Applications of Spectral Shift Functions. II: Index Theory and Non-Fredholm Operators

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Original Motivation to Study the Witten Index:

In the late 1970's to the mid 1980's, a number of papers on supersymmetric (SUSY) quantum mechanics computing the so-called **Witten index** appeared in the physics literature.

They showed two remarkable facts:

(1) In the context of supersymmetric scattering theoretic situations in one dimension, the **Witten index** was directly related to the **scattering phase shift**.

(2) The computed **Witten index** exhibited a certain **"topological invariance"** (i.e., it was invariant w.r.t. small deformations of the potential coefficients, etc.).

Our motivation in the late 1980's was:

(*i*) Understand and rigorously prove all this!

(*ii*) Since scattering phase shifts are special cases of appropriate **Lifshits-Krein spectral shift functions** ξ, establish the connection between the **Witten index** and ξ.

(*iii*) Prove the topological invariance of the **Witten index** in general.

Original Motivation to Study the Witten Index (contd.):

This story began in the late 1980's. In recent years we revisited this circle of ideas and applied it to **Model Operators of Dirac-type**.

In particular, we applied this to **massless Dirac-type** operators. The latter play a role in **graphene**, an object related to **Buckminsterfullerenes (buckyballs)**, leading to the Nobel Prize in Chemistry for **R. F. Curl, Sir H. W. Kroto**, and **R. E. Smalley** in 1996.

According to Wikipedia: A fullerene is a molecule of carbon in the form of a hollow sphere, ellipsoid, tube, and many other shapes. Spherical fullerenes, also referred to as Buckminsterfullerenes (buckyballs), resemble the balls used in football (soccer). The first fullerene molecule to be discovered, and the family's namesake, buckminsterfullerene (C60), was manufactured in 1985 by **Richard Smalley, Robert Curl, James Heath, Sean O'Brien, and Harold Kroto** at Rice University, "C60: Buckminsterfullerene", Nature, **318** (6042), 162–163 (1985).

A Bit of Notation:

- $\mathcal{H}$ denotes a (separable, complex) Hilbert space, $I_{\mathcal{H}}$ represents the identity operator in $\mathcal{H}$.
- If A is a closed (typically, self-adjoint) operator in $\mathcal{H}$, then
- $\rho(A) \subseteq \mathbb{C}$ denotes the **resolvent set** of A; $z \in \rho(A) \iff A - z\, I_{\mathcal{H}}$ is a bijection.
- $\sigma(A) = \mathbb{C} \backslash \rho(A)$ denotes the **spectrum** of A.
- $\sigma_p(A)$ denotes the **point spectrum** (i.e., the set of eigenvalues) of A.
- $\sigma_d(A)$ denotes the **discrete spectrum** of A (i.e., isolated eigenvalues of finite (algebraic) multiplicity).
- If A is closable in $\mathcal{H}$, then $\overline{A}$ denotes the **operator closure** of A in $\mathcal{H}$.

Note. All operators will be **linear** in this course.

A Bit of Notation (contd.):

- $\mathcal{B}(\mathcal{H})$ is the set of **bounded** operators defined on $\mathcal{H}$.

  $\mathcal{B}_p(\mathcal{H})$, $1 \leq p \leq \infty$ denotes the pth trace ideal of $\mathcal{B}(\mathcal{H})$,

  (i.e., $T \in \mathcal{B}_p(\mathcal{H}) \Longleftrightarrow \sum_{j\in\mathcal{J}} \lambda_j\big((T^*T)^{1/2}\big)^p < \infty$, where $\mathcal{J} \subseteq \mathbb{N}$ is an appropriate index set, and the eigenvalues $\lambda_j(T)$ of T are repeated according to their algebraic multiplicity),

  $\mathcal{B}_1(\mathcal{H})$ is the set of **trace class** operators,

  $\mathcal{B}_2(\mathcal{H})$ is the set of **Hilbert–Schmidt** operators,

  $\mathcal{B}_\infty(\mathcal{H})$ is the set of **compact** operators.

- $\operatorname{tr}_{\mathcal{H}}(A) = \sum_{j\in\mathcal{J}} \lambda_j(A)$ denotes the **trace** of $A \in \mathcal{B}_1(\mathcal{H})$.

- $\det_{\mathcal{H}}(I_{\mathcal{H}} - A) = \prod_{j\in\mathcal{J}}[1 - \lambda_j(A)]$ denotes the **Fredholm determinant**, defined for $A \in \mathcal{B}_1(\mathcal{H})$.

- $\det_{2,\mathcal{H}}(I_{\mathcal{H}} - B) = \prod_{j\in\mathcal{J}}[1 - \lambda_j(B)]e^{\lambda_j(B)}$ denotes the **modified Fredholm determinant**, defined for $B \in \mathcal{B}_2(\mathcal{H})$.

Basics of Fredholm Index Theory:

A few useful facts:

1. An operator A in $\mathcal{H}$ is called **nonnegative** (denoted by $A \geq 0$) if

$$(f, Af)_{\mathcal{H}} \geq 0 \text{ for all } f \in \text{dom}(A).$$

Similarly, A in $\mathcal{H}$ is called **strictly positive** if there exists $\varepsilon > 0$ such that

$$(f, Af)_{\mathcal{H}} \geq \varepsilon \|f\|_{\mathcal{H}}^2 \text{ for all } f \in \text{dom}(A).$$

This is denoted by $A \geq \varepsilon I_{\mathcal{H}}$.

2. **von Neumann's Theorem:** Suppose T is closed and densely defined in $\mathcal{H}$. Then T^*T (and hence TT^*) is **self-adjoint** and **nonnegative**, $T^*T \geq 0$.

Sketch of E. Nelson's short proof of this fact: Consider the self-adjoint Dirac-type operator,

$$D = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}$$

and just square it to get

$$D^2 = \begin{pmatrix} T^*T & 0 \\ 0 & TT^* \end{pmatrix} \geq 0.$$

Basics of Fredholm Index Theory (contd.):

3. **Compare spectra of T^*T and TT^*:** $\lambda > 0$ is an eigenvalue of T^*T with multiplicity $m(\lambda)$ **if and only if** λ is an eigenvalue of TT^* with the same multiplicity $m(\lambda)$.

In fact, one can even prove that **on the orthogonal complement of their respective kernels (null spaces)**, T^*T and TT^* are **unitarily equivalent** (**Deift, 1978**).

Note. **Nothing** has (and can) be said about $\lambda = 0$. This fact is precisely what lies at the **origin** of **Index Theory!**

Basics of Fredholm Index Theory (contd.):

Fredholm operators:

Definition.

Let T be a closed and densely defined operator in $\mathcal{H}$. Then T is **Fredholm** if and only if $\operatorname{ran}(T)$ is closed in $\mathcal{H}$ and $\dim(\ker(T)) + \dim(\ker(T^*)) < \infty$.

If T is **Fredholm**, its **index** (denoted by $\operatorname{ind}(T)$), is defined as

$$\begin{aligned}\operatorname{ind}(T) &= \dim(\ker(T)) - \dim(\ker(T^*)) \\ &= \dim(\ker(T^*T)) - \dim(\ker(TT^*)).\end{aligned}$$

Facts. Suppose T is a closed and densely defined operator in $\mathcal{H}$. Then,

(i) T is **Fredholm** if and only if T^* is.

(ii) T is **Fredholm** if and only if there exists $\varepsilon > 0$ such that $\inf(\sigma_{ess}(T^*T)) \geq \varepsilon$ **and** $\inf(\sigma_{ess}(TT^*)) \geq \varepsilon$. (Note. The **"and"** is crucial here!)

Basics of Fredholm Index Theory (contd.):

We recall that $\mathcal{B}_\infty(\mathcal{H})$ denotes the Banach space of **compact** operators on $\mathcal{H}$.

Theorem (Invariance w.r.t. Relatively Compact Perturbations).

T **Fredholm**, S relatively compact w.r.t. T (e.g., $S(T - z_0 I_{\mathcal{H}})^{-1} \in \mathcal{B}_\infty(\mathcal{H})$ for some $z_0 \in \rho(T)$), then $T + S$ is **Fredholm** and

$$\operatorname{ind}(T + S) = \operatorname{ind}(T).$$

$\longrightarrow$ **Stability** of the **Fredholm index** w.r.t. additive **relatively compact** perturbations.

Think, **“topological invariance”** ....

Basics of Fredholm Index Theory (contd.):

Another fundamental result, the **additivity** of the **Fredholm Index**: Extend the notion of Fredholm operators to a two-Hilbert space setting, i.e, $T : \mathrm{dom}(T) \to \mathcal{H}_2$, $\mathrm{dom}(T) \subseteq \mathcal{H}_1$, where $\mathcal{H}_j$, $j = 1, 2$, are complex, separable Hilbert spaces as follows: T is densely defined and closed, with $\dim(T) + \dim(T^*) < \infty$.

Define the product of two (unbounded) operators maximally in the usual sense: If T maps from $\mathcal{H}_1$ to $\mathcal{H}_2$ and S from $\mathcal{H}_2$ to $\mathcal{H}_3$, then

$$\mathrm{dom}(ST) = \{f \in \mathrm{dom}(T) \subseteq \mathcal{H}_1 \,|\, Tf \in \mathrm{dom}(S) \subseteq \mathcal{H}_2\},$$
$$STh = S(Th), \; h \in \mathrm{dom}(ST).$$

Theorem (Additivity of the Fredholm Index).

S and T **Fredholm**, such that ST is densely defined, then ST is **Fredholm** and

$$\mathrm{ind}(ST) = \mathrm{ind}(S) + \mathrm{ind}(T).$$

Basics of Fredholm Index Theory (contd.):

A brief **Summary on (Unbounded) Fredholm Operators**: We now take a slightly more general approach and permit a two-Hilbert space setting as follows: Suppose $\mathcal{H}_j$, $j \in \{1,2\}$, are complex, separable Hilbert spaces. Then $T\colon \operatorname{dom}(T) \subseteq \mathcal{H}_1 \to \mathcal{H}_2$, T is called a **Fredholm operator**, denoted by $T \in \Phi(\mathcal{H}_1, \mathcal{H}_2)$, if

(i) T is closed and densely defined in $\mathcal{H}_1$.

(ii) $\mathrm{ran}(T)$ is closed in $\mathcal{H}_2$.

(iii) $\dim(\ker(T)) + \dim(\ker(T^*)) < \infty$.

If T is Fredholm, its **Fredholm index** is given by

$$\operatorname{ind}(T) = \dim(\ker(T)) - \dim(\ker(T^*)).$$

If $T\colon \operatorname{dom}(T) \subseteq \mathcal{H}_1 \to \mathcal{H}_2$ is densely defined and closed, we associate with $\operatorname{dom}(T) \subset \mathcal{H}_1$ the **graph Hilbert subspace** $\mathcal{H}_T \subseteq \mathcal{H}_1$ induced by T defined by

$$\mathcal{H}_T = (\operatorname{dom}(T); (\,\cdot\,,\,\cdot\,)_{\mathcal{H}_T}), \quad (f,g)_{\mathcal{H}_T} = (Tf, Tg)_{\mathcal{H}_2} + (f,g)_{\mathcal{H}_1},$$
$$\|f\|_{\mathcal{H}_T} = \big[\|Tf\|^2_{\mathcal{H}_2} + \|f\|^2_{\mathcal{H}_1}\big]^{1/2}, \; f,g \in \operatorname{dom}(T).$$

Basics of Fredholm Index Theory (contd.):

There is, however, a slightly different and a more general approach based on **codimension**: Suppose $\mathcal{H}_j$, $j \in \{1, 2\}$, are complex, separable Hilbert spaces. Then $T\colon \operatorname{dom}(T) \subseteq \mathcal{H}_1 \to \mathcal{H}_2$, T is called a **Fredholm operator** if

$$(i)\ T \text{ is closed in } \mathcal{H}_1.$$
$$(ii)\ \operatorname{ran}(T) \text{ is closed in } \mathcal{H}_2.$$
$$(iii)\ \dim(\ker(T)) + \operatorname{codim}(T) < \infty.$$

Here,

$$\operatorname{codim}(T) = \dim(\mathcal{H}_2/\operatorname{ran}(T)).$$

Notes. (i) This does **not** assume that T is densely defined, so is more general. (ii) $\operatorname{codim}(T)$ is also called the **defect** of T. Sometimes it's also called the **corank** of T.

If T is Fredholm, its **Fredholm index** is then given by

$$\operatorname{ind}(T) = \dim(\ker(T)) - \operatorname{codim}(T).$$

Basics of Fredholm Index Theory (contd.):

Theorem.

Suppose $T : \text{dom}(T) \to \mathcal{H}_2$, $\text{dom}(T) \subseteq \mathcal{H}_1$ is closed and $\text{codim}(T) < \infty$. Then $\mathrm{ran}(T)$ is closed in $\mathcal{H}_2$.

Thus, if T is closed and $\mathrm{ran}(T)$ is not closed in $\mathcal{H}_2$, then $\text{codim}(T) = \infty$.

B.t.w., up to this point everything works for Banach spaces.

Example.

Suppose $T : \text{dom}(T) \to \mathcal{H}_2$, $\text{dom}(T) \subseteq \mathcal{H}_1$ is densely defined, closed, and $\mathrm{ran}(T)$ is dense but **not** closed in $\mathcal{H}_2$. Then

$$\text{codim}(T) = \infty.$$

On the other hand,

$$\ker(T^*) = \mathrm{ran}(T)^\perp = \{0\},$$

and hence,

$$0 = \dim(\ker(T^*)) \neq \text{codim}(T) = \infty.$$

Basics of Fredholm Index Theory (contd.):

Theorem.

Suppose $T : \mathrm{dom}(T) \to \mathcal{H}_2$, $\mathrm{dom}(T) \subseteq \mathcal{H}_1$ is densely defined, closed, and $\dim(\ker(T)) + \mathrm{codim}(T) < \infty$. Then $\mathrm{ran}(T)$ is closed in $\mathcal{H}_2$ and

$$\begin{aligned} \mathrm{ind}(T) &= \dim(\ker(T)) - \mathrm{codim}(T) \\ &= \dim(\ker(T)) - \dim(\ker(T^*)) \\ &= \dim(\ker(T^*T)) - \dim(\ker(TT^*)). \end{aligned}$$

At this point we return to the original definition of Fredholm operators as closed, densely defined operators, with closed range, satisfying,

$$\dim(\ker(T)) + \dim(\ker(T^*)) < \infty.$$

Basics of Fredholm Index Theory (contd.):

For $A_0, A_1 \in \Phi(\mathcal{H}_1, \mathcal{H}_2) \cap \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, A_0 and A_1 are called **homotopic** in $\Phi(\mathcal{H}_1, \mathcal{H}_2)$ if there exists $A\colon [0,1] \to \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ continuous, such that $A(t) \in \Phi(\mathcal{H}_1, \mathcal{H}_2)$, $t \in [0,1]$, with $A(0) = A_0$, $A(1) = A_1$.

