Isolated singularities in analytic spaces

José Seade

These are notes for my lectures at the meeting “On the Geometry and Topology of Singularities”, in honor of the 60th Birthday of Lê Dũng Tráng. The aim is to give an introduction to some aspects of singularity theory. The topics selected for this discussion were chosen having also the purpose of explaining to the participants of the School some of the material that will be used in other lecture courses.

The notes are divided into four sections, corresponding to my 4 lectures. Section one discusses the local conical structure of analytic sets. Section two looks more carefully at the case of complex surface singularities, and explains their relationship with plumbing and Waldhausen manifolds.

Section three discusses Milnor’s fibration theorem for holomorphic functions.

Finally, in section four we look at additional geometric structures one has on isolated singularity-germs, namely the contact, $Spin^c$ and (with some restrictions) spin structures on the link. We begin by explaining briefly what is the meaning of having these structures.

1 The local conical structure of analytic sets

Let us begin with an example; consider the Pham-Brieskorn polynomial

$$f(\mathbf{z}) = z_0^{a_0} + \cdots + z_n^{a_n}, \quad a_i > 1,$$

where $\mathbf{z} = (z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$. It is clear that the origin $\mathbf{0} \in \mathbb{C}^{n+1}$ is the only critical point of $f$, so the fibres $V_t = f^{-1}(t)$ are all complex n-manifolds for $t \neq 0$ and $V = f^{-1}(0)$ is a complex hypersurface with an isolated singularity at $\mathbf{0}$.
We want to study the topology of $V$ and of the $V_i$'s. For this let $d$ be the least common multiple of the $a_i$ and define an action $\Gamma$ of the non-zero complex numbers $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$ by:

$$\lambda \cdot (z_0, \cdots, z_n) \mapsto (\lambda^{d/a_0}z_0, \cdots, \lambda^{d/a_n}z_n).$$

Notice this action satisfies:

$$f(\lambda \cdot (z_0, \cdots, z_n)) = \lambda^d \cdot f(z_0, \cdots, z_n).$$

Hence $V$ is an invariant set of the action and one has the following property:

**Property 1.1** Restricting the action to $t \in \mathbb{R}^+$ we get a real analytic flow $\{\Phi_t\}$ (or a vector field) on $\mathbb{C}^{n+1}$ whose orbits are real lines (arcs) which converge to 0 when $t$ tends to 0, they escape to $\infty$ when $t \to \infty$, being transversal to all spheres around 0, and they leave $V$ invariant (i.e., $V$ is union of orbits).

Notice that this flow defines a 1-parameter group of diffeomorphisms of $\mathbb{C}^{n+1}$ that preserve $V$. In fact we can do more: consider the unit sphere $\mathbb{S}^{2n+1}$, and let $\mathbb{S}_r$ be some other sphere around the origin, of positive radius $r < 1$. Now define a map

$$\mathbb{S}^{2n+1} \xrightarrow{\phi_r} \mathbb{S}_r,$$

as follows: given a point $x \in \mathbb{S}^{2n+1}$, let it flow by $\{\Phi_t\}$ until it reaches the sphere $\mathbb{S}_r$. This is obviously a diffeomorphism. Furthermore, since $V$ is union of $\Gamma$-orbits, the intersection $K_1 = V \cap \mathbb{S}^{2n+1}$ is mapped diffeomorphically onto $K_r = V \cap \mathbb{S}_r$. Thus we get a diffeomorphism of pairs $(\mathbb{S}^{2n+1}, K_1) \cong (\mathbb{S}_r, K_r)$. Doing this for all $r$ within $(0,1)$ we get a 1-parameter family of diffeomorphisms that shrinks the sphere more and more, converging to the origin, and this family preserves the intersections of $V$ with the various spheres.

Thus we arrive to the following:

**Theorem 1.2** The variety $V$ intersects transversally every $(2n+1)$-sphere $\mathbb{S}_r$ around the origin; hence the intersection $K_r = V \cap \mathbb{S}_r$ is a smooth manifold of real dimension $2n - 1$ embedded as a codimension 2 submanifold of $\mathbb{S}_r$. Furthermore, for each $r$, $0 < r < 1$, we have a diffeomorphism of pairs $(\mathbb{S}^{2n+1}, K_1) \cong (\mathbb{S}_r, K_r)$. In fact, if $\mathbb{B}$ is the unit ball in $\mathbb{C}^{n+1}$, then the pair $(\mathbb{B} \setminus \{0\}, V \setminus \{0\} \cap \mathbb{B})$ is diffeomorphic to the cylinder over $(\mathbb{S}^{2n+1}, K_1)$, and $(\mathbb{B}, V \cap \mathbb{B})$ is homeomorphic to the cone over $(\mathbb{S}^{2n+1}, K_1)$, with vertex at 0.
In particular, the diffeomorphism type of the manifold $K_r$ does not depend on the choice of the sphere. This manifold is called the link of the singularity. It determines the topology of $V$; and the way $K_r$ is embedded in the sphere determines the embedding of $V$ in $\mathbb{C}^{n+1}$. These manifolds are nowadays know as Brieskorn manifolds, because E. Brieskorn [9, 10] studied them thoroughly, obtaining remarkable results about their topology when $n > 3$. There are also important results in this direction by Hirzebruch, Milnor and others (see for instance Chapter I in [58] for more about the topology of these manifolds).

However, the essence of 1.2 was already known to K. Brauner (1928) in
the case of two variables, and we briefly explain his results below in this section.

Theorem 1.2 is the paradigm of the following general result:

Every (real or complex) analytic space is locally a cone around each of its points.

The rest of this section is devoted to discussing this statement.

Let us consider first the isolated singularity case. The arguments in general are essentially the same, just more complicated technically.

Let \( V \) be a real or complex analytic space, of real dimension \( d > 0 \), and let \( 0 \) be an isolated singular point of \( V \). For simplicity we assume \( V \) is equidimensional and locally irreducible at \( 0 \). We assume also that \( V \) is embedded in some affine space \( \mathbb{R}^n \) and the singular point is the origin.

The aim is to construct a vector field on a sufficiently small ball \( B_r \) around \( 0 \) having the same properties as the above vector field (or \( \phi \)), namely:

i) It is everywhere transversal to all the spheres in \( B_r \) centered at \( 0 \). Thus its integral lines are transversal to all the spheres and converge to \( 0 \).

ii) It has \( V \) as an invariant set, i.e., \( V \) is union of orbits (integral lines).

We do it in two steps. First we consider the function \( \tilde{r} : \mathbb{R}^n \to [0, \infty) \) defined by \( x \mapsto \|x\|^2 \), whose gradient vector field \( \nabla \tilde{r} \) satisfies condition (i) above. The next step is to “adapt” this vector field to \( V \) in a neighbourhood of \( 0 \), so that it becomes tangent to \( V \setminus \{0\} \). For this we use the Curve Selection Lemma of Milnor:

**Lemma 1.3** (Curve Selection Lemma) Let \( U \) be an open neighbourhood of \( 0 \) in \( \mathbb{R}^n \) and let \( f_1, \ldots, f_k, g_1, \ldots, g_s \) be real analytic functions on \( U \) such that \( 0 \) is in the closure of the semi-analytic set:

\[
Z := \{ x \in U \mid f_1(z) = \ldots = f_k(x) \text{ and } g_i(x) > 0, \forall i = 1, \ldots, s \}
\]

Then there exists a real analytic curve \( \gamma : [0, \delta) \to U \) with \( \gamma(0) = 0 \) and \( \gamma(t) \in Z \) for all \( t \in (0, \delta) \).

In practice, \( f_1, \ldots, f_k \) are the functions that define \( V \), and \( g_1, \ldots, g_s \) define a semianalytic set \( H \) in \( V \), containing \( 0 \) in its closure. The lemma says that if that happens, then there is a whole analytic curve in \( H \) converging to \( 0 \).

This lemma is extremely useful, and using it one can easily prove the following lemma.
Lemma 1.4 Let \( g : V \cap U \to [0, \infty) \) be the restriction of a real analytic function \( \tilde{g} \) defined on \( U \), such that \( g^{-1}(0) = \{0\} \). Then \( \{0\} \) is not an accumulation point of critical values of \( g \) on \( V \cap U \setminus \{0\} \).

We refer to Milnor’s book [38] for the proof of the Curve Selection Lemma; he does it in the algebraic category, but his proof works in general with minor (obvious) modifications. And we refer to Looijenga’s book [34, p. 22] for a direct proof lemma 1.4 using the Curve Selection Lemma. In the complex analytic context, Lemma 1.4 is a particular case of the Bertini-Sard theorem. Milnor, in his book, proved this lemma for algebraic maps using a general (say “finiteness”) result of Whitney about the number of components of algebraic sets.

