\varepsilon)\big)\,d\lambda \overset{s}{\to} \frac{1}{2}\big(P_H([\lambda_1,\lambda_2])+P_H((\lambda_1,\lambda_2))\big)$$

strongly. Here $P_H(\Omega)=\chi_\Omega(H)$ denotes the orthogonal projection onto the set $\Omega\subseteq\mathbb{R}$ associated with H (and $\chi_\Omega(\lambda)$ is the characteristic function of Ω).

Remark: The integral is defined as a Riemann sum.

Proof.

By the spectral theorem we have $f_n(H) \stackrel{s}{\to} f(H)$ provided $f_n(\lambda) \to f(\lambda)$ pointwise and $\|f_n\|_\infty \le C$ is uniformly bounded. Hence the problem reduces to compute the limit of the associated functions:

$$\begin{aligned}\frac{1}{2\pi\mathrm{i}}\int_{\lambda_1}^{\lambda_2}\left(\frac{1}{x-\lambda-\mathrm{i}\varepsilon}-\frac{1}{x-\lambda+\mathrm{i}\varepsilon}\right)d\lambda &= \frac{1}{\pi}\int_{\lambda_1}^{\lambda_2}\frac{\varepsilon}{(x-\lambda)^2+\varepsilon^2}d\lambda\\ &= \frac{1}{\pi}\left(\arctan\left(\frac{\lambda_2-x}{\varepsilon}\right)-\arctan\left(\frac{\lambda_1-x}{\varepsilon}\right)\right)\\ &\to \frac{1}{2}\left(\chi_{[\lambda_1,\lambda_2]}(x)+\chi_{(\lambda_1,\lambda_2)}(x)\right).\end{aligned}$$

□

In a similar fashion one establishes the following extension:

Theorem

Let H be self-adjoint. Then

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}} f(\lambda)\big(\mathcal{R}_H(\lambda+\mathrm{i}\varepsilon)-\mathcal{R}_H(\lambda-\mathrm{i}\varepsilon)\big)d\lambda \xrightarrow{s} f(H)$$

for any bounded and continuous function $f \in C_b(\mathbb{R})$.

In particular,

$$\exp(-\mathrm{i}tH) = \lim_{\varepsilon\downarrow 0}\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}t\lambda}\big(\mathcal{R}_H(\lambda+\mathrm{i}\varepsilon)-\mathcal{R}_H(\lambda-\mathrm{i}\varepsilon)\big)d\lambda$$

and everything boils down to understanding the limits $\mathcal{R}_H(\lambda\pm\mathrm{i}0)$.

For differential (or difference) equations the resolvent is an integral (summation) operator whose kernel is known as Green's function. For our discrete Schrödinger operator H this kernel is explicitly given by

$$\mathcal{R}_{n,m}(z) = \mathcal{R}_{m,n}(z) = \frac{f_n^+(z) f_m^-(z)}{W(f^+(z), f^-(z))}, \qquad m \le n,$$

where $f^\pm(z)$ are the (weak) solutions of the underlying difference equation $Hf = zf$ which are square summable near $\pm\infty$ and

$$W_n(f,g) = f_n g_{n+1} - f_{n+1} g_n$$

is the discrete Wronski determinant which can be easily seen to be independent of n if f and g both solve $Hf = zf$.

To proceed further we will assume that q decays sufficiently fast. $q \in \ell^1_1(\mathbb{Z})$ will be sufficient for our purpose. Under this assumption we assume that there exist solutions of $Hf = zf$ which asymptotically look like the unperturbed solutions:

$$f_n^{\pm}(\theta) \sim \mathrm{e}^{\mp \mathrm{i} n\theta}, \quad n \to \pm\infty.$$

Here θ and z are related via

$$2 - 2\cos\theta = z$$

as before, normalized such that $\mathrm{Im}(\theta) < 0$.
If q is compactly supported this is obvious. In the general case use Duhamel's formula to obtain an integral equations for the perturbed solutions in terms of the unperturbed ones. The trick here is to start the summation not at a finite initial point but (at first formally) at $\pm\infty$. Then show existence of solutions using the standard fix point iteration procedure for Volterra-type equations.

If q is compactly supported, then $f_n^\pm(\theta)$ will start out equal to the unperturbed solutions and will start to pick up multiples of shifted perturbed solutions as soon as we hit the support of q. This motivates the following ansatz for the Jost solutions

$$f_n^\pm(\theta) = \mathrm{e}^{\mp\mathrm{i}\theta n} \left(1 + \sum_{m=\pm 1}^{\pm\infty} B_{n,m}^\pm \mathrm{e}^{\mp\mathrm{i} m\theta} \right),$$

where $B_{n,m}^\pm \in \mathbb{R}$. Moreover, inserting this into our equation one can get a corresponding difference equation for $B^\pm$ and then use induction to show

$$|B_{n,m}^\pm| \le C_n^\pm \sum_{k=n+\lfloor m/2 \rfloor}^{\pm\infty} |q_k|.$$

The estimate for $B^{\pm}$ shows that the decay of $B^{\pm}$ is directly related to the decay of q. Moreover, $f_n^{\pm}(\theta)$ is analytic for θ in the strip $\mathrm{Im}(\theta) < 0$, $-\pi \le \mathrm{Re}(\theta) \le \pi$ and continuously extends to the boundary $\mathrm{Im}(\theta) = 0$, $-\pi \le \mathrm{Re}(\theta) \le \pi$. If we require further decay of q we can also control derivatives on the boundary. For compact support we get even an analytic extension to $\mathrm{Im}(\theta) > 0$.