Theorem.

Let $\mathcal{H}_j$, $j = 1, 2, 3$, be complex, separable Hilbert spaces, then (i)–(vii) hold:

(i) If $T \in \Phi(\mathcal{H}_1, \mathcal{H}_2)$ and $S \in \Phi(\mathcal{H}_2, \mathcal{H}_3)$, such that ST is densely defined, then $ST \in \Phi(\mathcal{H}_1, \mathcal{H}_3)$ and

$$\operatorname{ind}(ST) = \operatorname{ind}(S) + \operatorname{ind}(T).$$

(ii) Assume that $T \in \Phi(\mathcal{H}_1, \mathcal{H}_2)$ and $K \in \mathcal{B}_\infty(\mathcal{H}_1, \mathcal{H}_2)$, then $(T + K) \in \Phi(\mathcal{H}_1, \mathcal{H}_2)$ and

$$\operatorname{ind}(T + K) = \operatorname{ind}(T).$$

(iii) Suppose that $T \in \Phi(\mathcal{H}_1, \mathcal{H}_2)$ and $K \in \mathcal{B}_\infty(\mathcal{H}_S, \mathcal{H}_2)$, then $(T + K) \in \Phi(\mathcal{H}_1, \mathcal{H}_2)$ and

$$\operatorname{ind}(T + K) = \operatorname{ind}(T).$$

Basics of Fredholm Index Theory (contd.):

Theorem (contd.).

(iv) Assume that $T \in \Phi(\mathcal{H}_1, \mathcal{H}_2)$. Then there exists $\varepsilon(T) > 0$ such that for any $R \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ with $\|R\|_{\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)} < \varepsilon(T)$, one has $(T + R) \in \Phi(\mathcal{H}_1, \mathcal{H}_2)$ and

$$\operatorname{ind}(T + R) = \operatorname{ind}(T), \quad \dim(\ker(T + R)) \leq \dim(\ker(T)).$$

(v) Let $T \in \Phi(\mathcal{H}_1, \mathcal{H}_2)$, then $T^* \in \Phi(\mathcal{H}_2, \mathcal{H}_1)$ and

$$\operatorname{ind}(T^*) = -\operatorname{ind}(T).$$

(vi) Assume that $T \in \Phi(\mathcal{H}_1, \mathcal{H}_2)$ and that the Hilbert space $\mathcal{V}_1$ is continuously embedded in $\mathcal{H}_1$, with $\operatorname{dom}(S)$ dense in $\mathcal{V}_1$. Then $T \in \Phi(\mathcal{V}_1, \mathcal{H}_2)$ with $\ker(T)$ and $\operatorname{ran}(T)$ the same whether T is viewed as an operator $T\colon \operatorname{dom}(T) \subseteq \mathcal{H}_1 \to \mathcal{H}_2$, or as an operator $T\colon \operatorname{dom}(T) \subseteq \mathcal{V}_1 \to \mathcal{H}_2$.

(vii) Assume that the Hilbert space $\mathcal{W}_1$ is continuously and densely embedded in $\mathcal{H}_1$. If $T \in \Phi(\mathcal{W}_1, \mathcal{H}_2)$ then $T \in \Phi(\mathcal{H}_1, \mathcal{H}_2)$ with $\ker(T)$ and $\operatorname{ran}(T)$ the same whether T is viewed as an operator $T\colon \operatorname{dom}(T) \subseteq \mathcal{H}_1 \to \mathcal{H}_2$, or as an operator $T\colon \operatorname{dom}(T) \subseteq \mathcal{W}_1 \to \mathcal{H}_2$.

Basics of Fredholm Index Theory (contd.):

Theorem (contd.).

(*viii*) Homotopic operators in $\Phi(\mathcal{H}_1, \mathcal{H}_2) \cap \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ have equal Fredholm index. More precisely, the set $\Phi(\mathcal{H}_1, \mathcal{H}_2) \cap \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is open in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, hence $\Phi(\mathcal{H}_1, \mathcal{H}_2)$ contains at most countably many connected components, on each of which the Fredholm index is constant. Equivalently, $\text{ind} \colon \Phi(\mathcal{H}_1, \mathcal{H}_2) \to \mathbb{Z}$ is locally constant, hence continuous, and homotopy invariant.

A prime candidate for the Hilbert spaces $\mathcal{V}_1, \mathcal{W}_1 \subseteq \mathcal{H}_1$ in items (*vi*) and (*vii*) of this Theorem (e.g., in applications to differential operators) is the graph Hilbert space $\mathcal{H}_T$ induced by T. Moreover, an immediate consequence of this Theorem is the following homotopy invariance of the Fredholm index for a family of Fredholm operators with fixed domain.

Basics of Fredholm Index Theory (contd.):

Corollary.

Let $T(s) \in \Phi(\mathcal{H}_1, \mathcal{H}_2)$, $s \in I$, where $I \subseteq \mathbb{R}$ is a connected interval, with $\operatorname{dom}(T(s)) := \mathcal{V}_T$ independent of $s \in I$. In addition, assume that $\mathcal{V}_T$ embeds densely and continuously into $\mathcal{H}_1$ (for instance, $\mathcal{V}_T = \mathcal{H}_{T(s_0)}$ for some fixed $s_0 \in I$) and that $T(\cdot)$ is continuous with respect to the norm $\|\cdot\|_{\mathcal{B}(\mathcal{V}_T, \mathcal{H}_2)}$. Then

$$\operatorname{ind}(T(s)) \in \mathbb{Z} \text{ is independent of } s \in I.$$

The corresponding case of unbounded operators with varying domains (and $\mathcal{H}_1 = \mathcal{H}_2$) is treated in detail in **CL63**.

Basics of Fredholm Index Theory (contd.):

Some literature this summary on (unbounded) Fredholm operator is taken from:

BB13 D. D. Bleecker and B. Booß-Bavnbek, *Index Theory with Applications to Mathematics and Physics*, International Press, Boston, 2013; Chs. 1, 3.

CL63 H. O. Cordes and J. P. Labrousse, *The invariance of the index in the metric space of closed operators*, J. Math. Mech. **12**, 693–719 (1963).

EE89 D. E. Edmunds and W. D. Evans, *Spectral Theory and Differential Operators*, Clarendon Press, Oxford, 1989; Sect. I.3.

GGK90 I. Gohberg, S. Goldberg, and M. A. Kaashoek, *Classes of Linear Operators, Vol. I*, Operator Theory: Advances and Applications, Vol. 49, Birkhäuser, Basel, 1990; Chs. XI, XVII.

GK92 I. Gohberg and N. Krupnik, *One-Dimensional Linear Singular Integral Equations. I. Introduction*, Operator Theory: Advances and Applications, Vol. 53, Birkhäuser, Basel, 1992; Sects. IV.6, IV.10.

Basics of Fredholm Index Theory (contd.):

MP86 S. G. Mikhlin and S. Prössdorf, *Singular Integral Operators*, Springer, Berlin, 1986; Sect. I.3.

Mu13 A. Mukherjee, *Atiyah–Singer Index Theorem. An Introduction*, Hindustan Book Agency, New Delhi, 2013; Ch. 2.

Sc02 M. Schechter, *Principles of Functional Analysis*, 2nd ed., Graduate Studies in Math., Vol. 36, Amer. Math. Soc., Providence, RI, 2002; Chs. 5, 7.

Now we start to look into situations where T is **not** necessarily **Fredholm**, and where the **Fredholm index** will have to be replaced by a **regularized** "index", the **Witten index**:

Witten Indices:

Definition.

Let T be a closed, linear, densely defined operator in $\mathcal{H}$ and suppose that for some (and hence for **all**) $z \in \mathbb{C} \backslash [0, \infty)$,

$$\big[(T^*T - z I_{\mathcal{H}})^{-1} - (TT^* - z I_{\mathcal{H}})^{-1}\big] \in \mathcal{B}_1(\mathcal{H}).$$

Introduce the **resolvent regularization**

$$\Delta(T, \lambda) = (-\lambda) \operatorname{tr}_{\mathcal{H}} \big((T^*T - \lambda I_{\mathcal{H}})^{-1} - (TT^* - \lambda I_{\mathcal{H}})^{-1}\big), \quad \lambda < 0.$$

Then the **Witten index** $W_r(T)$ of T is defined by

$$W_r(T) = \lim_{\lambda \uparrow 0} \Delta(T, \lambda),$$

whenever this limit exists.

The subscript "r" indicates the use of the **resolvent regularization.**

Witten Indices (contd.):

Definition.

Let T be a closed, linear, densely defined operator in $\mathcal{H}$ and suppose that for some $t_0 > 0$ (and hence for **all** $t > t_0$),

$$\left[e^{-t_0 T^* T} - e^{-t_0 T T^*}\right] \in \mathcal{B}_1(\mathcal{H}).$$

Introduce the **semigroup (heat kernel) regularization**

$$\operatorname{ind}_t(T) = \operatorname{tr}_{\mathcal{H}} \left(e^{-t T^* T} - e^{-t T T^*}\right), \quad t \geq t_0.$$

Then the **Witten index** $W_s(T)$ of T is defined by

$$W_s(T) = \lim_{t \to \infty} \operatorname{ind}_t(T),$$

whenever this limit exists.

The subscript "s" indicates the use of the **semigroup (heat kernel) regularization.**

Witten Indices (contd.):

Consistency with the **Fredholm index** and connection to the **spectral shift function**:

Theorem. (F.G., B. Simon, JFA 79 (1988).)

Suppose that T is a **Fredholm** operator in $\mathcal{H}$.

(i) Assume that $\big[(TT^* - z\, I_{\mathcal{H}})^{-1} - (T^*T - z\, I_{\mathcal{H}})^{-1}\big] \in \mathcal{B}_1(\mathcal{H})$ for some $z \in \mathbb{C}\backslash[0,\infty)$. Then the **Witten index** $W_r(T)$ exists, equals the **Fredholm index**, $\mathrm{ind}(T)$, of T, and

$$W_r(T) = \mathrm{ind}(T) = \xi(0_+; TT^*, T^*T).$$

(ii) Assume that $\big[e^{-t_0 T^*T} - e^{-t_0 TT^*}\big] \in \mathcal{B}_1(\mathcal{H})$ for some $t_0 > 0$. Then the **Witten index** $W_s(T)$ exists, equals the **Fredholm index**, $\mathrm{ind}(T)$, of T, and

$$W_s(T) = \mathrm{ind}(T) = \xi(0_+; TT^*, T^*T).$$

Witten Indices (contd.):

To settle existence of the limits we start with the following result:

Theorem.

Assume the resolvent, resp., semigroup trace class hypothesis and suppose that $\xi(\cdot; TT^*, T^*T)$ is continuous from above at $\lambda = 0$. Then $W_r(T)$, resp., $W_s(T)$ exist and

$$W_r(T) = \xi(0_+; TT^*, T^*T), \ \text{resp.,} \ W_s(T) = \xi(0_+; TT^*, T^*T).$$

Proof.

(i) Assume the resolvent trace class hypothesis. Then with $\delta > 0$ (and $-\lambda \int_{[0,\infty)} (\mu - \lambda)^{-2}\, d\mu = 1$)

$$\Delta(T, \lambda) = (-\lambda)\, \mathrm{tr}_{\mathcal{H}}\big((T^*T - \lambda)^{-1} - (TT^* - \lambda)^{-1}\big)$$

$$= -\lambda \int_{[0,\infty)} (\mu - \lambda)^{-2} \xi(\mu; TT^*, T^*T)\, d\mu$$

$$= \xi(0_+; TT^*, T^*T) - \lambda \int_{[0,\infty)} (\mu - \lambda)^{-2} [\xi(\mu; TT^*, T^*T) - \xi(0_+; TT^*, T^*T)] d\mu$$

Witten Indices (contd.):

Proof (contd.).

$$= \xi(0_+; TT^*, T^*T) - \underbrace{\lambda \int_{[\delta,\infty)} \frac{[\xi(\mu; TT^*, T^*T) - \xi(0_+; TT^*, T^*T)]}{(\mu-\lambda)^2}\, d\mu}_{\longrightarrow 0 \text{ as } \lambda \uparrow 0}$$

$$- \underbrace{\lambda \int_{[0,\delta]} \frac{[\xi(\mu; TT^*, T^*T) - \xi(0_+; TT^*, T^*T)]}{(\mu-\lambda)^2}\, d\mu}_{|\cdots| \leq \varepsilon \text{ by continuity of } \xi \text{ at 0 from above}}$$

$$\underset{t\to\infty}{\longrightarrow} \xi(0_+; TT^*, T^*T).$$

(ii) Assume the semigroup $\mathcal{B}_1$-hypothesis, $\delta > 0$, and use $t \int_{[0,\infty)} e^{-t\lambda}\, d\lambda = 1$:

$$\mathrm{ind}_t(T) = \mathrm{tr}_{\mathcal{H}}\Big(e^{-tT^*T} - e^{-tTT^*}\Big) = t \int_{[0,\infty)} e^{-t\mu} \xi(\lambda; TT^*, T^*T)\, d\mu$$

$$= \xi(0_+; TT^*, T^*T) + t \int_{[0,\infty)} e^{-t\mu} [\xi(\mu; TT^*, T^*T) - \xi(0_+; TT^*, T^*T)]\, d\mu$$

Witten Indices (contd.):

Proof (contd.).

$$= \xi(0_+; TT^*, T^*T) + \underbrace{t\int_{[\delta,\infty)} e^{-t\lambda}[\xi(\lambda; TT^*, T^*T) - \xi(0_+; TT^*, T^*T)]\, d\lambda}_{\longrightarrow 0 \text{ as } t \to \infty}$$

$$+ \underbrace{t\int_{[0,\delta]} e^{-t\lambda}[\xi(\lambda; TT^*, T^*T) - \xi(0_+; TT^*, T^*T)]\, d\lambda}_{|\cdots| \leq \varepsilon \text{ by continuity of } \xi \text{ at } 0 \text{ from above}}$$

$$\underset{t\to\infty}{\longrightarrow} \xi(0_+; TT^*, T^*T).$$

∎

We still need to prove consistency in the case where T is **Fredholm**! This follows next.