Now we return to our function \( \tilde{r} : \mathbb{R}^n \to [0, \infty) \) defined by \( x \mapsto \|x\|^2 \). The lemma above implies that there are no critical points of its restriction \( r = \tilde{r}|_V \) sufficiently near \( 0 \). Notice that the critical points of \( r \) on \( V \setminus \{0\} \) are the points where \( V \setminus \{0\} \) is tangent to a level surface of \( \tilde{r} \), i.e., tangent to a sphere around \( 0 \). Thus we get that the gradient vector field \( \nabla r \) of \( r \) has no critical points on \( B_r \cap (V \setminus \{0\}) \), for a sufficiently small ball \( B_r \) around \( 0 \). Hence \( \nabla r \) is transversal to all the spheres contained in \( B_r \) and it is tangent to \( V \setminus \{0\} \).

Let us now extend (parallel extension) \( \nabla r \) to a neighbourhood \( U_1 \) of \( V \cap B_r \setminus \{0\} \) in \( B_r \). We denote this extension by \( \nabla \tilde{r} \), and take a partition of unity to glue the vector fields \( \nabla \tilde{r} \) on \( U_1 \) and \( \nabla \tilde{r} \) on \( B_r \setminus V \).

The resulting vector field is singular only at \( 0 \) and satisfies properties (i) and (ii) above. We denote this vector field by \( v_{\text{rad}} \) and call it a radial vector field for \( V \) at \( 0 \).

Using this vector field we can now easily conclude that \( V \) has a local conical structure near \( 0 \). The manifold \( K = V \cap S_\varepsilon \), for \( \varepsilon > 0 \) sufficiently small, is called the link of the singularity. One has Milnor’s theorem:

**Theorem 1.5** The link \( K \) is a smooth manifold of dimension \( d - 1 \), whose diffeomorphism type is independent of the choice of \( \varepsilon \), and the pair \( (B_\varepsilon, B_\varepsilon \cap V) \) is homeomorphic to the cone over \( (S_\varepsilon, S_\varepsilon \cap V) \).

Now we consider the general case of (possibly) non-isolated singularities. For this we need to use Whitney stratifications, that will be studied in David Trottmann’s course (in this School). Here we recall briefly what this means (see for instance [33] for more on the topic). Let \( V \) be as above, a real
A stratification of $V$ means a locally finite partition of $V$ into subsets $V_\alpha$, called strata, such that:

i) Each $V_\alpha$ is a non-singular subanalytic space (in particular a manifold).

ii) The closure $\overline{V}_\alpha$ of each stratum in $V$, and the set $\overline{V}_\alpha \setminus V_\alpha$ are subanalytic spaces.

iii) If $V_\beta \cap \overline{V}_\alpha \neq \emptyset$ then $V_\beta \subset \overline{V}_\alpha$.

The stratification is said to be Whitney regular, or simply a Whitney stratification, if it further satisfies the,

**Whitney condition:** for all strata $V_\alpha$, $V_\beta$ such that $V_\beta \subset \overline{V}_\alpha$, and for each sequence $(x_n, y_n)$ of points in $V_\alpha \times V_\beta$ converging to a point $(y, y)$ in $V_\beta \times V_\beta$, for which exists the limit $T$ of the tangent spaces $T_{x_n}$ (tangent to the stratum), and also exists the limit $\mathcal{L}$ of the secant lines $x_n y_n$, one has an inclusion $\mathcal{L} \subset T$.

A theorem of Whitney (in the complex analytic case) and Hironaka says that given $V$ as above, and an arbitrary locally finite family $\{A_\beta\}$ of subanalytic sets of $V$, there is a Whitney stratification of $V$ for which each $A_\beta$ is union of strata. In this article, $\{A_\beta\}$ is the singular set of $V$, possibly intersected with a small ball around the singularity we are looking at.

Let us now return to our study of the local conical structure of analytic sets. We need the following definitions.
Definition 1.6  Let $V$ be a real analytic space equipped with a Whitney stratification $\{V_\alpha\}$. A stratified vector field on $V$ means a continuous section $v$ of the tangent bundle $T\mathbb{R}^n|_V$ such that at each point $x \in V$, the vector $v(x)$ is contained in the stratum that contains $x$.

Definition 1.7  A stratified vector field, defined on a neighbourhood of a point $p \in V$, is radial at $p$ if it is transversal to the intersection of $V$ with all sufficiently small spheres in $\mathbb{R}^n$ centered at $p$.

One has the following lemma; the first part of it is due to M. H. Schwartz [53, 54] in the complex analytic case, but her arguments (the radial extension technique) also work in the real analytic setting.