Note: The limit $\mathrm{Im}(\theta) \to 0$ corresponds to the limit $z \to [0, 4]$ which is precisely the continuous spectrum of H.

The case $\mathrm{Im}(\theta) = 0$ corresponds to the continuous spectrum $z \in [0, 4]$. In particular, in this case the differential equation is real-valued and the complex conjugate of a solution will again be a solution. But then $f^{\pm}(\theta)$ and $\overline{f^{\pm}(\theta)} = f^{\pm}(-\theta)$ are four solutions of a second order equation which cannot be linearly independent! This leads to the scattering relations

$$T(\theta) f_m^{\pm}(\theta) = R^{\mp}(\theta) f_m^{\mp}(\theta) + f_m^{\mp}(-\theta), \quad \theta \in [-\pi, \pi],$$

Physically $f_m^{\mp}(-\theta)$ can be interpreted as an incoming (from $\mp\infty$) plane wave which splits into a reflected wave $R^{\mp}(\theta) f_m^{\mp}(\theta)$ and a transmitted wave $T(\theta) f_m^{\pm}(\theta)$. The transmission and reflection coefficients can be expressed in terms of Wronskians:

$$T(\theta) = \frac{2\mathrm{i}\sin\theta}{W(\theta)}, \quad R^{\pm}(\theta) = \pm\frac{W^{\pm}(\theta)}{W(\theta)}.$$

with $W(\theta) = W(f^{+}(\theta), f^{-}(\theta))$ and $W^{\pm}(\theta) = W(f^{\mp}(\theta), f^{\pm}(-\theta))$.

Inserting everything into our abstract formula for $\exp(-\mathrm{i}tH)P_H([0,4])$ we obtain the following formula for its kernel

$$K_{n,k}(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}t\phi_v(\theta)} h_n^+(\theta) h_k^-(\theta) T(\theta) d\theta, \quad k \le n,$$

where $\phi_v(\theta) = 2 - 2\cos\theta + v\theta$, $v = \frac{|n-k|}{t}$, is the phase function we already know from the free case and $h^\pm$ capture the difference between the free and perturbed solutions:

$$f_n^\pm(\theta) = \mathrm{e}^{\mp\mathrm{i}n\theta} h_n^\pm(\theta).$$

If $q = 0$ we have $h_n^\pm(\theta) = 1$ as well as $T(\theta) = 1$ and we recover our formula for the free case. Note that here we have added the projector $P_H([0,4])$ onto the continuous spectrum $[0,4]$ and consequently ignored the contribution from any possible eigenvalues (which are stationary solutions and hence cannot exhibit decay).

However, in the perturbed case our oscillatory integral has a nontrivial amplitude and we cannot apply the van der Corput lemma directly. The key observation is that the amplitude is in the Wiener algebra, that is, the set of all continuous functions

$$A(\theta) = \sum_{n\in\mathbb{Z}} \hat{A}_n \mathrm{e}^{\mathrm{i}\theta n}$$

whose Fourier coefficients

$$\hat{A}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} A(\theta)\mathrm{e}^{-\mathrm{i}\theta n} d\theta$$

are summable.

This becomes a Banach space upon defining the norm to be

$$\|\hat{A}\|_{\ell_1} = \sum_{n\in\mathbb{Z}} |\hat{A}_n|.$$

Moreover, we even have a Banach algebra together with the usual pointwise product of functions. Recall that this corresponds to convolution of the Fourier coefficients.

Theorem (Wiener)

Suppose A is in the Wiener algebra and does not vanish. Then A^{-1} is in the Wiener algebra as well.

The Wiener algebra

For $h_n^{\pm}(\theta)$ summability of the Fourier coefficients is immediate from our estimate

$$|B_{n,m}^{\pm}| \le C_n^{\pm} \sum_{k=n+\lfloor m/2 \rfloor}^{\pm\infty} |q_k|.$$

This also shows that $W(\theta)$ is in the Wiener algebra and so will be $W(\theta)^{-1}$ by Wiener's theorem provided $W(\theta)$ does not vanish on $[-\pi,\pi]$. In fact energy conservation $|T|^2 + |R_\pm|^2 = 1$ shows that this can only happen on the boundary of the spectrum when $\sin(\theta) = 0$. In this case, when $W(\theta)$ vanishes at one of the boundaries, one speaks of a resonance. Moreover, varying just one value of q will immediately destroy a resonance and hence this situation is not generic. Nevertheless, one can show:

Theorem

Suppose $q \in \ell^1_1(\mathbb{Z})$. The transmission T as well the reflection $R_\pm$ coefficients are in the Wiener algebra.

Lemma

Consider the oscillatory integral

$$I(t) = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}t\phi(\theta)} A(\theta) d\theta,$$

where $\phi(\theta)$ is real-valued. If $|\phi^{(l)}(\theta)| \geq 1$ for some $l \geq 2$ and A is in the Wiener algebra, then

$$|I(t)| \leq \frac{c_l \|\hat{A}\|_{\ell^1}}{t^{1/l}},$$

where c_l is the same constant as in the van der Corput lemma.

Proof.