Witten Indices (contd.):

To settle consistency between $W_r(T)$, $W_s(T)$, and $\mathrm{ind}(T)$ we state the following result:

Theorem.

Assume the resolvent, resp., semigroup trace class hypothesis and suppose that T is **Fredholm**. Then $W_r(T)$, resp., $W_s(T)$ exist and

$$\mathrm{ind}(T) = W_{r,\,\text{resp.},\,s}(T) = \xi(0_+; TT^*, T^*T).$$

Proof.

We'll illustrate the semigroup case and recall that

$$T \textbf{ Fredholm} \iff \{T^*T \text{ and } TT^* \text{ are } \textbf{Fredholm}\}$$

and since

$$\sigma(T^*T)\backslash\{0\} = \sigma(T^*T)\backslash\{0\},$$

one has for some $\delta > 0$,

$$\xi(\lambda; TT^*, T^*T) = \xi(0_+; TT^*, T^*T) \text{ for } \lambda \in (0, \delta).$$

Witten Indices (contd.):

Proof (contd.).

Moreover, by general properties of $\xi(\cdot; TT^*, T^*T)$, one also has *ab initio*

$$\mathrm{ind}(T) = \xi(0_+; TT^*, T^*T).$$

Thus,

$$\begin{aligned}
\mathrm{ind}_t(T) &= t \int_{[0,\infty)} e^{-t\lambda} \xi(\lambda; TT^*, T^*T)\, d\lambda \\
&= \xi(0_+; TT^*, T^*T) + t \int_{[0,\infty)} e^{-t\lambda} [\xi(\lambda; TT^*, T^*T) - \xi(0_+; TT^*, T^*T)]\, d\lambda \\
&= \xi(0_+; TT^*, T^*T) + \underbrace{t \int_{[(\delta/2),\infty)} e^{-t\lambda} [\xi(\lambda; TT^*, T^*T) - \xi(0_+; TT^*, T^*T)]\, d\lambda}_{\longrightarrow 0 \text{ as } t \to \infty} \\
&\underset{t\to\infty}{\longrightarrow} \xi(0_+; TT^*, T^*T).
\end{aligned}$$

∎

Stability Properties of the Witten Index:

Remarks. (i) *Ab initio,* the **Witten index**, $W_{r,\,\text{resp.},\,s}(T)$, let alone the **regularized Witten index**, $\Delta(T,\lambda)$, resp., $ind_t(T)$, have no "business" to be invariant w.r.t. "small" perturbations; after all, they're **NOT** an index!

(ii) In general (i.e., if T is not Fredholm), $W_r(T)$, resp., $W_s(T)$, are **not integer-valued**; in fact, they can be **any real number**. In concrete 2d magnetic field systems they can have the meaning of (non-quantized) **magnetic flux** $F \in \mathbb{R}$, an arbitrary real number (see the upcoming example below).

Still, one can prove a stability result:

F.G. and B. Simon, *Topological invariance of the Witten index*, J. Funct. Anal. **79**, 91–102 (1988),

showed that $W_{r,\,\text{resp.},\,s}(T)$ has **stability properties** w.r.t. additive perturbations similar to the Fredholm index, replacing the **relative compactness** assumption on the perturbation by "appropriate" **relative trace class** conditions as discussed in the following:

Stability Properties of the Witten Index (contd.):

Basic setup for semigroups: S closed in $\mathcal{H}$, S infinitesimally bounded w.r.t. T,

$$T_\beta = T + \beta S, \quad \beta \in \mathbb{R}.$$

Theorem.

Let $\beta \in \mathbb{R}$ and assume that T is densely defined and closed in $\mathcal{H}$ and that S is densely defined in $\mathcal{H}$ s.t.

$$\text{for some } \gamma \in (0,1/2),\ S(T^*T+1)^{-\gamma},\, S^*(TT^*+1)^{-\gamma} \in \mathcal{B}(\mathcal{H}),$$
$$\text{for some } \tau > 0,\ S(T^*T+1)^{-\tau},\, S^*(TT^*+1)^{-\tau} \in \mathcal{B}_1(\mathcal{H}),$$
$$\text{for all } t \in \mathbb{R},\ \left[e^{-tT^*T} - e^{-tTT^*}\right] \in \mathcal{B}_1(\mathcal{H}).$$

Then, $\mathrm{ind}_t(T+\beta S)$ and $\mathrm{ind}_t(T)$ exist for all $t > 0$ and the heat kernel regularized index is invariant w.r.t. the perturbation βS,

$$\mathrm{ind}_t(T+\beta S) = \mathrm{ind}_t(T), \quad t > 0,\ \beta \in \mathbb{R}.$$

Moreover, if $W_s(T)$ exists, then also $W_s(T+\beta S) = W_s(T)$, $\beta \in \mathbb{R}$.

Stability Properties of the Witten Index (contd.):

Algebraic idea behind the proof:

Consider

$$Q = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix}, \quad \Sigma_3 = \begin{pmatrix} I_{\mathcal{H}} & 0 \\ 0 & -I_{\mathcal{H}} \end{pmatrix},$$

in $\mathcal{H} \oplus \mathcal{H}$.

Note. Q is also called a **supersymmetric Dirac-type operator** (sometimes it is also called a super charge).

Introduce the off-diagonally **perturbed Dirac-type** operator

$$Q(\beta) = Q + \beta R, \quad \beta \in \mathbb{R}.$$

Then

$$Q^2 = \begin{pmatrix} T^*T & 0 \\ 0 & TT^* \end{pmatrix}, \quad Q(\beta)^2 = \begin{pmatrix} (T+\beta S)^*(T+\beta S) & 0 \\ 0 & (T+\beta S)(T+\beta S)^* \end{pmatrix}$$

in $\mathcal{H} \oplus \mathcal{H}$, and hence,

Stability Properties of the Witten Index (contd.):

Algebraic idea behind the proof (contd.):

$$\frac{d}{d\beta}\operatorname{tr}_{\mathcal{H}\oplus\mathcal{H}}\Big(\Sigma_3\Big[e^{-tQ(\beta)^2}-e^{-tQ^2}\Big]\Big)$$
$$=-\int_0^t ds\,\operatorname{tr}_{\mathcal{H}\oplus\mathcal{H}}\Big(\Sigma_3 e^{-sQ(\beta)^2}[Q(\beta)R-RQ(\beta)]e^{-(t-s)Q(\beta)^2}\Big)=0$$

(using $\Sigma_3 Q(\beta)+Q(\beta)\Sigma_3\subseteq 0$, $\Sigma_3 R+R\Sigma_3\subseteq 0$, and cyclicity of the trace),

since (trivially),

$$Q(\beta)Q(\beta)^2=Q(\beta)^2Q(\beta),\quad \Sigma_3 Q(\beta)=-Q(\beta)\Sigma_3,$$

and hence,

$$\operatorname{tr}_{\mathcal{H}\oplus\mathcal{H}}\Big(\Sigma_3\Big[e^{-tQ(1)^2}-e^{-tQ^2}\Big]\Big)=\operatorname{tr}_{\mathcal{H}\oplus\mathcal{H}}\Big(\Sigma_3\Big[e^{-tQ(0)^2}-e^{-tQ^2}\Big]\Big)\equiv 0,\quad t>0.$$

Stability Properties of the Witten Index (contd.):

Algebraic idea behind the proof (contd.):

This immediately yields

$$\mathrm{ind}_t(T + \beta S) = \mathrm{ind}_t(T), \quad t > 0, \ \beta \in \mathbb{R}. \qquad (*)$$

Note. The invariance of this index regularization in $(*)$ is very interesting even in the classical situation where T is **Fredholm** !

We recall the **semigroup (heat kernel) regularization**

$$\mathrm{ind}_t(T) = \mathrm{tr}_{\mathcal{H}}\left(e^{-tT^*T} - e^{-tTT^*}\right), \quad t \geq t_0,$$

and emphasize that $(*)$ implies **invariance** of the **regularized Witten index** **without** even taking the limit $t \to \infty$! (These supersymmetric constructions exhibit a remarkable rigidity .....)

The actual proof of $(*)$ follows this route, but is forced to painstakingly justify all trace class properties and the applicability of cyclicity of the trace at each step.

Stability Properties of the Witten Index (contd.):

Resolvent regularizations can be treated analogously:

Assume that S closed in $\mathcal{H}$, S infinitesimally bounded w.r.t. T,

$$T_\beta = T + \beta S, \quad \beta \in \mathbb{R},$$

and introduce

$$\Delta(T_\beta, z) = (-z)\, \mathrm{tr}_{\mathcal{H}} \left((T_\beta^* T_\beta - z\, I_{\mathcal{H}})^{-1} - (T_\beta T_\beta^* - z\, I_{\mathcal{H}})^{-1} \right), \quad z \in \mathbb{C} \backslash [0, \infty).$$

Stability Properties of the Witten Index (contd.):

Theorem. (F.G., B. Simon, JFA 79 (1988), BGGSS, JMP 28 (1987).)

Suppose that for some $z_0 \in \mathbb{C}\backslash[0,\infty)$,

$$\big[(T_\beta T_\beta^* - z_0 I_{\mathcal{H}})^{-1} - (T_\beta^* T_\beta - z_0 I_{\mathcal{H}})^{-1}\big] \in \mathcal{B}_1(\mathcal{H}) \text{ for all } \beta \in \mathbb{R},$$

$$S^*S(T^*T - z_0 I_{\mathcal{H}})^{-1},\ SS^*(TT^* - z_0 I_{\mathcal{H}})^{-1} \in \mathcal{B}_\infty(\mathcal{H}),$$

$$[T^*S + S^*T](T^*T - z_0 I_{\mathcal{H}})^{-1}, [ST^* + TS^*](TT^* - z_0 I_{\mathcal{H}})^{-1} \in \mathcal{B}_\infty(\mathcal{H}),$$

$$(T^*T - z_0 I_{\mathcal{H}})^{-1} S^*S (T^*T - z_0 I_{\mathcal{H}})^{-1} \in \mathcal{B}_1(\mathcal{H}),$$

$$(TT^* - z_0 I_{\mathcal{H}})^{-1} SS^* (TT^* - z_0 I_{\mathcal{H}})^{-1} \in \mathcal{B}_1(\mathcal{H}),$$

$$(T^*T - z_0 I_{\mathcal{H}})^{-1}[T^*S + S^*T](T^*T - z_0 I_{\mathcal{H}})^{-1} \in \mathcal{B}_1(\mathcal{H}),$$

$$(TT^* - z_0 I_{\mathcal{H}})^{-1}[TS^* + ST^*](TT^* - z_0 I_{\mathcal{H}})^{-1} \in \mathcal{B}_1(\mathcal{H}),$$

$$(T^*T - z_0 I_{\mathcal{H}})^{-m} S^* (TT^* - z_0 I_{\mathcal{H}})^{-m} \in \mathcal{B}_1(\mathcal{H}) \text{ for some } m \in \mathbb{N}.$$

(This can be improved a bit ....) Then, for all $\beta \in \mathbb{R}$,

$$\Delta(T + \beta S, z) = \Delta(T, z), \quad W_r(T + \beta S) = W_r(T) \text{ (if } W_r(T) \text{ exists).}$$

Stability Properties of the Witten Index (contd.):

Algebraic idea behind the proof:

Introduce for some $z_0 \in \mathbb{C}\backslash[0,\infty)$,

$$F_\beta(z) := \operatorname{tr}_{\mathcal{H}}\big((T_\beta T_\beta^* - z\,I_{\mathcal{H}})^{-1} - (T_\beta^* T_\beta - z\,I_{\mathcal{H}})^{-1}\big), \quad z \in \mathbb{C}\backslash[0,\infty).$$

Then,

$$\begin{aligned}\frac{\partial F_\beta(z)}{\partial \beta} &= \operatorname{tr}_{\mathcal{H}}\big((T_\beta^* T_\beta - z\,I_{\mathcal{H}})^{-1}[T_\beta^* S + S^* T_\beta](T_\beta^* T_\beta - z\,I_{\mathcal{H}})^{-1} \\ &\qquad - (T_\beta T_\beta^* - z\,I_{\mathcal{H}})^{-1}[T_\beta S^* + S\,T_\beta^*](T_\beta T_\beta^* - z\,I_{\mathcal{H}})^{-1}\big) \\ &= 0, \quad \beta \in \mathbb{R}, \ \ z \in \mathbb{C}\backslash[0,\infty),\end{aligned}$$

using the **commutation formulas**

$$\begin{aligned}&(T_\beta^* T_\beta - z\,I_{\mathcal{H}})^{-1} T_\beta^* \subseteq T_\beta^* (T_\beta T_\beta^* - z\,I_{\mathcal{H}})^{-1}, \\ &(T_\beta T_\beta^* - z\,I_{\mathcal{H}})^{-1} T_\beta \subseteq T_\beta (T_\beta^* T_\beta - z\,I_{\mathcal{H}})^{-1}, \quad \beta \in \mathbb{R}, \ \ z \in \mathbb{C}\backslash[0,\infty),\end{aligned}$$

and **cyclicity** of the trace.

A $2d$ Example:

A $2d$ Magnetic Field Example (D. Bolle, F.G., H. Grosse, W. Schweiger, B. Simon '87; F.G., B. Simon '88).