Lemma 1.8  Let $V$ be as above, equipped with a Whitney stratification $\{V_\alpha\}$, and let $p$ be a point in $V$. Then there exist a stratified radial vector field on a neighbourhood of $p$ in the ambient space, whose solutions are transversal to all sufficiently small spheres around $p$, and converge to $p$.

The M. H. Schwartz’ technique of radial extension allows us to construct a stratified, radial vector field on a neighbourhood of each point of $V$. To get it to be integrable and so that its solutions are transversal to all sufficiently small spheres around $p$ (and converge to $p$), one needs some extra work: as in the isolated singularity case, the Curve Selection lemma allows us to prove this claim on each stratum; then one uses the Whitney conditions to glue all these vector fields into a single one, having the properties we want.

We summarize this discussion in the following well-known theorem (see for instance [33]).

Theorem 1.9  Let $V$ be a real analytic space, $p$ a point in $V$, and $\{V_\alpha\}$ a Whitney stratification of $V$. Then:

i) Every sufficiently small sphere $S_r(p)$ centered at $p$ meets every Whitney stratum transversally.

ii) For $r > 0$ sufficiently small, the pair $(B_r(p), B_r(p) \cap V)$ is homeomorphic to the cone over the pair $(S_r(p), S_r(p) \cap V)$. Thus,

iii) the homeomorphism type of the intersection $V \cap S_r(p)$ does not depend on the choice of the sphere (provided this is small enough).
As before, the variety $K_r = S_r(p) \cap V$ is called the link of $p$ in $V$. It determines the topology of $V$ near $p$, and its embedding in $S_r(p)$ determines (locally) the embedding of $V$ in the ambient space.

To finish this section, let us envisage briefly the cases of low dimensional complex analytic varieties.

Suppose first $V$ has complex dimension 1. Then its link $K$ is a union of circles, one for each branch of $V$. For instance, if $V$ is defined by a single equation $f : \mathbb{C}^2 \to \mathbb{C}$ and

$$f = \prod_{i=1}^{s} f_i^{n_i},$$

is its decomposition into irreducible factors, then $V$ has $s$ irreducible components, or branches. The link $K$ is then a union of $s$ knots in a 3-sphere $S^3$. The study of this type of knots is the topic of the course given by Anne Pichon in this School. For instance, if $f$ is the polynomial $z_1^p + z_2^q$ with $p, q$ being relative prime, then $V$ is irreducible and its link $K$ is a torus knot of type $(p, q)$, i.e. it is contained in a torus $S^1 \times S^1$ wrapped so that it goes around the torus giving $p$ turns in one direction and $q$ turns in the other direction. The cases $(2, 5)$ and $(3, 4)$ are depicted in Figure 3.

![Figure 3: Toral knots of types (2, 5) and (3, 4).](image)

If $V$ has complex dimension 1 and it has an isolated singularity at a point 0, then the link $K$ is an oriented 3-manifold, which is a graph manifold. This
situation will be the topic of next lecture, and much more on the subject will be said in the course of Walter Neumann, next week in this School.
2 On the Topology of Complex Surface Singularities

In this section we study germs of complex analytic surfaces with an isolated singularity.

Consider first a singular (reduced) complex curve $C$ in some smooth complex surface $X$. An important result for plane curves, due to Max Noether (1883), is that by a finite sequence of blowing ups we can always resolve the singularities of $C$. Let us explain this with a little more care (see [5] for a clear account on the subject). Given a smooth point $x$ in a complex surface $X$, take local coordinates so that we identify the germ of $X$ at $x$ with that of $\mathbb{C}^2$ at 0. Let us take a small disc $U$ around $x$ and consider the map:

$$
\gamma : U - \{x\} \rightarrow \mathbb{C}P^1
$$

which associates to each $y \in U - \{x\}$ the point in $\mathbb{C}P^1$ represented by the line determined by $x$ and $y$. The graph of $\gamma$ is an analytic subset of $(U - \{x\}) \times \mathbb{C}P^1$, whose closure

$$
\tilde{X} = \overline{\text{graph}(\gamma)} \subset (U - \{x\}) \times \mathbb{C}P^1
$$

turns out to be a smooth complex surface. Notice that $\tilde{X}$ is obtained by removing $x$ from $X$ and replacing it by the limits of lines converging to $x$. Thus we have replaced $x$ by a copy of $\mathbb{C}P^1$. There is a projection map $\pi : \tilde{X} \rightarrow X$ which is biholomorphic away from $E \cong \mathbb{C}P^1 = \pi^{-1}(x)$. This transformation is called the blow up of $X$ at $x$ (in the literature this is called sometimes a $\sigma$-process or a monoidal transformation).