We insert $A(\theta) = \sum_{n\in\mathbb{Z}} \hat{A}_n \mathrm{e}^{\mathrm{i}n\theta}$ and use Fubini to obtain

$$I(t) = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}t\phi(\theta)} \sum_{n\in\mathbb{Z}} \hat{A}_n \mathrm{e}^{\mathrm{i}\theta n} d\theta = \sum_{p\in\mathbb{Z}} \hat{f}_n I_{n/t}(t), \quad I_v(t) = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}t(\phi(\theta)+v\theta)} d\theta.$$

By the van der Corput lemma we have $|I_v(t)| \le c_l t^{-1/l}$ and hence

$$|I(t)| \le \sum_{n\in\mathbb{Z}} |\hat{f}_n| |I_{n/t}(t)| \le \frac{c_l \|\hat{A}\|_{\ell^1}}{t^{1/l}}$$

as claimed. □

A few remarks about this lemma:

- It cannot hold for $l = 1$ as the Fourier coefficients of an element in the Wiener algebra can have arbitrary slow decay (consider lacunary Fourier coefficients).
- One can show that it does not hold for continuous functions.
- This lemma is typically found for absolutely continuous functions, that is,

$$A(\theta) = A(-\pi) + \int_{-\pi}^{\theta} A'(t)dt, \qquad A' \in L^1[-\pi, \pi].$$

Neither version implies the other.

As a consequence we obtain:

Theorem

Let $q \in \ell^1_1$. Then the following dispersive decay estimates hold

$$\|\mathrm{e}^{-\mathrm{i}tH} P_c(H)\|_{\ell^1 \to \ell^\infty} = \mathcal{O}(t^{-1/3}), \quad t \to \infty,$$

and

$$\|\mathrm{e}^{-\mathrm{i}tH} P_c(H)\|_{\ell^2_\sigma \to \ell^2_{-\sigma}} = \mathcal{O}(t^{-1/2}), \quad t \to \infty, \quad \sigma > 1/2,$$

where $P_c(H) = P_H([0,4])$ is the projection onto the continuous spectrum.

Note: The proof requires in fact one more trick! We need the Wiener norm of $h_n^\pm$ to be uniformly bounded with respect to $n \in \mathbb{Z}$. However, this is only true for $n \in \mathbb{Z}_\pm$. To get it for our amplitude one has to use the scattering relations to turn $h_n^\pm(\theta)$ into linear combinations of $h_n^\mp(\theta)$ and $h_n^\mp(-\theta)$.

Fix some (σ-finite) measure space (X,μ) and abbreviate $L^p = L^p(X, d\mu)$ for notational simplicity. If $f \in L^{p_0} \cap L^{p_1}$ for some $p_0 < p_1$ then applying generalized Hölder's inequality to $f = f^{1-\theta} f^{\theta}$ it is not hard to see that $f \in L^p$ for every $p \in [p_0, p_1]$ and we have the **Lyapunov inequality**

$$\|f\|_p \le \|f\|_{p_0}^{1-\theta} \|f\|_{p_1}^{\theta},$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\theta \in (0,1)$.

Denote by $L^{p_0} + L^{p_1}$ the space of (equivalence classes) of measurable functions f which can be written as a sum $f = f_0 + f_1$ with $f_0 \in L^{p_0}$ and $f_1 \in L^{p_1}$ (clearly such a decomposition is not unique and different decompositions will differ by elements from $L^{p_0} \cap L^{p_1}$). Then we have

$$L^p \subseteq L^{p_0} + L^{p_1}, \qquad p_0 < p < p_1.$$

So if we have an operator $L^{p_0} \cap L^{p_1} \to L^{q_0} \cap L^{q_1}$ which is bounded both from $L^{p_0} \to L^{q_0}$ and $L^{p_1} \to L^{q_1}$ there is a unique extension to $L^{p_0} + L^{p_1} \to L^{q_0} + L^{q_1}$. Consequently we also get a restriction to $L^p \to L^q$ which is again continuous according to the Riesz–Thorin interpolation theorem.

Theorem (Riesz–Thorin)

Let $(X, d\mu)$ *and* $(Y, d\nu)$ *be* σ*-finite measure spaces and* $1 \le p_0, p_1, q_0, q_1 \le \infty$. *If* A *is a linear operator on*

$$A : L^{p_0}(X, d\mu) + L^{p_1}(X, d\mu) \to L^{q_0}(Y, d\nu) + L^{q_1}(Y, d\nu)$$

satisfying

$$\|Af\|_{q_0} \le M_0 \|f\|_{p_0}, \qquad \|Af\|_{q_1} \le M_1 \|f\|_{p_1},$$

then A *has continuous restrictions*

$$A_\theta : L^{p_\theta}(X, d\mu) \to L^{q_\theta}(Y, d\nu), \qquad \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \ \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

satisfying $\|A_\theta\| \le M_0^{1-\theta} M_1^\theta$ *for every* $\theta \in (0,1)$.

By self-adjointness of H we obtain that $\exp(-\mathrm{i}tH) : \ell^2 \to \ell^2$ is unitary and in particular

$$\|\mathrm{e}^{-\mathrm{i}tH} P_c(H)\|_{\ell^2 \to \ell^2} \le 1.$$

Interpolating between this and our $\ell^1 \to \ell^\infty$ estimate the Riesz–Thorin theorem gives us the following estimate

$$\|\mathrm{e}^{-\mathrm{i}tH} P_c(H)\|_{\ell^{p'} \to \ell^p} = \mathcal{O}(t^{-1/3(1/p'-1/p)})$$

for any $p' \in [1, 2]$ with $\frac{1}{p} + \frac{1}{p'} = 1$.