$\mathcal{H} = L^2(\mathbb{R}^2)$, $\quad T = \overline{[(-i\partial_{x_1} - a_1(x)) + i(i\partial_{x_2} + a_2(x))]|_{C_0^\infty(\mathbb{R}^2)}}$, $\quad$ fix some $\varepsilon > 0$,

$a = (a_1, a_2) = (\partial_{x_2}\phi, -\partial_{x_1}\phi)$, $\quad b = \partial_{x_1} a_2 - \partial_{x_2} a_1 = -\Delta\phi$,

$\phi \in C^2(\mathbb{R}^2)$, $\quad \phi(x) \underset{|x|\to\infty}{=} -F\ln(|x|) + C + O\big(|x|^{-\varepsilon}\big)$, $\quad F, C \in \mathbb{R}$,

$(\nabla\phi)(x) \underset{|x|\to\infty}{=} -F|x|^{-2}x + o\big(|x|^{-1-\varepsilon}\big)$, $\quad (\Delta\phi)^{1+\varepsilon}, \ \big(1 + |\cdot|^{\varepsilon}\big)(\Delta\phi) \in L^1\big(\mathbb{R}^2\big)$,

$(\Delta\phi)^{1+\varepsilon}, \ \big(1 + |\cdot|^{\varepsilon}\big)(\Delta\phi) \in L^1\big(\mathbb{R}^2\big)$,

$\mathbf{H}_1 = \mathbf{T}^*\mathbf{T} = \big[(-i\nabla - a)^2 + b\big]\big|_{H^2(\mathbb{R}^2)}$, $\quad \mathbf{H}_2 = \mathbf{T}\mathbf{T}^* = \big[(-i\nabla - a)^2 - b\big]\big|_{H^2(\mathbb{R}^2)}$.

The magnetic flux F is given by $F = \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}^2} d^2x\, b(x)$,

$\sigma(\mathbf{H}_j) = [0,\infty) \implies T, \mathbf{H}_j$ are **not** Fredholm, $j = 1, 2$.

A $2d$ Example (contd.):

A $2d$ Magnetic Field Example (contd.).

$$\Delta_r(T,z) = z\,\mathrm{tr}_{L^2(\mathbb{R})}\big((\mathbf{H}_2 - z)^{-1} - (\mathbf{H}_1 - z)^{-1}\big) = -F, \quad z \in \mathbb{C}\backslash[0,\infty),$$

$W_r(T) = -F$ (can be **any** prescribed real number !!!!!),

$$i(T) := \dim(\ker(T)) - \dim(\ker(T^*)) = \dim(\ker(\mathbf{H}_1)) - \dim(\ker(\mathbf{H}_2)),$$

$$i(T)\,\mathrm{sgn}(F) = \theta(-F)\dim(\ker(T)) - \theta(F)\dim(\ker(T^*)) = \begin{cases} -N, & |F| = N + \varepsilon,\ 0 < \varepsilon < 1, \\ -(N-1), & |F| = N,\ N \in \mathbb{N}, \end{cases}$$

$$\xi(\lambda; \mathbf{H}_2, \mathbf{H}_1) = F\,\theta(\lambda),\ \lambda \in \mathbb{R}.$$

Here, $\theta(x) = 1$, $x \geq 0$, $\theta(x) = 0$, $x < 0$, and $\mathrm{sgn}(x) = 1$, $x > 0$, $\mathrm{sgn}(x) = 0$, $x = 0$, $\mathrm{sgn}(x) = -1$, $x < 0$.

Idea of Proof: Use a decomposition w.r.t. angular momenta $\longrightarrow$ reduce this to an infinite sequence of 1d problems.

A Model Fredholm Operator:

Consider the **model (Fredholm) operator**,

$$\mathbf{D_A} = (d/dt) + \mathbf{A}, \quad \operatorname{dom}(\mathbf{D_A}) = \operatorname{dom}(d/dt) \cap \operatorname{dom}(\mathbf{A_-}) \text{ in } L^2(\mathbb{R}; \mathcal{H})$$

($\mathcal{H}$ a complex, separable Hilbert space), where $\operatorname{dom}(d/dt) = W^{2,1}(\mathbb{R}; \mathcal{H})$, and

$$\mathbf{A} = \int_{\mathbb{R}}^{\oplus} A(t)\, dt, \quad \mathbf{A_-} = \int_{\mathbb{R}}^{\oplus} A_-\, dt \text{ in } L^2(\mathbb{R}; \mathcal{H}) \simeq \int_{\mathbb{R}}^{\oplus} \mathcal{H}\, dt,$$

$A_\pm = \lim_{t \to \pm\infty} A(t)$ exist in norm resolvent sense and are boundedly invertible, i.e., $0 \in \rho(A_\pm)$, $\longleftarrow$ **Fredholm property**

and where we consider the case of **relative trace class** perturbations $[A(t) - A_-]$,

$$A(t) = A_- + B(t), \quad t \in \mathbb{R},$$

$$B(t)(A_- - zI_{\mathcal{H}})^{-1} \in \mathcal{B}_1(\mathcal{H}), \quad t \in \mathbb{R} \quad \text{(plus quite a bit more} \longrightarrow \text{next 2 pages)},$$

such that $\mathbf{D_A}$ becomes a Fredholm operator in $L^2(\mathbb{R}; \mathcal{H})$.

A Model Fredholm Operator (contd.):

Just for clarity:

The Hilbert space $L^2(\mathbb{R};\mathcal{H})$ consists of equivalence classes f of weakly (and hence strongly) Lebesgue measurable $\mathcal{H}$-valued functions $f(\cdot) \in \mathcal{H}$ (whose elements are equal a.e. on $\mathbb{R}$), such that $\|f(\cdot)\|_{\mathcal{H}} \in L^2(\mathbb{R}; dt)$. The norm and scalar product on $L^2(\mathbb{R};\mathcal{H})$ are then given by

$$\|f\|^2_{L^2(\mathbb{R};\mathcal{H})} = \int_{\mathbb{R}} \|f(t)\|^2_{\mathcal{H}}\, dt,$$
$$(f,g)_{L^2(\mathbb{R};\mathcal{H})} = \int_{\mathbb{R}} (f(t),g(t))_{\mathcal{H}}\, dt, \quad f,g \in L^2(\mathbb{R};\mathcal{H}).$$

Of course,

$$L^2(\mathbb{R};\mathcal{H}) \simeq \int_{\mathbb{R}}^{\oplus} \mathcal{H}\, dt \quad (\textbf{constant fiber direct integral}).$$

A Model Fredholm Operator (contd.):

Operators $\mathbf{A}$ in $L^2(\mathbb{R};\mathcal{H})$:

$$(\boldsymbol{A}f)(t) = A(t)f(t) \text{ for a.e. } t \in \mathbb{R},$$

$$f \in \operatorname{dom}(\boldsymbol{A}) = \bigg\{ g \in L^2(\mathbb{R};\mathcal{H}) \,\bigg|\, g(t) \in \operatorname{dom}(A(t)) \text{ for a.e. } t \in \mathbb{R},$$

$$t \mapsto A(t)g(t) \text{ is (weakly) measurable, } \int_{\mathbb{R}} \|A(t)g(t)\|_{\mathcal{H}}^2 \, dt < \infty \bigg\}.$$

Thus, if in addition, $\{A(t)\}_{t\in\mathbb{R}}$ is **N-measurable**, $\mathbf{A}$ is the **direct integral** of the family $\{A(t)\}_{t\in\mathbb{R}}$ over $\mathbb{R}$,

$$\mathbf{A} = \int_{\mathbb{R}}^{\oplus} A(t)\, dt \text{ in } L^2(\mathbb{R};\mathcal{H}) \simeq \int_{\mathbb{R}}^{\oplus} \mathcal{H}\, dt, \quad \mathcal{H} \text{ a separable, complex } H\text{-space.}$$

Note. $\{T(t)\}_{t\in\mathbb{R}}$ is **N-measurable** (**A. E. Nussbaum,** DMJ **31**, 33–44 (1964)) if

$$\big\{(|T(t)|^2 + I_{\mathcal{H}})^{-1}\big\}_{t\in\mathbb{R}}, \quad \big\{T(t)(|T(t)|^2 + I_{\mathcal{H}})^{-1}\big\}_{t\in\mathbb{R}}, \quad \big\{(|T(t)^*|^2 + I_{\mathcal{H}})^{-1}\big\}_{t\in\mathbb{R}},$$

are weakly measurable.

Notation: $A(t)$, $B(t)$, etc., "act" in $\mathcal{H}$, but $\mathbf{A}$, $\mathbf{B}$, etc., "act" in $L^2(\mathbb{R};\mathcal{H})$.

A Model Fredholm Operator (contd.):

Note. We also require that $A(\cdot)$ has **limiting operators**

$$A_+ = \lim_{t\to+\infty} A(t), \quad A_- = \lim_{t\to-\infty} A(t)$$

in an appropriate (= norm resolvent convergence) sense.

J. Robbin and D. Salamon, *The spectral flow and the Maslov index*, Bull. London Math. Soc. **27**, 1–33 (1995).

The **Fredholm index**, $\operatorname{ind}(\mathbf{D_A})$, of $\mathbf{D_A}$ **equals** the **spectral flow**, $\mathrm{SpFlow}(\{A(t)\}_{t=-\infty}^{\infty})$, of the operator family $\{A(t)\}_{t=-\infty}^{+\infty}$.

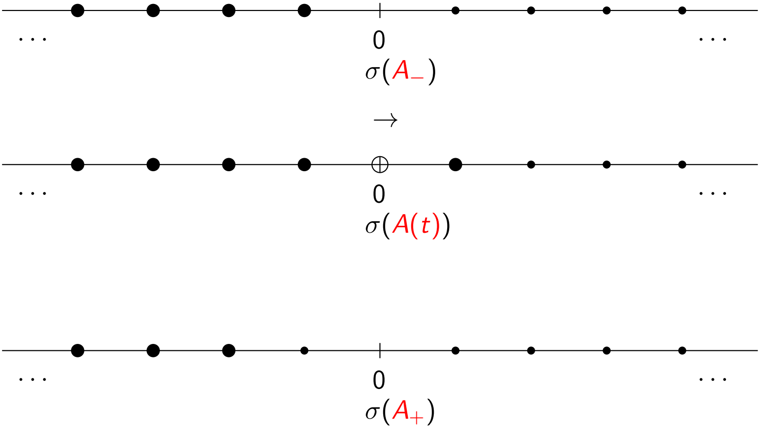
In their paper, $A(t)$ are unbounded self-adjoint operators in a Hilbert space $\mathcal{H}$ with **compact resolvent** (thus **discrete spectrum**) and t-constant domains.

$A_\pm = \lim_{t\to\pm\infty} A(t)$ **exist and are boundedly invertible**, i.e., $0 \in \rho(A_\pm)$.

Note. A_+ and A_- are also assumed boundedly invertible in our approach, as long as we consider **Fredholm** situations.

A Model Fredholm Operator (contd.):

Spectral flow "=" (the number of eigenvalues of $A(t)$ that cross 0 rightward) − (the number of eigenvalues of $A(t)$ that cross 0 leftward) as t runs from $-\infty$ to $+\infty$

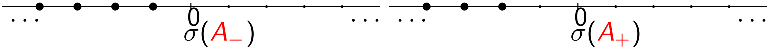


A Model Fredholm Operator (contd.):

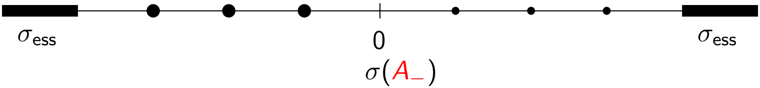
We prove: **Fredholm index = spectral shift function at 0 = spectral flow.**

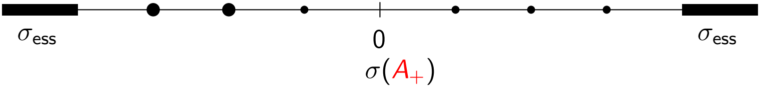
This allows us to handle more general families of operators than before:

Before:



After:





A Model Fredholm Operator (contd.):

A $1d$ Example (D. Bolle, F.G., H. Grosse, W. Schweiger, B. Simon '87).

The simplest possible example in this context:

$$\mathcal{H} = \mathbb{C}, \quad L^2(\mathbb{R}; \mathcal{H}) = L^2(\mathbb{R}),$$

$$A(\cdot) \in C^1(\mathbb{R}), \quad A(t) \underset{t\to\pm\infty}{\longrightarrow} A_\pm \in \mathbb{R}, \quad A'(t) \underset{t\to\pm\infty}{\longrightarrow} 0,$$

$$\mathbf{D_A} = (d/dt) + \mathbf{A}, \quad \mathbf{D_A^*} = -(d/dt) + \mathbf{A},$$

$$\operatorname{dom}(\mathbf{D_A}) = \operatorname{dom}(\mathbf{D_A^*}) = \operatorname{dom}(d/dt) = W^{2,1}(\mathbb{R}),$$

$$(\mathbf{A}f)(t) = A(t)f(t) \text{ for a.e. } t \in \mathbb{R},\ f \in L^2(\mathbb{R}),$$

$$\mathbf{H}_1 = \mathbf{D_A^*}\mathbf{D_A} = -\frac{d^2}{dt^2} + \mathbf{A}^2 - \mathbf{A}', \quad \mathbf{H}_2 = \mathbf{D_A}\mathbf{D_A^*} = -\frac{d^2}{dt^2} + \mathbf{A}^2 + \mathbf{A}',$$

$$\operatorname{dom}(\mathbf{H}_1) = \operatorname{dom}(\mathbf{H}_2) = W^{2,2}(\mathbb{R}),$$

$$\sigma_{\rm ess}(\mathbf{H}_1) = \sigma_{\rm ess}(\mathbf{H}_2) = \big[\min\big(A_-^2, A_+^2\big), \infty\big),$$

$\mathbf{D_A}$ is Fredholm if and only if $A_\pm \in \mathbb{R}\backslash\{0\}$.

A Model Fredholm Operator (contd.):

A $1d$ Example (contd.).

$$\Delta(\mathbf{D_A}, z) = z\,\mathrm{tr}_{L^2(\mathbb{R})}\big((\mathbf{H}_2 - z)^{-1} - (\mathbf{H}_1 - z)^{-1}\big) = [g_z(A_+) - g_z(A_-)]/2,$$

$$g_z(x) = x(x^2 - z)^{-1/2}, \quad z \in \mathbb{C}\backslash[0,\infty),\ x \in \mathbb{R},$$

$$\begin{aligned}
\mathrm{ind}(\mathbf{D_A}) &= \dim(\ker(\mathbf{D_A})) - \dim(\ker(\mathbf{D_A^*})) \\
&= \dim(\ker(\mathbf{H}_1)) - \dim(\ker(\mathbf{H}_2)) \\
&= \frac{1}{2}[\mathrm{sgn}(A_+) - \mathrm{sgn}(A_-)] = \begin{cases} +1, & A_- < 0 < A_+, \\ -1, & A_+ < 0 < A_-, \\ \ \ 0, & A_\pm > 0 \text{ or } A_\pm < 0 \end{cases} \\
&= \lim_{z \to 0} \Delta(\mathbf{D_A}, z) \\
&= \xi(0_+; \mathbf{H}_2, \mathbf{H}_1) \ \text{(the spectral shift function w.r.t. the pair } (\mathbf{H}_2, \mathbf{H}_1)).
\end{aligned}$$

Topological invariance: $\mathrm{ind}(\mathbf{D_A})$ does **not** depend on $A(t)$, $t \in \mathbb{R}$, **only** on its **asymptotes** $A_\pm = \lim_{t\to\pm\infty} A(t)$ **!** $\longleftrightarrow$ One of our principal motivations.......