Now, given the reduced (maybe reducible) singular curve $C \subset X$, with $X$ a smooth complex surface, let $x$ be a point in $C_{\text{sing}}$, the singular set of $C$, and look at the blow up of $X$ at $x$, $\pi : \tilde{X} \rightarrow X$. The closure $\pi^{-1}(C - x)$ in $\tilde{X}$ is called the proper (or strict) transform of $C$ under the blow up, and denoted $\tilde{C}$. Notice that $\tilde{C}$ is obtained by removing $x$ from $C$ and replacing it by the limits of lines which are tangent to $C - \{x\}$. This curve $\tilde{C}$ is analytic in $\tilde{X}$ and projects to $C$ under $\pi$; this curve may still be singular, but somehow its singularities are simpler. We may now repeat the process, choosing a singular point in $\tilde{C}$, blowing up $\tilde{X}$ at this point to get $\pi_2 : \tilde{X}_2 \rightarrow \tilde{X}$ and then consider the proper transform of $C$ in $\tilde{X}_2$, which is the closure of $(\pi_2 \circ \pi)^{-1}(C - x)$, and so on. The theorem is (see [3, II.7.1] for a short proof):

**Theorem 2.1** Let $X$ be a smooth complex surface and $C \subset X$ an embedded reduced curve. Then there is a smooth complex surface $Y$ and a proper map $\pi : Y \rightarrow X$.
\( \tau : Y \rightarrow X \) obtained by a finite sequence of blow ups, such that the proper transform \( \tilde{C} \) of \( C \) in \( Y \) is smooth.

The curve \( E = \tau^{-1}(C) \), is called the total transform of \( C \) in \( Y \). It consists of the proper transform \( \tilde{C} \) and the divisor \( \tau^{-1}(C_{\text{sing}}) \). The theorem above can be refined to the following theorem, which will be used later in the text.

**Theorem 2.2** Let \( X \) and \( C \) be as above. Then by performing finitely many blow ups more, if necessary, we can assume that the whole total transform \( E = \tau^{-1}(C) \) of \( C \) in \( Y \) has only ordinary normal double points as singularities, and these are all away from the proper transform \( \tilde{C} \) of \( C \).

We recall that an ordinary normal double point is locally defined by the equation \( xy = 0 \).

Now consider the germ \((V, 0)\) of a normal complex surface singularity. The following important theorem has a long history. This was first stated by Jung but his proof was not complete, and was completed later by Hirzebruch. The first complete proof of 2.3 is due to O. Zariski; this was done using blowing ups and it was based on a previous proof by R. Walker (1935) which was not correct either. We refer to [3, Ch. III] for a proof. This result was later generalized by Hironaka to all dimensions and for (real or complex) analytic spaces with arbitrary singularities.

**Theorem 2.3** Let \((V, 0)\) be a normal complex surface singularity. For simplicity assume it is defined in a sufficiently small ball around the origin in some \( \mathbb{C}^n \), so that \( V^* = V - 0 \) is non-singular. Then there exists a non-singular complex surface \( \tilde{V} \) and a proper analytic map \( \pi : \tilde{V} \rightarrow V \), such that:

i) the inverse image of 0, \( E = \pi^{-1}(0) \), is a (connected, reduced) divisor in \( \tilde{V} \), i.e. a union of 1-dimensional compact curves in \( \tilde{V} \); and

ii) the restriction of \( \pi \) to \( \pi^{-1}(V^*) \) is a biholomorphic map between \( \tilde{V} - E \) and \( V^* \).

The surface \( \tilde{V} \) is called a resolution of the singularity of \( V \), and \( \pi : \tilde{V} \rightarrow V \) is the resolution map. Sometimes these are called desingularizations of the singularities instead of resolutions. The divisor \( E \) is called the exceptional divisor.
Notice that “the” resolution of \((V, 0)\) is not unique: given a resolution \(\tilde{V}\) we can obtain new resolutions by performing blow ups at points in \(E\). By Theorem 2.2 above, given a resolution, we can make blow ups on it, if necessary, so that the divisor \(E\) in Theorem 2.3 is good, i.e.:

\[\begin{align*}
\text{iii) each irreducible component } E_i \text{ of } E \text{ is non-singular; and} \\
\text{iv) } E \text{ has normal crossings, i.e. } E_i \text{ intersects } E_j, i \neq j, \text{ in at most one point,}
\end{align*}\]

where they meet transversally, and no three of them intersect.