Next we look at average decay in an L^p sense in stead of pointwise estimates with respect to t. To this end we introduce the following space-time norms

$$\|F\|_{L^q_t \ell^p_n} = \left(\int \|F(t)\|^q_{\ell^p_n} dt \right)^{1/q}.$$

Here we regard $F : \mathbb{R} \to \ell^p$ and the subscripts t and n are supposed to remind you that the norm is to be taken with respect to the time and spatial variable, respectively. In our case time is continuous and the spatial variable is discrete, so there is no danger of confusion (however, in the case where the spatial variable is also continuous there of course is). Then our previous estimates imply:

Theorem

Suppose $q \in \ell^1_1$. Then we have the following estimates:

$$\|\mathrm{e}^{-\mathrm{i}tH} P_c(H) f\|_{L^r_t \ell^p_n} \le C \|f\|_{\ell^2_n},$$

$$\|\int \mathrm{e}^{-\mathrm{i}tH} P_c(H) F(s) ds\|_{\ell^2_n} \le C \|F\|_{L^{r'}_t \ell^{p'}_n},$$

$$\|\int \mathrm{e}^{-\mathrm{i}(t-s)H} P_c(H) F(s) ds\|_{L^r_t \ell^p_n} \le C \|F\|_{L^{r'}_t \ell^{p'}_n},$$

where $p, r \ge 2$,

$$\frac{1}{r} + \frac{1}{3p} \le \frac{1}{6},$$

and a prime denotes the corresponding dual index.

Consider $L^p(X, d\mu)$ with if $1 \le p \le \infty$ and let p' be the corresponding dual index, $\frac{1}{p} + \frac{1}{p'} = 1$. Then for $f \in L^p(X, d\mu)$ and $g \in L^{p'}(X, d\mu)$ we have Hölder's inequality

$$\|f\, g\|_1 \le \|f\|_p \|g\|_{p'}$$

and its "converse" (variational characterization of the norm)

$$\|f\|_p = \sup_{\|g\|_{p'}=1} \left| \int_X fg\, d\mu \right|.$$

Suppose, μ and ν are two σ-finite measures and f is $\mu \otimes \nu$ measurable. Let $1 \le p \le \infty$. Then we have Minkovski's inequality

$$\left\| \int_Y f(.,y) d\nu(y) \right\|_p \le \int_Y \|f(.,y)\|_p d\nu(y),$$

where the p-norm is computed with respect to μ.

Theorem (Hardy–Littlewood–Sobolev inequality)

Let $0 < \alpha < 1$, $p \in (1, \frac{1}{\alpha})$, and $q = \frac{p}{1-p\alpha} \in (\frac{1}{1-\alpha}, \infty)$ (i.e, $\frac{\alpha}{1} = \frac{1}{p} - \frac{1}{q}$). Then the Riesz potential of order α,

$$(\mathcal{I}_\alpha f)(t) = \frac{\Gamma(\frac{1-\alpha}{2})}{2^\alpha \pi^{1/2} \Gamma(\frac{\alpha}{2})} \int_{\mathbb{R}} \frac{1}{|t-s|^{1-\alpha}} f(s) ds,$$

satisfies

$$\|\mathcal{I}_\alpha f\|_q \le C_{p,\alpha} \|f\|_p.$$

Proof.

To begin with we use the following variational characterization of our space-time norms:

$$\|F\|_{L^r_t \ell^p_n} = \sup_{\|G\|_{L^{r'}_t \ell^{p'}_n}=1} \left| \int \sum_n F(n,t) G(n,t) dt \right|$$

Then, using self-adjointness of H and Fubini

$$\int \sum_n (\exp(-\mathrm{i}tH) P_c f)_n F(n,t) dt = \sum_n f_n \int (\exp(-\mathrm{i}tH) P_c F(t))_n dt,$$

which shows that the first and second estimate are equivalent upon using the above characterization.

Proof (Cont.).

Similarly, using again self-adjointness

$$
\begin{aligned}
&\left\| \int \exp(-\mathrm{i}tH) P_c F(.,t) dt \right\|_{\ell^2_n}^2 \\
&\quad = \sum_n \int \exp(-\mathrm{i}tH) P_c F(n,t) dt \int \overline{\exp(-\mathrm{i}sH) P_c F(n,s)} ds \\
&\quad = \sum_n \int F(n,t) \int \exp(-\mathrm{i}(t-s)H) P_c \overline{F(n,s)} ds\, dt,
\end{aligned}
$$

which shows that the second and the third estimate are equivalent with a similar argument as before.

Proof (Cont.).

Hence it remains to prove the last one. Applying Minkowski's inequality and our interpolation estimate we obtain

$$\begin{aligned}\| \int \mathrm{e}^{-\mathrm{i}(t-s)H} P_c(H) F(n,s) ds \|_{\ell^p_n} &\leq \int \| \mathrm{e}^{-\mathrm{i}(t-s)H} P_c(H) F(n,s) \|_{\ell^p_n} ds \\ &\leq C \int \frac{1}{|t-s|^{\alpha}} \| F(n,s) \|_{\ell^{p'}_n} ds,\end{aligned}$$

where $\alpha = (1/3)(1/p' - 1/p)$. Now taking the $\|.\|_{L^r_t}$ norm on both sides and using the Hardy–Littlewood–Sobolev inequality finishes the proof. □

Note: There is also an abstract result by

- M. Keel and T. Tao, *Endpoint Strichartz estimates*, Amer. J. Math. **120** (1998), 955–980.

which applies here. Note that one can apply the *truncated version* since our kernels are bounded near $t = 0$.

An important physical model is the discrete nonlinear Schrödinger (dNLS) equation

$$\mathrm{i}\dot{u}(t) = Hu(t) \pm |u(t)|^{2p} u(t), \quad t \in \mathbb{R},$$

which arises in many physical applications.