A Model Fredholm Operator (contd.):

A $1d$ Example (contd.).

The **non-Fredholm** case: W.l.o.g., $A_- = 0 \Longrightarrow \sigma_{ess}(\mathbf{H}_1) = \sigma_{ess}(\mathbf{H}_2) = [0, \infty)$.

$$
\begin{aligned}
i(\mathbf{D_A}) &:= \dim(\ker(\mathbf{D_A})) - \dim(\ker(\mathbf{D_A^*})) \\
&= \dim(\ker(\mathbf{H}_1)) - \dim(\ker(\mathbf{H}_2)) \\
&= \begin{cases} 0, & A_- = 0,\ A_+ \neq 0, \\ 0, & A_- = A_+ = 0, \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
W_r(\mathbf{D_A}) &= \lim_{z \to 0} \Delta(\mathbf{D_A}, z) \\
&= \begin{cases} \frac{1}{2}\,\mathrm{sgn}(A_+), & A_- = 0,\ A_+ \neq 0, \\ 0, & A_- = A_+ = 0, \end{cases} \quad \textbf{Levinson's theorem,} \\
&= \xi(0_+; \mathbf{H}_2, \mathbf{H}_1) \ \text{(the spectral shift function w.r.t. the pair } (\mathbf{H}_2, \mathbf{H}_1)).
\end{aligned}
$$

Topological invariance: $W_r(\mathbf{D_A})$ again does **not** depend on $A(t)$, $t \in \mathbb{R}$, **only** on its **asymptotes** $A_\pm = \lim_{t \to \pm\infty} A(t)$ **!**

A Model Fredholm Operator (contd.):

A $1d$ Example (contd.).

The **Fredholm** case: $A_\pm \neq 0$.

$$\xi(\lambda; \boldsymbol{H}_2, \boldsymbol{H}_1) = \pi^{-1}\{\theta(\lambda - A_+^2)\arctan\left((\lambda - A_+^2)/A_+\right) - \theta(\lambda - A_-^2)\arctan\left((\lambda - A_-^2)/A_-\right)\} + \theta(\lambda)[\mathrm{sgn}(A_-) - \mathrm{sgn}(A_+)]/2, \quad \lambda \in \mathbb{R}.$$

The **Non-Fredholm** case: W.l.o.g., $A_- = 0$, $A_+ \neq 0$.

$$\xi(\lambda; \boldsymbol{H}_2, \boldsymbol{H}_1) = \pi^{-1}\{\theta(\lambda - A_+^2)\arctan\left((\lambda - A_+^2)/A_+\right) - \theta(\lambda)[\mathrm{sgn}(A_+)]/2, \quad \lambda \in \mathbb{R},$$

$$\theta(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0, \end{cases} \quad \mathrm{sgn}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

A Model Fredholm Operator (contd.):

F.G., Y. Latushkin, K. Makarov, F. Sukochev, and Y. Tomilov, Adv. Math. **227**, 319–420 (2011),

abbreviated as **GLMST '11** from now on, generalized results regarding $\mathbf{D_A} = (d/dt) + \mathbf{A}$ in $L^2(\mathbb{R}; \mathcal{H})$ in

A. Pushnitski, *The spectral flow, the Fredholm index, and the spectral shift function*, in *Spectral Theory of Differential Operators: M. Sh. Birman 80th Birthday Collection*, AMS, 2008, pp. 141–155.

Pushnitski studied the case of **trace class** perturbations $[A(t) - A_-]$, i.e.,

$$A(t) = A_- + B(t), \quad t \in \mathbb{R},$$

$$B(t) \in \mathcal{B}_1(\mathcal{H}), \quad t \in \mathbb{R}.$$

GLMST '11 studied the case of **relative** **trace class** perturbations $[A(t) - A_-]$, i.e.,

$$A(t) = A_- + B(t), \quad t \in \mathbb{R},$$

$$B(t)(A_- - z)^{-1} \in \mathcal{B}_1(\mathcal{H}), \quad t \in \mathbb{R} \quad \text{(plus quite a bit more .....).}$$

A Model Fredholm Operator (contd.):

Main Hypotheses (recall, we're aiming at $A(t) = A_- + B(t)$ ....):

- A_- – **self-adjoint** on $\operatorname{dom}(A_-) \subseteq \mathcal{H}$, $\mathcal{H}$ a complex, separable Hilbert space.
- $B(t)$, $t \in \mathbb{R}$, – **closed, symmetric,** in $\mathcal{H}$, $\operatorname{dom}(B(t)) \supseteq \operatorname{dom}(A_-)$.
- There exists a family $B'(t)$, $t \in \mathbb{R}$, – **closed, symmetric,** in $\mathcal{H}$, with

  $\operatorname{dom}(B'(t)) \supseteq \operatorname{dom}(A_-)$, such that

  $B(t)(|A_-| + I_{\mathcal{H}})^{-1}$, $t \in \mathbb{R}$, is **weakly locally a.c.** and for a.e. $t \in \mathbb{R}$,

  $$\frac{d}{dt}\big(g, B(t)(|A_-| + I_{\mathcal{H}})^{-1}h\big)_{\mathcal{H}} = \big(g, B'(t)(|A_-| + I_{\mathcal{H}})^{-1}h\big)_{\mathcal{H}}, \quad g, h \in \mathcal{H}.$$

- $B'(\cdot)(|A_-| + I_{\mathcal{H}})^{-1} \in \mathcal{B}_1(\mathcal{H})$, $t \in \mathbb{R}$, $\quad \int_{\mathbb{R}} \big\|B'(t)(|A_-| + I_{\mathcal{H}})^{-1}\big\|_{\mathcal{B}_1(\mathcal{H})}\, dt < \infty$.
- $\big\{(|B(t)|^2 + I_{\mathcal{H}})^{-1}\big\}_{t\in\mathbb{R}}$ and $\big\{(|B'(t)|^2 + I_{\mathcal{H}})^{-1}\big\}_{t\in\mathbb{R}}$ are **weakly measurable.**

A Model Fredholm Operator (contd.):

Consequences of these hypotheses:

$A(t) = A_- + B(t)$, $\operatorname{dom}(A(t)) = \operatorname{dom}(A_-)$, $t \in \mathbb{R}$, is self-adjoint.

There exists $A_+ = A(+\infty) = A_- + B(+\infty)$, $\operatorname{dom}(A_+) = \operatorname{dom}(A_-)$,

$\text{n-lim}_{t\to\pm\infty}(A(t) - zI_{\mathcal{H}})^{-1} = (A_\pm - zI_{\mathcal{H}})^{-1}$,

$(A_+ - A_-)(A_- - zI_{\mathcal{H}})^{-1} \in \mathcal{B}_1(\mathcal{H})$,

$\big[(A(t) - zI_{\mathcal{H}})^{-1} - (A_\pm - zI_{\mathcal{H}})^{-1}\big] \in \mathcal{B}_1(\mathcal{H})$, $t \in \mathbb{R}$,

$\big[(A_+ - zI_{\mathcal{H}})^{-1} - (A_- - zI_{\mathcal{H}})^{-1}\big] \in \mathcal{B}_1(\mathcal{H})$,

$\sigma_{\text{ess}}(A(t)) = \sigma_{\text{ess}}(A_+) = \sigma_{\text{ess}}(A_-)$, $t \in \mathbb{R}$.

A Model Fredholm Operator (contd.):

Theorem. (A. Carey, F.G., D. Potapov, F. Sukochev, Y. Tomilov '13.)

Under these hypotheses, $\mathbf{D_A} = \frac{d}{dt} + \mathbf{A}$ on $\operatorname{dom}(\mathbf{D_A}) = \operatorname{dom}(d/dt) \cap \operatorname{dom}(\mathbf{A}_-)$, is **closed** in $L^2(\mathbb{R}; \mathcal{H})$.

Moreover, $\mathbf{D_A}$ is **Fredholm** if and only if $0 \in \rho(A_+) \cap \rho(A_-)$.

The fact that $\mathbf{D_A}$ is closed was known since our **GLMST '11** paper.

Also, **sufficiency** of the condition $0 \in \rho(A_+) \cap \rho(A_-)$ for the **Fredholm** property of $\mathbf{D_A}$ has been proved in **GLMST '11**.

What was new then was that the condition $0 \in \rho(A_+) \cap \rho(A_-)$ is also **necessary** for the **Fredholm** property of $\mathbf{D_A}$.

A Model Fredholm Operator (contd.):

Next, for T a linear operator in the Hilbert space $\mathcal{K}$, introduce

$$\sigma_{ess}(T) = \{\lambda \in \mathbb{C} \,|\, T - \lambda I_{\mathcal{K}} \text{ is not Fredholm}\}.$$

Corollary. (A. Carey, F.G., D. Potapov, F. Sukochev, Y. Tomilov '13.)

Under these hypotheses, (**without** assuming $0 \in \rho(A_+) \cap \rho(A_-)$),

$$\sigma_{ess}(\mathbf{D_A}) = (\sigma(A_+) + i\,\mathbb{R}) \cup (\sigma(A_-) + i\,\mathbb{R}).$$

Note. Suppose $\sigma(A_\pm) = \mathbb{R}$ (e.g., massless Dirac operators in any space dimension), then this yields examples with the curious property that

$$\sigma_{ess}(\mathbf{D_A}) = \mathbb{C}, \ \text{ i.e., } \ \rho(\mathbf{D_A}) = \emptyset.$$

It so happens that massless Dirac operators $A_\pm$, $A(\,\cdot\,)$ are indeed the prime examples in which we're interested. (Note, $\mathbf{A}_\pm$, $\mathbf{A}$ are self-adjoint, but $\mathbf{D_A}$, of course, is **not**!)

A Glimpse at the Literature on the Model Operator:

Note: $\mathbf{D_A} = (d/dt) + \mathbf{A}$ in $L^2(\mathbb{R}; \mathcal{H})$ is a **model operator**: It arises in connection with **Dirac-type operators** (on compact and noncompact manifolds), the Maslov index, Morse theory (index), Floer homology, winding numbers, Sturm oscillation theory, dynamical systems, evolution operators, etc.

The literature on the **spectral flow and index theory** alone is endless:

M. F. Atiyah, N. Azamov, M.-T. Benameur, B. Booss-Bavnbek,

N. V. Borisov, C. Callias, A. Carey, P. Dodds, S. Dostoglou, A. Floer,

K. Furutani, P. Kirk, M. Lesch, W. Müller, N. Nicolaescu, J. Phillips,

V. Patodi, A. Pushnitski, P. Rabier, A. Rennie, J. Robbin, D. Salamon,

R. Schrader, I. Singer, F. Sukochev, C. Wahl, E. Witten,

K. P. Wojciechowski, etc.

Just scratching the surface ...... apologies for inevitable omissions ......

More Literature on Fredholm (Witten) Indices of the Model Operator:

Just a few selections:

C. Callias, *Axial anomalies and index theorems on open spaces*, Commun. Math. Phys. **62**, 213–234 (1978). **Started an avalanche in supersymmetric QM.**

D. Bolle, F.G., H. Grosse, W. Schweiger, and B. Simon, *Witten index, axial anomaly, and Krein's spectral shift function in supersymmetric quantum mechanics*, J. Math. Phys. **28**, 1512–1525 (1987).

This treats the scalar case when $A(t)$ is a scalar function and hence $\dim(\mathcal{H}) = 1$ (very humble beginnings!). The **Krein–Lifshitz spectral shift function** is **linked to index theory**.

F.G. and B. Simon, *Topological invariance of the Witten index*, J. Funct. Anal. **79**, 91–102 (1988). $\longleftarrow$ **proved topological invariance .....**

In this context, see also,

R. W. Carey and J. D. Pincus, Proc. Symp. Pure Math. 44, 149–161 (1986).

W. Müller, Springer Lecture Notes in Math. Vol. **1244** (1987).

More Literature on Fredholm (Witten) Indices of the Model Operator (contd.):

More references:

S. Dostoglou and D. A. Salamon, *Cauchy–Riemann operators, self-duality, and the spectral flow*, 1st European Congress of Mathematics, Vol. I, Invited Lectures (Part 1), A. Joseph, F. Mignot, F. Murat, B. Prum, R. Rentschler (eds.), Progress Math., Vol. 119, Birkhäuser, Basel, 1994, pp. 511–545.

J. Robbin and D. Salamon, *The spectral flow and the Maslov index*, Bull. London Math. Soc. **27**, 1–33 (1995).

Very influential papers.

A. Pushnitski, *The spectral flow, the Fredholm index, and the spectral shift function*, in *Spectral Theory of Differential Operators: M. Sh. Birman 80th Birthday Collection*, AMS, 2008, pp. 141–155.

This motivated our work in GLMST '11.