The above estimates are typically used to show solvability of this equation. However, in the discrete case the situation is simpler. To this end we recall some basic techniques from analysis in Banach spaces.

Let X, Y be Banach spaces with X real and $U \subseteq X$ open.
Then a function $F: U \to Y$ is called differentiable at $x \in U$ if there exists a bounded linear function $dF(x) \in \mathscr{L}(X,Y)$ such that

$$F(x+u) = F(x) + dF(x)\,u + o(u),$$

where o, O are the **Landau symbols**. Explicitly

$$\lim_{u\to 0} \frac{|F(x+u) - F(x) - dF(x)\,u|}{|u|} = 0.$$

The linear map $dF(x)$ is called the Fréchet derivative of F at x. If $dF \in C(U, \mathscr{L}(X,Y))$ we write $F \in C^1(U,Y)$ as usual.
Notes: The derivative (if it exists) is unique. If we take a complex Banach space for X we get a version of complex differentiability.

Differentiability implies existence of directional derivatives

$$\delta F(x,u) := \lim_{\varepsilon\to 0} \frac{F(x+\varepsilon u) - F(x)}{\varepsilon}, \qquad \varepsilon \in \mathbb{R}\setminus\{0\},$$

which are also known as Gâteaux derivative. The converse is not true!
We will only look at Fréchet derivatives and hence drop "Fréchet".

Some easy observations (proof as in the finite dimensional case):

- The derivative is linear.
- F differentiable at x implies F continuous at x.
- Let $Y := \bigtimes_{j=1}^{m} Y_j$ and let $F : X \to Y$ be given by $F = (F_1, \dots, F_m)$ with $F_j : X \to Y_j$. Then $F \in C^1(X, Y)$ if and only if $F_j \in C^1(X, Y_j)$, $1 \le j \le m$, and in this case $dF = (dF_1, \dots, dF_m)$.
- If $X = \bigtimes_{i=1}^{n} X_i$, then one can define the partial derivative $\partial_i F \in \mathscr{L}(X_i, Y)$, which is the derivative of F considered as a function of the i-th variable alone (the other variables being fixed). We have $dF\, u = \sum_{i=1}^{n} \partial_i F\, u_i$, $u = (u_1, \dots, u_n) \in X$, and $F \in C^1(X, Y)$ if and only if all partial derivatives exist and are continuous.
- Chain rule
- Mean value theorem: $|F(x) - F(y)| \le M|x - y|$ with $M := \sup_{0 \le t \le 1} \|dF((1-t)x + ty)\|$.

Suppose $f \in C^1(\mathbb{R})$ with $f(0) = 0$. Let $X := \ell^p(\mathbb{N})$, then

$$F : X \to X, \qquad (x_n)_{n\in\mathbb{N}} \mapsto (f(x_n))_{n\in\mathbb{N}}$$

is differentiable for every $x \in X$ with derivative given by the multiplication operator

$$(dF(x)u)_n = f'(x_n)u_n.$$

First of all note that the mean value theorem implies $|f(t)| \le M_R|t|$ for $|t| \le R$ with $M_R := \sup_{|t|\le R} |f'(t)|$. Hence, since $\|x\|_\infty \le \|x\|_p$, we have $\|F(x)\|_p \le M_{\|x\|_\infty} \|x\|_p$ and F is well defined. This also shows that multiplication by $f'(x_n)$ is a bounded linear map.

Example 1 (good guy)

To establish differentiability we use

$$f(t+s) - f(t) - f'(t)s = s \int_0^1 \big(f'(t+s\tau) - f'(t)\big) d\tau$$

and since f' is uniformly continuous on every compact interval, we can find a $\delta > 0$ for every given $R > 0$ and $\varepsilon > 0$ such that

$$|f'(t+s) - f'(t)| < \varepsilon \quad \text{if} \quad |s| < \delta,\ |t| < R.$$

Now for $x, u \in X$ with $\|x\|_\infty < R$ and $\|u\|_\infty < \delta$ we have $|f(x_n + u_n) - f(x_n) - f'(x_n)u_n| < \varepsilon |u_n|$ and hence

$$\|F(x+u) - F(x) - dF(x)u\|_p < \varepsilon \|u\|_p$$

which establishes differentiability. Moreover, using uniform continuity of f on compact sets a similar argument shows that dF is continuous (observe that the operator norm of a multiplication operator by a sequence is the sup norm of the sequence) and hence we even have $F \in C^1(X, X)$.

Example 2 (bad guy)

Let $X := L^2(0,1)$ and consider

$$F : X \to X, \qquad x \mapsto \sin(x).$$

First of all note that by $|\sin(t)| \leq |t|$ our map is indeed from X to X and since sine is Lipschitz continuous we get the same for F: $\|F(x) - F(y)\|_2 \leq \|x - y\|_2$. Moreover, F is Gâteaux differentiable at $x = 0$ with derivative given by

$$\delta F(0) = \mathbb{I}$$

but it is not differentiable at $x = 0$.

To see that the Gâteaux derivative is the identity note that

$$\lim_{\varepsilon\to 0} \frac{\sin(\varepsilon u(t))}{\varepsilon} = u(t)$$

pointwise and hence

$$\lim_{\varepsilon\to 0} \left\| \frac{\sin(\varepsilon u(.))}{\varepsilon} - u(.) \right\|_2 = 0$$

by dominated convergence since $|\frac{\sin(\varepsilon u(t))}{\varepsilon}| \le |u(t)|$.
To see that F is not differentiable let

$$u_n = \pi\chi_{[0,1/n]}, \qquad \|u_n\|_2 = \frac{\pi}{\sqrt{n}}$$

and observe that $F(u_n) = 0$, implying that

$$\frac{\|F(u_n) - u_n\|_2}{\|u_n\|_2} = 1$$

does not converge to 0.