Fredholm Indices of the Model Operator:

The following result is proved in **GLMST '11**:

Theorem. (GLMST '11)

Under these hypotheses, and if $\boxed{0 \in \rho(A_+) \cap \rho(A_-)}$ $\Longleftrightarrow$ **Fredholm Property**

$$\operatorname{ind}(\mathbf{D_A}) = \dim(\ker(\mathbf{D_A})) - \dim(\ker(\mathbf{D_A^*})) \qquad \textbf{Fredholm Index}$$

$$= \xi(0_+; \mathbf{H}_2, \mathbf{H}_1) \qquad \mathbf{H}_1 = \mathbf{D_A^*}\mathbf{D_A}, \; \mathbf{H}_2 = \mathbf{D_A}\mathbf{D_A^*}$$

$$= \operatorname{SpFlow}(\{A(t)\}_{t=-\infty}^{\infty}) \qquad \textbf{Spectral Flow}$$

$$= \xi(0; A_+, A_-) \quad \textbf{internal SSF} \qquad A_\pm = A(\pm\infty), \quad 0 \in \rho(A_\pm)$$

$$= \pi^{-1} \lim_{\varepsilon \downarrow 0} \operatorname{Im}\big(\ln\big(\det_{\mathcal{H}}\big((A_+ - i\varepsilon I_{\mathcal{H}})(A_- - i\varepsilon I_{\mathcal{H}})^{-1}\big)\big)\big) \quad \textbf{Path Independence}$$

$\boldsymbol{H}_1 = \boldsymbol{D}_{\boldsymbol{A}}^* \boldsymbol{D}_{\boldsymbol{A}} = -\frac{d^2}{dt^2} +_q \boldsymbol{V}_1, \; \boldsymbol{V}_1 = \boldsymbol{A}^2 - \boldsymbol{A}'$,

"$+_q$" abbreviates the form sum, $\mathbf{D_A} = (d/dt) + \mathbf{A}$ in $L^2(\mathbb{R}; \mathcal{H})$,

$\boldsymbol{H}_2 = \boldsymbol{D}_{\boldsymbol{A}} \boldsymbol{D}_{\boldsymbol{A}}^* = -\frac{d^2}{dt^2} +_q \boldsymbol{V}_2, \; \boldsymbol{V}_2 = \boldsymbol{A}^2 + \boldsymbol{A}'$. $\qquad \mathbf{T} = \int_{\mathbb{R}}^{\oplus} T(t)\, dt$.

Fredholm Indices of the Model Operator (contd.):

Two key elements in the proof: The **Trace identity** and **Pushnitski's formula**

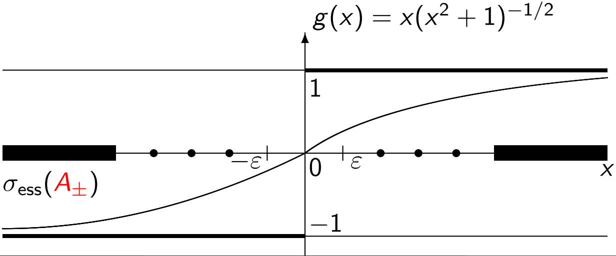
Theorem (Trace Identity).

Given our hypotheses,

$$\operatorname{tr}_{L^2(\mathbb{R};\mathcal{H})}\big((\mathbf{H}_2 - z\,\boldsymbol{I})^{-1} - (\mathbf{H}_1 - z\,\boldsymbol{I})^{-1}\big) = \frac{1}{2z}\operatorname{tr}_{\mathcal{H}}(g_z(A_+) - g_z(A_-)),$$

where $g_z(x) = \frac{x}{\sqrt{x^2-z}}$, $z \in \mathbb{C}\backslash[0,\infty)$, $x \in \mathbb{R}$, a "smoothed-out" sign fct.

By far the biggest and most fascinating headache in this context ......



Fredholm Indices of the Model Operator (contd.):

There's nothing special about **resolvent** differences of $\mathbf{H}_2$ and $\mathbf{H}_1$ on the l.h.s. of the trace identity!

Under appropriate conditions on f one obtains

$$\operatorname{tr}_{L^2(\mathbb{R};\mathcal{H})}\big(f(\boldsymbol{H}_2) - f(\boldsymbol{H}_1)\big) = \operatorname{tr}_{\mathcal{H}}\big(F(A_+) - F(A_-)\big),$$

where F is determined by f via an Abel-type transformation

$$F(\nu) = \frac{\nu}{2\pi} \int_{[\nu^2,\infty)} \frac{[f(\lambda) - f(0)]\, d\lambda}{\lambda(\lambda - \nu^2)^{1/2}}, \text{ resp., by } F'(\nu) = \frac{1}{\pi} \int_{[\nu^2,\infty)} \frac{f'(\lambda)\, d\lambda}{(\lambda - \nu^2)^{1/2}}.$$

Fredholm Indices of the Model Operator (contd.):

A Besov space consideration shows that

$$\big[F(A_+) - F(A_-)\big] \in \mathcal{B}_1(\mathcal{H}) \text{ if}$$

$$(1+\nu^2)^{-3/4} F \in L^2(\mathbb{R}; d\nu), \quad (1+\nu^2)^{3/4} F' \in L^2(\mathbb{R}; d\nu),$$

$$(1+\nu^2)^{9/4} \big| F'' + 3\nu(1+\nu^2)^{-1} F' \big| \in L^2(\mathbb{R}; d\nu),$$

$$\text{and } [g_{-1}(A_+) - g_{-1}(A_-)] \in \mathcal{B}_1(\mathcal{H}), \quad g_{-1}(\nu) = \frac{\nu}{(\nu^2+1)^{1/2}}, \quad \nu \in \mathbb{R}.$$

Here's the corresponding **heat kernel** version:

$$f(\lambda) = e^{-s\lambda}, \quad F(\nu) = -\frac{1}{2} \operatorname{erf}\big(s^{1/2}\nu\big), \;\; s \in (0,\infty),$$

where

$$\operatorname{erf}(x) = \frac{2}{\pi^{1/2}} \int_0^x dy \, e^{-y^2}, \quad x \in \mathbb{R}.$$

Fredholm Indices of the Model Operator (contd.):

Theorem (Pushnitski's Formula, an Abel-Type Transform).

Given our hypotheses,

$$\xi(\lambda; \mathbf{H}_2, \mathbf{H}_1) = \frac{1}{\pi} \int_{-\lambda^{1/2}}^{\lambda^{1/2}} \frac{\xi(\nu; A_+, A_-)\, d\nu}{(\lambda - \nu^2)^{1/2}} \text{ for a.e. } \lambda > 0.$$

Relating the **external** SSF, $\xi(\,\cdot\,; \mathbf{H}_2, \mathbf{H}_1)$ and **internal** SSF, $\xi(\,\cdot\,; A_+, A_-)$.

Recalling,

$$\mathbf{D}_{\mathbf{A}} = (d/dt) + \mathbf{A}, \quad \mathbf{A} = \int_{\mathbb{R}}^{\oplus} A(t)\, dt \text{ in } L^2(\mathbb{R}; \mathcal{H}),$$

$$\mathbf{H}_1 = \mathbf{D}_{\mathbf{A}}^* \mathbf{D}_{\mathbf{A}}, \quad \mathbf{H}_2 = \mathbf{D}_{\mathbf{A}} \mathbf{D}_{\mathbf{A}}^*,$$

$$A(t) \underset{t \pm \infty}{\longrightarrow} A_\pm,$$

etc.

Fredholm Indices of the Model Operator (contd.):

Given our hypotheses, especially, assuming the **Fredholm** case, where A_- and A_+ are **boundedly invertible**, i.e., $0 \in \rho(A_\pm)$. Then,

- $\xi(\lambda; A_+, A_-)$ is constant for a. e. λ near 0.
- $\mathbf{H}_1$ and $\mathbf{H}_2$ have no essential spectrum near zero.
- therefore, $\xi(\lambda; \mathbf{H}_2, \mathbf{H}_1)$ is constant for a.e. $\lambda > 0$ near 0.

By Pushnitski's formula, $\xi(\lambda; A_+, A_-) = \xi(\lambda; \mathbf{H}_2, \mathbf{H}_1)$ for a.e. $\lambda > 0$ near 0.

But $\mathbf{H}_1 = \mathbf{D}_\mathbf{A}^* \mathbf{D}_\mathbf{A}$ and $\mathbf{H}_2 = \mathbf{D}_\mathbf{A} \mathbf{D}_\mathbf{A}^*$ imply:

- $\mathrm{ind}(\mathbf{D}_\mathbf{A}) = \dim(\ker(\mathbf{D}_\mathbf{A})) - \dim(\ker(\mathbf{D}_\mathbf{A}^*)) = \dim(\ker(\mathbf{H}_1)) - \dim(\ker(\mathbf{H}_2))$.

On the other hand, properties of the ξ-function imply:

- $\xi(\lambda; \mathbf{H}_2, \mathbf{H}_1) = \dim(\ker(\mathbf{H}_1)) - \dim(\ker(\mathbf{H}_2))$ for a.e. $\lambda > 0$ near 0_+.

Putting all this together:

- $\boxed{\mathrm{ind}(\mathbf{D}_\mathbf{A}) = \xi(0_+; \mathbf{H}_2, \mathbf{H}_1) = \xi(0; A_+, A_-).}$

Fredholm Indices of the Model Operator (contd.):

The spectral shift function $\xi(\lambda; A_+, A_-)$, $\lambda \in \mathbb{R}$, "roughly" equals

$$\xi(\lambda; A_+, A_-) = \#\,\{\text{eigenvalues of } A(t) \text{ that cross } \lambda \text{ rightward}\} \\ -\#\,\{\text{ eigenvalues of } A(t) \text{ that cross } \lambda \text{ leftward}\}.$$

Corollary.

Index = Spectral flow.

The actual details are a bit involved, and employ the continuity of the path $\{A(t)\}_{t=-\infty}^{\infty}$ w.r.t. the **Riesz metric** $\|g(A_+) - g(A_-)\|_{\mathcal{B}(\mathcal{H})}$, with $g(x) = x(x^2+1)^{-1/2}$ (cf. **M. Lesch '05**).

A complete treatment of spectral flow appeared in **GLMST '11**.

Fredholm Indices of the Model Operator (contd.):

In the end, it boils down to proving

$$\big[g(A_+) - g(A_-)\big] \in \mathcal{B}_1(\mathcal{H}).$$

This is by far the hardest problem in this context since

$$g(+\infty) = 1 \text{ and } g(-\infty) = -1$$

for

$$g(x) = \frac{x}{\sqrt{x^2+1}}, \quad x \in \mathbb{R}.$$

One needs a new technique: Double Operator Integrals (DOI) to show the following:

Main Lemma.

$g(A_+) - g(A_-) = T(K)$, where $T : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ is a bounded operator and

$$K = \overline{(|A_+| + I)^{-1/2}(A_+ - A_-)(|A_-| + I)^{-1/2}} \in \mathcal{B}_1(\mathcal{H}).$$

A 5 min. Course on Double Operator Integrals (DOI):

Daletskij and S. G. Krein (1960'), Birman and Solomyak (1960–70'), Peller, dePagter, Sukochev (1990–05), and others.

Our main goal: Given self-adjoint operators A_- and A_+ and a Borel function f, represent $f(A_+) - f(A_-)$ as a **double Stieltjes integral** with respect to the spectral measures $dE_{A_+}(\lambda)$ and $dE_{A_-}(\mu)$.

A 5 min. Course on DOI (contd.):

If $A_\pm$ are **self-adjoint** matrices in $\mathbb{C}^n$, then $A_+ = \sum_{j=1}^n \lambda_j E_{A_+}(\{\lambda_j\})$ and $A_- = \sum_{k=1}^n \mu_k E_{A_-}(\{\mu_k\})$ imply:

$$
\begin{aligned}
f(A_+) - f(A_-) &= \sum_{j=1}^n \sum_{k=1}^n [f(\lambda_j) - f(\mu_k)] E_{A_+}(\{\lambda_j\}) E_{A_-}(\{\mu_k\}) \\
&= \sum_{j=1}^n \sum_{k=1}^n \frac{f(\lambda_j) - f(\mu_k)}{\lambda_j - \mu_k} E_{A_+}(\{\lambda_j\})(\lambda_j - \mu_k) E_{A_-}(\{\mu_k\}) \\
&= \sum_{j=1}^n \sum_{k=1}^n \frac{f(\lambda_j) - f(\mu_k)}{\lambda_j - \mu_k} \\
&\quad \times E_{A_+}(\{\lambda_j\}) \Bigg(\sum_{j'=1}^n \lambda_{j'} E_{A_+}(\{\lambda_{j'}\}) - \sum_{k'=1}^n \mu_{k'} E_{A_-}(\{\mu_{k'}\}) \Bigg) E_{A_-}(\{\mu_k\}) \\
&= \sum_{j=1}^n \sum_{k=1}^n \frac{f(\lambda_j) - f(\mu_k)}{\lambda_j - \mu_k} E_{A_+}(\{\lambda_j\})(A_+ - A_-) E_{A_-}(\{\mu_k\}).
\end{aligned}
$$

A 5 min. Course on DOI (contd.):

The **Birman–Solomyak formula**:

$$f(A_+) - f(A_-) = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{f(\lambda) - f(\mu)}{\lambda - \mu} \, dE_{A_+}(\lambda) \, (A_+ - A_-) \, dE_{A_-}(\mu).$$

More generally: For a bounded Borel function $\phi(\lambda, \mu)$ we would like to define a **bounded transformer** $T_\phi : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ so that

$$T_\phi(K) = \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(\lambda, \mu) \, dE_{A_+}(\lambda) \, K \, dE_{A_-}(\mu), \quad K \in \mathcal{B}_1(\mathcal{H}).$$

$$T_\phi(K) = \int_{\mathbb{R}} \alpha(\lambda) \, dE_{A_+}(\lambda) \, K \int_{\mathbb{R}} \beta(\mu) \, dE_{A_-}(\mu) \text{ for } \phi(\lambda, \mu) = \alpha(\lambda) \, \beta(\mu),$$

$$T_\phi(K) = \int_{\mathbb{R}} \alpha_s(A_+) \, K \, \beta_s(A_-) \, \nu(s) ds \text{ for } \boxed{\phi(\lambda, \mu) = \int_{\mathbb{R}} \alpha_s(\lambda) \, \beta_s(\mu) \, \nu(s) \, ds,}$$

where α_s, β_s are bounded Borel functions, $\int_{\mathbb{R}} \|\alpha_s\|_\infty \, \|\beta_s\|_\infty \, \nu(s) \, ds < \infty$.

The (**Wiener**) class of such ϕ's is denoted by $\mathfrak{A}_0$.

Back to the Main Lemma:

Recall that $(A_+ - A_-)(A_-^2 + I)^{-1/2} \in \mathcal{B}_1(\mathcal{H})$ by hypotheses.

Interpolation Lemma.

$\overline{K} \in \mathcal{B}_1(\mathcal{H}), \quad K = (A_+^2 + I)^{-1/4}(A_+ - A_-)(A_-^2 + I)^{-1/4}, \;\; \mathrm{dom}(K) = \mathrm{dom}(A_-).$

Consider the function

$$\phi(\lambda,\mu) = (1+\lambda^2)^{1/4}\frac{g(\lambda)-g(\mu)}{\lambda-\mu}(1+\mu^2)^{1/4}, \quad g(x) = x(1+x^2)^{-1/2}.$$

Double Operator Integral Lemma.

$\phi(\lambda,\mu) \in \mathfrak{A}_0$ and thus $T_\phi : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ is bounded. In addition,

$$g(A_+) - g(A_-) = T_\phi(\overline{K}) \text{ and thus } \big[g(A_+) - g(A_-)\big] \in \mathcal{B}_1(\mathcal{H}).$$

Back to the Main Lemma (contd.):

Main Lemma (again).

$g(A_+) - g(A_-) = T(K)$, where $T : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ is a bounded operator and

$$K = \overline{(|A_+| + I)^{-1/2}(A_+ - A_-)(|A_-| + I)^{-1/2}} \in \mathcal{B}_1(\mathcal{H}).$$

Formally:

$$\begin{aligned} T_\phi(K) &= \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(\lambda, \mu)\, dE_{A_+}(\lambda)\, K\, dE_{A_-}(\mu) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{g(\lambda) - g(\mu)}{\lambda - \mu}\, dE_{A_+}(\lambda)\, (A_+ - A_-)\, dE_{A_-}(\mu) \\ &= g(A_+) - g(A_-). \end{aligned}$$

Back to the Main Lemma (contd.):

To see that $\phi \in \mathfrak{A}_0$ we split:

$$\begin{aligned}\phi(\lambda, \mu) &= (1+\lambda^2)^{1/4} \frac{g(\lambda) - g(\mu)}{\lambda - \mu} (1+\mu^2)^{1/4} \\ &= \psi(\lambda, \mu) + \frac{\psi(\lambda, \mu)}{(1+\lambda^2)^{1/2}(1+\mu^2)^{1/2}} + \frac{\lambda \psi(\lambda, \mu) \mu}{(1+\lambda^2)^{1/2}(1+\mu^2)^{1/2}},\end{aligned}$$

where

$$\psi(\lambda, \mu) := \frac{(1+\lambda^2)^{1/4}(1+\mu^2)^{1/4}}{(1+\lambda^2)^{1/2} + (1+\mu^2)^{1/2}} = \zeta\big(\log(1+\lambda^2)^{1/2} - \log(1+\mu^2)^{1/2}\big),$$

$$\zeta(\lambda - \mu) := \big[e^{(\lambda-\mu)/2} + e^{-(\lambda-\mu)/2}\big]^{-1} = \frac{1}{2\pi} \int_{\mathbb{R}} e^{is\lambda} e^{-is\mu}\, \widehat{\zeta}(s)\, ds.$$

Since $\widehat{\zeta} \in L^1(\mathbb{R})$, $\psi \in \mathfrak{A}_0$ due to

$$\psi(\lambda, \mu) = \frac{1}{2\pi} \int_{\mathbb{R}} (1+\lambda^2)^{is/2} (1+\mu^2)^{-is/2}\, \widehat{\zeta}(s)\, ds.$$

Witten Indices, $\dim(\mathcal{H}) < \infty$:

Now we return to the **model operator** $\mathbf{D_A} = (d/dt) + \mathbf{A}$ in $L^2(\mathbb{R}; \mathcal{H})$:

(1) **Special Case:** $\dim(\mathcal{H}) < \infty$.

• Assume A_- is a self-adjoint matrix in $\mathcal{H}$.

• Suppose there exist families of self-adjoint matrices $\{B(t)\}_{t\in\mathbb{R}}$ such that $B(\cdot)$ is locally absolutely continuous on $\mathbb{R}$.

• Assume that $\int_{\mathbb{R}} dt \, \|B'(t)\|_{\mathcal{B}(\mathcal{H})} < \infty$.

Recall, $\mathbf{A} = \mathbf{A}_- + \mathbf{B}$, and $A(t) = A_- + B(t)$, $t \in \mathbb{R}$, with $A(t) \underset{t\to\pm\infty}{\longrightarrow} A_\pm$ in norm.

In the special case $\dim(\mathcal{H}) < \infty$ a **complete picture** emerges:

First, we recall (this has been known for a long time .....):

Lemma.

Under the new set of hypotheses for $\dim(\mathcal{H}) < \infty$, $\boldsymbol{D_A}$ (equivalently, $\boldsymbol{D_A^*}$) is **Fredholm** if and only if $0 \notin \{\sigma(A_+) \cup \sigma(A_-)\}$.

Witten Indices, $\dim(\mathcal{H}) < \infty$ (contd.):

The **non-Fredholm** case if $\dim(\mathcal{H}) < \infty$: We **no longer** assume $0 \notin \{\sigma(A_+) \cup \sigma(A_-)\}$:

Theorem. (A. Carey, F.G., D. Potapov, F. Sukochev, Y. Tomilov '13.)

Assume the new set of hypotheses for $\dim(\mathcal{H}) < \infty$. Then $\xi(\,\cdot\,; \boldsymbol{H}_2, \boldsymbol{H}_1)$ has a continuous representative on the interval $(0, \infty)$, $\xi(\,\cdot\,; A_+, A_-)$ is piecewise constant a.e. on $\mathbb{R}$, the Witten index $W_r(\boldsymbol{D_A})$ exists, and

$$\begin{aligned} W_r(\boldsymbol{D_A}) &= \xi(0_+; \boldsymbol{H}_2, \boldsymbol{H}_1) \\ &= [\xi(0_+; A_+, A_-) + \xi(0_-; A_+, A_-)]/2 \\ &= \tfrac{1}{2}[\#_>(A_+) - \#_>(A_-)] - \tfrac{1}{2}[\#_<(A_+) - \#_<(A_-)]. \end{aligned}$$

In particular, in the finite-dimensional context, $\dim(\mathcal{H}) < \infty$, $W_r(\boldsymbol{D_A})$ is either an **integer**, or a **half-integer** (a **Levinson-type theorem** in scattering theory).

Here $\#_>(A)$ (resp. $\#_<(A)$) denotes the **number of strictly positive** (resp., **strictly negative**) **eigenvalues** of a self-adjoint operator A in $\mathcal{H}$, counting multiplicity.

Details rely on scattering theory (matrix-valued Jost functions and solutions .....).

Witten Indices, $\dim(\mathcal{H}) = \infty$:

(2) **General Case:** $\dim(\mathcal{H}) = \infty$. Back to our **Main Hypotheses:**

- A_- – **self-adjoint** on $\operatorname{dom}(A_-) \subseteq \mathcal{H}$, $\mathcal{H}$ a complex, separable Hilbert space.
- $B(t)$, $t \in \mathbb{R}$, – **closed, symmetric,** in $\mathcal{H}$, $\operatorname{dom}(B(t)) \supseteq \operatorname{dom}(A_-)$.
- There exists a family $B'(t)$, $t \in \mathbb{R}$, – **closed, symmetric,** in $\mathcal{H}$, with

  $\operatorname{dom}(B'(t)) \supseteq \operatorname{dom}(A_-)$, such that

  $B(t)(|A_-| + I_{\mathcal{H}})^{-1}$, $t \in \mathbb{R}$, is **weakly locally a.c.** and for a.e. $t \in \mathbb{R}$,

$$\frac{d}{dt}\big(g, B(t)(|A_-| + I_{\mathcal{H}})^{-1}h\big)_{\mathcal{H}} = \big(g, B'(t)(|A_-| + I_{\mathcal{H}})^{-1}h\big)_{\mathcal{H}}, \quad g, h \in \mathcal{H}.$$

- $B'(\cdot)(|A_-| + I_{\mathcal{H}})^{-1} \in \mathcal{B}_1(\mathcal{H})$, $t \in \mathbb{R}$, $\quad \int_{\mathbb{R}} \big\|B'(t)(|A_-| + I_{\mathcal{H}})^{-1}\big\|_{\mathcal{B}_1(\mathcal{H})}\, dt < \infty$.
- $\big\{(|B(t)|^2 + I_{\mathcal{H}})^{-1}\big\}_{t\in\mathbb{R}}$ and $\big\{(|B'(t)|^2 + I_{\mathcal{H}})^{-1}\big\}_{t\in\mathbb{R}}$ are **weakly measurable.**

Principal objects: $A(t) = A_- + B(t)$, $t \in \mathbb{R}$, and $\mathbf{A} = \int_{\mathbb{R}}^{\oplus} A(t)\, dt$ in $L^2(\mathbb{R}; \mathcal{H})$.

Witten Indices, $\dim(\mathcal{H}) = \infty$ (contd.):

The **non-Fredholm** case if $\dim(\mathcal{H}) = \infty$: Again, we **no longer** assume $0 \notin \{\sigma(A_+) \cup \sigma(A_-)\}$:

A first fact:

For $\varphi \in (0, \pi/2)$ we introduce the sector

$$S_\varphi := \{z \in \mathbb{C} \,|\, |\arg(z)| < \varphi\}.$$

Theorem. (A. Carey, F.G., D. Potapov, F. Sukochev, Y. Tomilov '13.)

Assume the general hypotheses for $\dim(\mathcal{H}) = \infty$ and let $\varphi \in (0, \pi/2)$ be fixed. If 0 is a **right and left Lebesgue point** of $\xi(\,\cdot\,; A_+, A_-)$ (denoted by $\xi_L(0_\pm; A_+, A_-)$), then

$$\begin{aligned} W_r(\boldsymbol{D_A}) &= \lim_{z \to 0,\, z \in \mathbb{C} \setminus S_\varphi} z \operatorname{tr}_{L^2(\mathbb{R};\mathcal{H})} \big((\boldsymbol{H}_2 - z\,\boldsymbol{I})^{-1} - (\boldsymbol{H}_1 - z\,\boldsymbol{I})^{-1} \big) \\ &= - \lim_{z \to 0,\, z \in \mathbb{C} \setminus S_\varphi} z \int_{\mathbb{R}} \frac{\xi(\nu; A_+, A_-)\, d\nu}{(\nu^2 - z)^{3/2}} \\ &= [\xi_L(0_+; A_+, A_-) + \xi_L(0_-; A_+, A_-)]/2. \end{aligned}$$

Witten Indices, $\dim(\mathcal{H}) = \infty$ (contd.):

Naturally, the proof relies on a series of careful estimates employing Pushnitski's formula,

$$\xi(\lambda; \mathbf{H}_2, \mathbf{H}_1) = \frac{1}{\pi} \int_{-\lambda^{1/2}}^{\lambda^{1/2}} \frac{\xi(\nu; A_+, A_-)\, d\nu}{(\lambda - \nu^2)^{1/2}} \text{ for a.e. } \lambda > 0,$$

and the trace identity,

$$\operatorname{tr}_{L^2(\mathbb{R};\mathcal{H})} \big((\mathbf{H}_2 - z\, \boldsymbol{I})^{-1} - (\mathbf{H}_1 - z\, \boldsymbol{I})^{-1} \big) = \frac{1}{2z} \operatorname{tr}_{\mathcal{H}} (g_z(A_+) - g_z(A_-)),$$

where $g_z(x) = \frac{x}{\sqrt{x^2 - z}}$, $\quad z \in \mathbb{C} \backslash [0, \infty)$, $x \in \mathbb{R}$, a "smoothed-out" sign fct.

Neither formula depends on the Fredholm property of $\boldsymbol{D_A}$.

Witten Indices, $\dim(\mathcal{H}) = \infty$ (contd.):

Lebesgue points: Let $f \in L^1_{\text{loc}}(\mathbb{R}; dx)$.

Then $x \in \mathbb{R}$ is a **right Lebesgue point of f** if there exists an $\alpha_+ \in \mathbb{C}$ such that

$$\lim_{h \downarrow 0} \frac{1}{h} \int_x^{x+h} |f(y) - \alpha_+| \, dy = 0. \qquad \text{One then denotes } f_L(x_+) = \alpha_+.$$

Similarly, $x \in \mathbb{R}$ is a **left Lebesgue point of f** if there exists an $\alpha_- \in \mathbb{C}$ such that

$$\lim_{h \downarrow 0} \frac{1}{h} \int_{x-h}^{x} |f(y) - \alpha_-| \, dy = 0. \qquad \text{One then denotes } f_L(x_-) = \alpha_-.$$

Finally, $x \in \mathbb{R}$ is a **Lebesgue point of f** if there exist $\alpha \in \mathbb{C}$ such that

$$\lim_{h \downarrow 0} \frac{1}{2h} \int_{x-h}^{x+h} |f(y) - \alpha| \, dy = 0. \qquad \text{One then denotes } f_L(x_0) = \alpha.$$

That is, $x \in \mathbb{R}$ is a **Lebesgue point of f** if and only if it is a **left and a right Lebesgue point** and $\alpha_+ = \alpha_- = \alpha$.

These definitions are **not** universally accepted, but very common these days.

Witten Indices, $\dim(\mathcal{H}) = \infty$ (contd.):

A second fact:

Theorem. (A. Carey, F.G., D. Potapov, F. Sukochev, Y. Tomilov '13.)

Assume the general hypotheses for $\dim(\mathcal{H}) = \infty$. If 0 is a **right and left Lebesgue point** of $\xi(\,\cdot\,; A_+, A_-)$, then it is a **right Lebesgue point** of $\xi(\,\cdot\,; \boldsymbol{H}_2, \boldsymbol{H}_1)$ and

$$\xi_L(0_+; \boldsymbol{H}_2, \boldsymbol{H}_1) = [\xi_L(0_+; A_+, A_-) + \xi_L(0_-; A_+, A_-)]/2.$$

The proof employs Pushnitski's formula,

$$\xi(\lambda; \mathbf{H}_2, \mathbf{H}_1) = \frac{1}{\pi} \int_{-\lambda^{1/2}}^{\lambda^{1/2}} \frac{\xi(\nu; A_+, A_-)\, d\nu}{(\lambda - \nu^2)^{1/2}} \text{ for a.e. } \lambda > 0,$$

and combines the **right/left Lebesgue point** property of $\xi(\,\cdot\,; A_+, A_-)$ at 0 with Fubini's theorem as follows:

Witten Indices, $\dim(\mathcal{H}) = \infty$ (contd.):

Sketch of Proof.