Given $F \in C^1(U, Y)$ we have $dF \in C(U, \mathscr{L}(X, Y))$ and we can define the second derivative (provided it exists) via

$$dF(x + v) = dF(x) + d^2F(x)v + o(v).$$

In this case $d^2F : U \to \mathscr{L}(X, \mathscr{L}(X, Y))$ which maps x to the linear map $v \mapsto d^2F(x)v$ which for fixed v is a linear map $u \mapsto (d^2F(x)v)u$.
Equivalently, we could regard $d^2F(x)$ as a map $d^2F(x) : X^2 \to Y$, $(u, v) \mapsto (d^2F(x)v)u$ which is linear in both arguments. That is, $d^2F(x)$ is a bilinear map $X^2 \to Y$.
Continuing like this one can define derivatives of any order.

Example 1 (good guy) cont.

Suppose $f \in C^2(\mathbb{R})$ with $f(0) = 0$. Let $X := \ell^p(\mathbb{N})$, then we have $F \in C^2(X, X)$ with $d^2 F(x)v$ the multiplication operator by the sequence $f''(x_n)v_n$, that is,

$$(d^2 F(x)(u, v))_n = f''(x_n) v_n u_n.$$

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Theorem

Let X, Y, and Z be Banach spaces and let U, V be open subsets of X, Y, respectively. Let $F \in C^r(U \times V, Z)$, $r \geq 0$, and fix $(x_0, y_0) \in U \times V$. Suppose $\partial_x F \in C(U \times V, Z)$ exists (if $r = 0$) and $\partial_x F(x_0, y_0) \in \mathscr{L}(X, Z)$ is an isomorphism. Then there exists an open neighborhood $U_1 \times V_1 \subseteq U \times V$ of (x_0, y_0) such that for each $y \in V_1$ there exists a unique point $(\xi(y), y) \in U_1 \times V_1$ satisfying $F(\xi(y), y) = F(x_0, y_0)$. Moreover, ξ is in $C^r(V_1, Z)$ and fulfills (for $r \geq 1$)

$$d\xi(y) = -(\partial_x F(\xi(y), y))^{-1} \circ \partial_y F(\xi(y), y).$$

Note: Proof as in the finite dimensional case.

Theorem

Let I be an open interval, U an open subset of a Banach space X and Λ an open subset of another Banach space. Suppose $F \in C^r(I \times U \times \Lambda, X)$, $r \geq 1$, then the initial value problem

$$\dot{x} = F(t, x, \lambda), \qquad x(t_0) = x_0, \quad (t_0, x_0, \lambda) \in I \times U \times \Lambda, \tag{1}$$

has a unique solution $x(t, t_0, x_0, \lambda) \in C^r(I_1 \times I_2 \times U_1 \times \Lambda_1, X)$, where $I_{1,2}$, U_1, and Λ_1 are open subsets of I, U, and Λ, respectively. The sets I_2, U_1, and Λ_1 can be chosen to contain any point $t_0 \in I$, $x_0 \in U$, and $\lambda_0 \in \Lambda$, respectively.

ODE (global existence)

Lemma

Suppose $F \in C^1(\mathbb{R} \times X, X)$ *and let* $x(t)$ *be a maximal solution of the initial value problem* (1). *Suppose* $|F(t, x(t))|$ *is bounded on finite* t*-intervals. Then* $x(t)$ *is a global solution.*

Theorem

The Cauchy problem for the discrete nonlinear Schrödinger equation

$$\mathrm{i}\dot{u}(t) = Hu(t) \pm |u(t)|^{2p} u(t), \quad t \in \mathbb{R},$$

has a unique global norm preserving solution $u \in C^1(\mathbb{R}, \ell^2(\mathbb{Z}))$. *Moreover, the Cauchy problem is well-posed in the sense that the solution depends continuously on the initial condition.*

Proof.

First of all we can regard the right-hand side as a C^1 vector field in the Hilbert space $\ell^2(\mathbb{Z})$. Hence we get existence of a unique local solution. To show that this solution is in fact global observe

$$\frac{d}{dt}\|u(t)\|_{\ell^2}^2 = 0.$$

Hence the ℓ^2 norm is preserved and cannot blow up. $\square$

Note that in the above argument ℓ^2 can be replaced by weighted versions (as long as the weight is at most exponential) but the corresponding norms will no longer be preserved. However, using

$$\frac{d}{dt}|u_n(t)|^2 = 2\mathrm{Im}\big(u_n\overline{(u_{n+1}+u_{n-1})}\big)$$

plus Gronwall's inequality these norms can grow at most linearly.

The stationary dNLS equation

Our next aim is to establish existence of a stationary solution of the form

$$u_n(t) = \mathrm{e}^{-\mathrm{i}t\omega}\phi_n(\omega)$$

Plugging this ansatz into the dNLS we get the stationary dNLS

$$H\phi_n - \omega\phi_n \pm |\phi_n|^{2p}\phi_n = 0.$$

Of course we always have the trivial solution $\phi_n = 0$, but it is not clear if there are any nontrivial solutions.

Idea: Regard this equation as an equation

$$F(\omega, \phi) = (H - \omega)\phi_n \pm |\phi_n|^{2p}\phi_n = 0, \quad p > 0,$$

in the Hilbert space $\mathbb{R} \oplus \ell^2(\mathbb{Z})$. We have the trivial solution $F(\omega, 0) = 0$ and, by the implicit function theorem, the zero set can change locally only at points $\omega_0 \in \mathbb{R}$, where

$$\partial_\phi F(\omega_0, 0) = H - \omega_0$$

is not invertible. Hence a nontrival solution can only branch off from an eigenvalue ω_0 of the self-adjoint operator H.
Of course this is only a necessary condition and to see that a notrivial solution indeed branches off from an eigenvalue one has to work harder.

Suppose we have an abstract problem $F(\omega, x) = 0$ with $\omega \in \mathbb{R}$ and $x \in X$ some Banach space. We assume that $F \in C^1(\mathbb{R} \times X, X)$ and that there is a trivial solution $x = 0$, that is, $F(\omega, 0) = 0$.
The first step is to split off the trivial solution and reduce it effectively to a finite-dimensional problem. To this end we assume that we have found a point $\omega_0 \in \mathbb{R}$ such that the derivative $A := \partial_x F(\omega_0, 0)$ is not invertible.
Moreover, we will assume that A is a Fredholm operator:

- $\dim \operatorname{Ker}(A) < \infty$.
- $\dim \operatorname{Coker}(A) < \infty$.
- $\operatorname{Ran}(A)$ is closed.

Then there exists continuous projections

$$P : X = \operatorname{Ker}(A) \dot{+} X_0 \to \operatorname{Ker}(A), \qquad Q : X = X_1 \dot{+} \operatorname{Ran}(A) \to X_1$$

Note: The assumption that above dimensions are finite is crucial for the above projections to be continuous!

Important things to observe:

- $\dim \mathrm{Ker}(A) < \infty$.
- $\dim X_1 = \dim \mathrm{Coker}(A) < \infty$.
- With respect to the above splitting of the space A has a block structure
$$A = \begin{pmatrix} 0 & 0 \\ 0 & A_0 \end{pmatrix},$$
where A_0 is the restriction of A to $X_0 \to \mathrm{Ran}(A)$. Then A_0 is an isomorphism by the open mapping theorem! Note: The assumption that $\mathrm{Ran}(A)$ is closed is crucial here.

The Lyapunov–Schmidt reduction

Now split our equation into a system of two equations according to the above splitting of the underlying Banach space:

$$F(\omega, x) = 0 \qquad \Leftrightarrow \quad F_1(\omega, u, v) = 0,\ F_2(\omega, u, v) = 0,$$

where $x = u + v$ with $u = Px \in \mathrm{Ker}(A)$, $v = (1 - P)x \in X_0$ and $F_1(\omega, u, v) = Q\,F(\omega, u + v)$, $F_2(\omega, u, v) = (1 - Q)F(\omega, u + v)$.
Observations:

- Since P, Q are bounded this system is still C^1.
- The derivatives are given by (recall the block structure of A)

$$\partial_u F_1(\omega_0, 0, 0) = 0, \quad \partial_v F_1(\omega_0, 0, 0) = 0,$$

$$\partial_u F_2(\omega_0, 0, 0) = 0, \quad \partial_v F_2(\omega_0, 0, 0) = A_0$$

In particular, since A_0 is an isomorphism, the implicit function theorem tells us that we can (locally) solve F_2 for v. That is, there exists a neighborhood U of $(\omega_0, 0)$ and a unique function $\psi \in C^1(U, X_0)$ such that

$$F_2(\omega, u, \psi(\omega, u)) = 0, \qquad (\omega, u) \in U.$$

In particular, by the uniqueness part we have $\psi(\omega, 0) = 0$. Moreover, $\partial_u \psi(\omega_0, 0) = A_0^{-1} \partial_u F_2(\omega_0, 0, 0) = 0$.
Plugging this into the first equation reduces to original system to the finite dimensional system

$$\tilde{F}_1(\omega, u) = F_1(\omega, u, \psi(\omega, u)) = 0.$$

Of course the chain rule tells us that $\tilde{F} \in C^1$. Moreover, we still have $\tilde{F}_1(\omega, 0) = F_1(\omega, 0, \psi(\omega, 0)) = QF(\omega, 0) = 0$ as well as

$$\partial_u \tilde{F}_1(\omega_0, 0) = \partial_u F_1(\omega_0, 0, 0) + \partial_v F_1(\omega_0, 0, 0) \partial_u \psi(\omega_0, 0) = 0.$$

Now we reduced the problem to a finite-dimensional system, it remains to find conditions such that the finite dimensional system has a nontrivial solution. Our first assumption will be the requirement

$$\dim \operatorname{Ker}(A) = \dim \operatorname{Coker}(A) = 1$$

such that we actually have a problem in $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$. (In terms of our original problem this means that the eigenvalue of H is simple. Moreover, since H is self-adjoint the remaining assumptions for $H - \omega_0$ to be Fredholm of index zero will come for free provided ω_0 is an isolated eigenvalue.)