Since $\chi_{[-\sqrt{\lambda},\sqrt{\lambda}]}(\nu)\frac{1}{\lambda-\nu^2}$ is **even** w.r.t. $\nu \in \mathbb{R}$, and thus

$$\begin{aligned}
&\xi(\lambda; \boldsymbol{H}_2, \boldsymbol{H}_1) - \frac{1}{2}[\xi_L(0_+; A_+, A_-) + \xi_L(0_-; A_+, A_-)] \\
&= \frac{1}{\pi}\int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \frac{\xi(t; A_+, A_-)\, d\nu}{\sqrt{\lambda-\nu^2}} - \frac{1}{2}[\xi_L(0_+; A_+, A_-) + \xi_L(0_-; A_+, A_-)] \\
&= \frac{1}{\pi}\int_0^{\sqrt{\lambda}} \frac{[\xi(\nu; A_+, A_-) - \xi_L(0_+; A_+, A_-)]}{\sqrt{\lambda-\nu^2}}\, d\nu \\
&\quad + \frac{1}{\pi}\int_0^{\sqrt{\lambda}} \frac{[\xi(-\nu, A_+, A_-) - \xi_L(0_-; A_+, A_-)]}{\sqrt{\lambda-\nu^2}}\, d\nu.
\end{aligned}$$

Next, let 0 be a **right** and a **left Lebesgue point** of $\xi(\,\cdot\,; A_+, A_-)$, then abbreviating

$$f_\pm(\nu) := \xi(\pm\nu; A_+, A_-) - \xi_L(0_\pm; A_+, A_-), \quad \nu \in \mathbb{R},$$

and applying Fubini's theorem yields,

Witten Indices, $\dim(\mathcal{H}) = \infty$ (contd.):

$$\lim_{h\downarrow 0+} \frac{1}{h}\int_0^h \bigg| \xi(\lambda; \boldsymbol{H}_2, \boldsymbol{H}_1) - \frac{1}{2}[\xi_L(0_+; A_+, A_-) + \xi_L(0_-; A_+, A_-)] \bigg| \, d\lambda$$

$$= \lim_{h\downarrow 0+} \frac{1}{\pi h}\int_0^h \bigg| \int_0^{\sqrt{\lambda}} \frac{[f_+(\nu) + f_-(\nu)] \, d\nu}{\sqrt{\lambda - \nu^2}} \bigg| d\lambda$$

$$\leq \lim_{h\downarrow 0+} \frac{1}{\pi h}\int_0^h \bigg(\int_0^{\sqrt{\lambda}} \frac{[|f_+(\nu)| + |f_-(\nu)|] \, d\nu}{\sqrt{\lambda - \nu^2}} \bigg) d\lambda$$

$$= \lim_{h\downarrow 0+} \frac{1}{\pi h}\int_0^{\sqrt{h}} [|f_+(\nu)| + |f_-(\nu)|] \bigg(\int_{\nu^2}^h \frac{d\lambda}{\sqrt{\lambda - \nu^2}} \bigg) d\nu$$

$$= \lim_{h\downarrow 0+} \frac{2}{\pi h}\int_0^{\sqrt{h}} [|f_+(\nu)| + |f_-(\nu)|] \sqrt{h - \nu^2} \, d\nu$$

$$= \lim_{h\downarrow 0+} \frac{2}{\pi \sqrt{h}}\int_0^{\sqrt{h}} [|f_+(\nu)| + |f_-(\nu)|] \sqrt{1 - [\nu^2/h]} \, d\nu$$

$$\leq \lim_{h\downarrow 0+} \frac{2}{\pi \sqrt{h}}\int_0^{\sqrt{h}} [|f_+(\nu)| + |f_-(\nu)|] \, d\nu = 0 \quad \text{by the } \textbf{right/left L-point} \text{ hyp.}$$

Witten Indices, $\dim(\mathcal{H}) = \infty$ (contd.):

Combining these results yields:

Theorem. (A. Carey, F.G., D. Potapov, F. Sukochev, Y. Tomilov '13.)

Assume the general hypotheses for $\dim(\mathcal{H}) = \infty$ and that 0 is a **right and left Lebesgue point** of $\xi(\,\cdot\,; A_+, A_-)$ (and hence a **right Lebesgue point** of $\xi(\,\cdot\,; \boldsymbol{H_2}, \boldsymbol{H_1})$). Then, for fixed $\varphi \in (0, \pi/2)$,

$$\begin{aligned} W_r(\boldsymbol{D_A}) &= \lim_{z \to 0,\, z \in \mathbb{C}\backslash S_\varphi} z \operatorname{tr}_{L^2(\mathbb{R};\mathcal{H})} \big((\boldsymbol{H_2} - z\,\boldsymbol{I})^{-1} - (\boldsymbol{H_1} - z\,\boldsymbol{I})^{-1} \big) \\ &= \xi_L(0_+; \boldsymbol{H_2}, \boldsymbol{H_1}) \\ &= -\lim_{z \to 0,\, z \in \mathbb{C}\backslash S_\varphi} z \int_{\mathbb{R}} \frac{\xi(\nu; A_+, A_-)\, d\nu}{(\nu^2 - z)^{3/2}} \\ &= [\xi_L(0_+; A_+, A_-) + \xi_L(0_-; A_+, A_-)]/2. \end{aligned}$$

Applications to Massless Dirac-Type Operators:

(I) The case $\mathcal{H} = L^2(\mathbb{R})$:

Hypothesis

Suppose the real-valued functions ϕ, θ satisfy

$$\phi \in AC_{\rm loc}(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}), \ \phi' \in L^\infty(\mathbb{R}),$$
$$\theta \in AC_{\rm loc}(\mathbb{R}) \cap L^\infty(\mathbb{R}), \ \theta' \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}),$$
$$\lim_{t\to\infty} \theta(t) = 1, \ \lim_{t\to-\infty} \theta(t) = 0.$$

Given this hypothesis one introduces the family of self-adjoint operators $A(t)$, $t \in \mathbb{R}$, in $L^2(\mathbb{R})$,

$$A(t) = -i\frac{d}{dx} + \theta(t)\phi, \quad \mathrm{dom}(A(t)) = W^{1,2}(\mathbb{R}), \quad t \in \mathbb{R},$$

with asymptotes $A_\pm$ in $L^2(\mathbb{R})$ as $t \to \pm\infty$,

$$A_+ = -i\frac{d}{dx} + \phi, \quad A_- = -i\frac{d}{dx}, \quad \mathrm{dom}(A_\pm) = W^{1,2}(\mathbb{R}).$$

Appls. to Massless Dirac-Type Operators (contd.):

Introduce the operator d/dt in $L^2\big(\mathbb{R}; dt; L^2(\mathbb{R}; dx)\big)$ by

$$\left(\frac{d}{dt}f\right)(t) = f'(t) \text{ for a.e. } t \in \mathbb{R},$$
$$f \in \operatorname{dom}(d/dt) = \big\{g \in L^2\big(\mathbb{R}; dt; L^2(\mathbb{R})\big) \,\big|\, g \in AC_{\mathrm{loc}}\big(\mathbb{R}; L^2(\mathbb{R})\big), \; g' \in L^2\big(\mathbb{R}; dt; L^2(\mathbb{R})\big)\big\}$$
$$= W^{1,2}\big(\mathbb{R}; dt; L^2(\mathbb{R}; dx)\big).$$

Turning to the pair $(\boldsymbol{H}_2, \boldsymbol{H}_1)$ and identifying

$$L^2\big(\mathbb{R}; dt; L^2(\mathbb{R}; dx)\big) = L^2(\mathbb{R}^2; dtdx) \equiv L^2(\mathbb{R}^2),$$

we introduce the model operator $\boldsymbol{D_A}$ in $L^2(\mathbb{R}^2)$ by

$$\boldsymbol{D_A} = \frac{d}{dt} + \boldsymbol{A}, \quad \operatorname{dom}(\boldsymbol{D_A}) = W^{1,2}(\mathbb{R}^2),$$

with

$$\boldsymbol{D_A^*} = -\frac{d}{dt} + \boldsymbol{A}, \quad \operatorname{dom}(\boldsymbol{D_A^*}) = W^{1,2}(\mathbb{R}^2).$$

Appls. to Massless Dirac-Type Operators (contd.):

This finally yields

$$\begin{aligned} \boldsymbol{H}_1 = \boldsymbol{D}_{\boldsymbol{A}}^* \boldsymbol{D}_{\boldsymbol{A}} &= -\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - 2i\theta(t)\phi(x)\frac{\partial}{\partial x} \\ &\quad - \theta'(t)\phi(x) - i\theta(t)\phi'(x) + \theta^2(t)\phi(x)^2, \\ \boldsymbol{H}_2 = \boldsymbol{D}_{\boldsymbol{A}} \boldsymbol{D}_{\boldsymbol{A}}^* &= -\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - 2i\theta(t)\phi(x)\frac{\partial}{\partial x} \\ &\quad + \theta'(t)\phi(x) - i\theta(t)\phi'(x) + \theta^2(t)\phi(x)^2, \\ \operatorname{dom}(\boldsymbol{H}_1) &= \operatorname{dom}(\boldsymbol{H}_2) = W^{2,2}(\mathbb{R}^2). \end{aligned}$$

Theorem

For (Lebesgue) a.e. $\lambda > 0$ and a.e. $\nu \in \mathbb{R}$,

$$\xi(\lambda; \boldsymbol{H}_2, \boldsymbol{H}_1) = \xi(\nu; A_+, A_-) = \frac{1}{2\pi} \int_{\mathbb{R}} dx\, \phi(x).$$

Appls. to Massless Dirac-Type Operators (contd.):

Note. Both ξ's are **constant**! **WHY?**

This has to do with the simple fact A_- generates translations, that is,

$$\big(e^{\pm itA_-}u\big)(x) = u(x \pm t), \quad u \in L^2(\mathbb{R}),$$

and introducing U_+, the unitary operator in $L^2(\mathbb{R})$ of multiplication by

$$U_+ = e^{-i\int_0^x dx'\,\phi(x')},$$

one obtains the unitary equivalence of A_- and A_+,

$$e^{\pm itA_+} = U_+ e^{\pm itA_-} U_+^{-1},$$

and similarly, introducing the unitary operator $U(t)$ of multiplication in $L^2(\mathbb{R})$ by

$$U(t) = e^{-i\theta(t)\int_0^x dx'\,\phi(x')}, \quad t \in \mathbb{R},$$

one obtains

$$A(t) = U(t) A_- U(t)^{-1}, \quad t \in \mathbb{R}.$$

Appls. to Massless Dirac-Type Operators (contd.):

Theorem

The resolvent regularized Witten index $W_r(\boldsymbol{D_A})$ exists and equals

$$W_r(\boldsymbol{D_A}) = \xi(0_+; \boldsymbol{H}_2, \boldsymbol{H}_1) = \xi(0; A_+, A_-) = \frac{1}{2\pi} \int_{\mathbb{R}} dx\, \phi(x).$$

All of this quickly extends to the case where ϕ is an $m \times m$ matrix, $m \in \mathbb{N}$.

This settles the $(1+1)$-dimensional case (i.e., variables $(t, x) \in \mathbb{R}^2$), where $\mathcal{H} = L^2(\mathbb{R})$ and $A(\,\cdot\,), A_\pm$ are one-dimensional, massless Dirac-type operators.

Appls. to Massless Dirac-Type Operators (contd.):

(*II*) The case $\mathcal{H} = \big[L^2(\mathbb{R})\big]^N$: It took 7 years to make real progress on multi-dimensional situations. At this point we think we can handle the case where A_- is a **massless, *n*-dimensional Dirac-type operator** in $\big[L^2(\mathbb{R}^n)\big]^N$ of the type,

$$A_- = \boldsymbol{\alpha} \cdot \boldsymbol{P} = \sum_{j=1}^{n} \alpha_j P_j, \quad \operatorname{dom}(A_-) = \big[H^1(\mathbb{R}^n)\big]^N,$$

with $\boldsymbol{P} = -i\nabla$ denoting the momentum operator in $\mathbb{R}^n$ with components P_j, $1 \leq j \leq n$,

$$\boldsymbol{P} = (P_1, \dots, P_n), \quad P_j = \frac{1}{i}\frac{\partial}{\partial x_j}, \quad 1 \leq j \leq n.$$

Here $N = 2^{\lfloor (n+1)/2 \rfloor}$, $n \in \mathbb{N}$, and α_j, $1 \leq j \leq n$, $\alpha_{n+1} := \beta$, denote $n+1$ anti-commuting Hermitian $N \times N$ matrices with squares equal to I_N:

$$\alpha_j^* = \alpha_j, \quad \alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{j,k} I_N, \quad 1 \leq j, k \leq n+1$$

(where $\lfloor \cdot \rfloor$ denotes the floor function on $\mathbb{R}$, that is, $\lfloor x \rfloor$ characterizes the largest integer less or equal to $x \in \mathbb{R}$).

Appls. to Massless Dirac-Type Operators (contd.):

We note in passing that the corresponding massive free Dirac operator in $\big[L^2(\mathbb{R}^n)\big]^N$ associated with the mass parameter $m > 0$ then would be of the form

$$A_-(m) = A_- + m\,\beta = \boldsymbol{\alpha} \cdot \boldsymbol{P} + m\,\beta, \quad m > 0, \; \beta = \alpha_{n+1}.$$

The asymptote A_+ in $\big[L^2(\mathbb{R}^n)\big]^N$ then is of the form

$$A_+ = A_- + Q = \boldsymbol{\alpha} \cdot \boldsymbol{P} + Q, \quad \mathrm{dom}(A_+) = \mathcal{H}^1,$$

where $Q = \{Q_{\ell,m}\}_{1 \leq \ell,m \leq N}$ is a self-adjoint, $N \times N$ matrix-valued electrostatic potential satisfying for some fixed $\rho > 1$,

$$Q \in \big[L^\infty(\mathbb{R}^n)\big]^{N \times N}, \quad |Q_{\ell,m}(x)| \leq C[1 + |x|]^{-\rho}, \; x \in \mathbb{R}^n, \; 1 \leq \ell, m \leq N.$$

Similarly, the path $\{A(t)\}_{t \in \mathbb{R}}$ in $\big[L^2(\mathbb{R}^n)\big]^N$ reads,

$$A(t) = A_- + \theta(t) Q, \quad \mathrm{dom}(A(t)) = \big[H^1(\mathbb{R}^n)\big]^N, \; t \in \mathbb{R},$$

with $\lim_{t \to \infty} \theta(t) = 1$, $\lim_{t \to -\infty} \theta(t) = 0$, etc.

A Bit of Literature:

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More material is in preparation.