Explicitly, let u_0 span $\operatorname{Ker}(A)$ and let u_1 span X_1. Then we can write

$$\tilde{F}_1(\omega, \lambda u_0) = f(\omega, \lambda) u_1,$$

where $f \in C^1(V, \mathbb{R})$ with $V = \{(\omega, \lambda) | (\omega, \lambda u_0) \in U\} \subseteq \mathbb{R}^2$ a neighborhood of $(\omega_0, 0)$. Of course we still have $f(\omega, 0) = 0$ for $(\omega, 0) \in V$ as well as

$$\partial_\lambda f(\omega_0, 0) u_1 = \partial_u \tilde{F}_1(\omega_0, 0) u_0 = 0.$$

It remains to investigate f. To split off the trivial solution it suggests itself to write

$$f(\omega, \lambda) = \lambda\, g(\omega, \lambda)$$

We already have

$$g(\omega_0, 0) = \partial_\lambda f(\omega_0, 0) = 0$$

and in order to be able to solve g for ω using the implicit function theorem we clearly need

$$0 \neq \partial_\omega g(\omega_0, 0) = \partial_\omega \partial_\lambda f(\omega_0, 0).$$

Of course this last condition is a bit problematic since up to this point we only have $f \in C^1$. However, if we change our original assumption to $F \in C^2$ we get $f \in C^2$.

So all we need to do is to trace back our definitions and compute

$$\begin{aligned}\partial_\lambda^2 f(\omega_0,0)u_1 &= \partial_\lambda^2 \tilde{F}_1(\omega_0,\lambda u_0)\Big|_{\lambda=0} = \partial_\lambda^2 F_1(\omega_0,\lambda u_0,\psi(\omega_0,\lambda u_0))\Big|_{\lambda=0}\\ &= \partial_u^2 F_1(\omega_0,0,0)(u_0,u_0) = Q\partial_x^2 F(\omega_0,0)(u_0,u_0)\end{aligned}$$

and

$$\begin{aligned}\partial_\omega\partial_\lambda f(\omega_0,0)u_1 &= \partial_\omega\partial_\lambda \tilde{F}_1(\omega_0,\lambda u_0)\Big|_{\lambda=0} = \partial_\omega\partial_\lambda F_1(\omega_0,\lambda u_0,\psi(\omega_0,\lambda u_0))\Big|_{\lambda=0}\\ &= Q\partial_\omega\partial_x F(\omega_0,0)u_0.\end{aligned}$$

Theorem

Let $F \in C^2(\mathbb{R} \times X, X)$ *with* $F(\omega, x) = 0$ *for all* $\omega \in \mathbb{R}$*. Suppose that for some* $\omega_0 \in \mathbb{R}$ *we have that* $\partial_x F(\omega_0, 0)$ *is a Fredholm operator of index zero with a one-dimensional kernel spanned by* $u_0 \in X$*. Then, if*

$$\partial_\omega \partial_x F(\omega_0, 0) u_0 \notin \mathrm{Ran}(\partial_x F(\omega_0, 0))$$

there are open neighborhoods $I \subseteq \mathbb{R}$ *of* 0*,* $J \subseteq \mathbb{R}$ *of* ω_0*, and* $U \subseteq X$ *of* 0 *plus corresponding functions* $\omega \in C^1(I, J)$ *and* $\psi(J \times U, \mathrm{Ran}(\partial_x F(\omega_0, 0)))$ *such that the only nontrivial solution of* $F(\omega(\lambda), x) = 0$ *in a neighborhood of* 0 *is given by*

$$x(\lambda) = \lambda u_0 + \psi(\omega(\lambda), \lambda u_0).$$

Moreover,

$$\omega(\lambda) = \omega_0 - \frac{\partial_x^2 F(\omega_0, 0)(u_0, u_0)}{2\partial_\omega \partial_x F(\omega_0, 0) u_0} \lambda + o(\lambda), \quad x(\lambda) = \lambda u_0 + o(\lambda).$$

Applying this to the stationary dNLS equation

$$F(\omega,\phi) = (H-\omega)\phi_n \pm |\phi_n|^{2p}\phi_n, \qquad p > \frac{1}{2},$$

we have

$$\partial_\phi F(\omega,\phi) = H-\omega$$

and hence ω must be an eigenvalue of H. In fact, if ω_0 is a discrete eigenvalue, then self-adjointness implies that $H-\omega_0$ is Fredholm of index zero. Moreover, since the Wronskian of two square summable solutions must vanish, there can be at most one square summable solution, that is, eigenvalues are always simple for our discrete Schrödinger operator H. Finally, if u_0 is the eigenfunction corresponding to ω_0 we have

$$\partial_\omega\partial_\phi F(\omega_0,\phi)u_0 = -u_0 \notin \operatorname{Ran}(H-\omega_0) = \operatorname{Ker}(H-\omega_0)^\perp$$

and the Crandall–Rabinowitz theorem ensures existence of a stationary solution ϕ for ω in a neighborhood of ω_0.

Moreover, on can show that these solutions $\mathrm{e}^{-\mathrm{i}t\omega}\phi_n(\omega)$ are asymptotically stable. Idea: Make the ansatz

$$u(t) = \mathrm{e}^{-\mathrm{i}\theta(t)}\big(\phi(\omega(t)) + z(t)\big).$$

Then the linearized system for $v = \mathrm{Re}(z)$ and $w = \mathrm{Im}(z)$ can be shown to have a double eigenvalue and the solution can be decomposed with respect to the generalized eigenvectors. Then, using the Strichartz and some additional estimates following from them, one can show that the resulting system has a global solution provided the initial conditions are close to the stationary solution. Moreover, $\|z(t)\|_\infty \to 0$ and the parameters $\theta(t)$ and $\omega(t)$ are asymptotically linear, constant, respectively. The details are however quite tedious.

- P. G. Kevrekidis, D. E. Pelinovsky, and A. Stefanov, *Asymptotic stability of small bound states in the discrete nonlinear Schrödinger equation* SIAM J. Math. Anal. **41** (2009), 2010–2030.