**Definition 2.4** A resolution \(\pi : \tilde{V} \to V\) is good if its exceptional divisor is good, i.e. if it satisfies conditions (iii) and (iv) above.

Some authors allow good resolutions to have irreducible components of \(E\) intersecting transversally in more than one point, and they reserve the name very good for resolutions as in 2.4. This makes no big difference and we prefer to keep the notation of 2.4.

We recall that given a non-singular complex surface \(\tilde{V}\) and a Riemann surface \(S\) in it, the self-intersection of \(S\), usually denoted by \(S \cdot S\) or simply by \(S^2\), is the Euler class of its normal bundle \(\nu(S)\) in \(\tilde{V}\) (which coincides with its Chern class) evaluated in the fundamental cycle \([S]\). Equivalently, \(S \cdot S\) is the number of zeroes, counted with signs, of a generic section of the normal bundle \(\nu(S)\). It is an exercise to see that every time we make a blow up on a smooth complex 2-manifold, we get a copy of \(\mathbb{CP}^1\) with self-intersection -1. It is remarkable that the converse is true: recall that a non-singular curve in a smooth complex surface is said to be exceptional of the first kind if it is a copy of \(\mathbb{CP}^1\) embedded with self-intersection -1, one has:

**Theorem 2.5** (Castelnuovo’s criterium) Let \(\tilde{X}\) be a non-singular complex surface and \(C\) an exceptional curve of the first kind. Then \(S\) can be blown down analytically and we still get a non-singular surface \(X\).

This result is in fact a special case of a more general theorem of Castelnuovo for exceptional divisors of the first kind. We refer to [19, Ch. 3] for a proof. Notice that 2.4 has the very important consequence of giving us a minimal model:

**Definition 2.6** A resolution \(\pi : X \to V\) is minimal if given any other resolution \(X' \xrightarrow{p} V\), there is a proper analytic map \(X' \xrightarrow{p'} X\) such that \(\pi' = \pi \circ p\).
One has (see [3, III.6.2]):

**Theorem 2.7** Up to isomorphism, there exists a unique minimal resolution of $V$, and this is characterized by not containing non-singular rational curves with self intersection -1.

We remark that these statements are false in dimensions more than 2: there are not minimal resolutions in general.

Notice that the minimal resolution may not be good. For instance (c.f. [15]), for the Brieskorn singularity

$$z_1^2 + z_2^3 + z_3^7 = 0,$$

the minimal resolution has an exceptional divisor consisting of three non-singular rational curves meeting at one point, so it is not good; making one blow up at that point we obtain a good resolution, which has now a central curve which is a 2-sphere with self-intersection -1, and three other spheres meeting each the central curve in one point and with self intersections -2, -3, -7.

Something similar happens in general: we can make the minimal resolution good by performing blow ups, if necessary, and there is a unique (up to isomorphism) minimal good resolution.

Consider now a divisor $E = \bigcup_{i=1}^r E_i$ in a complex 2-manifold $X$, whose irreducible components $E_i$ are non-singular, they all meet transversally and no three of them intersect. To such a divisor we can associate an $r \times r$ integral matrix $A = ((E_{ij}))$, called the intersection matrix of $E$, as follows: on the diagonal $\Delta$ of $A$ we put the self intersection numbers $E_i^2$; and if a curve $E_i$ meets $E_j$ at $E_{ij}$ points, we put this number as the corresponding coefficient of $A$. So this is necessarily a symmetric matrix, whose coefficients away from the diagonal $\Delta$ are integers $\geq 0$ and in $\Delta$ we have the self-intersection numbers of the $E_i$, called the weights of these curves.

We have the following remarkable theorems of Mumford and Grauert (see [3, III.2]):

**Theorem 2.8** If $E$ is the exceptional divisor of a resolution $X \to V$, where $V$ is a normal surface, then the intersection matrix $A$ is negative definite (and the weights of the $E_i$ are all negative numbers).
Theorem 2.9 Conversely, if the divisor $E$ in $X$ is such that the intersection matrix $A$ is negative definite, then we can blow down $E$ analytically; we get a normal complex surface $V$, in general with a singularity at the image $0$ of $E$, and the projection $\pi : X \to V$ is a good resolution of $(V, 0)$ with exceptional divisor $E$.

A divisor $E$ as above is usually called an exceptional divisor, meaning by this that it can be blown down. It is said to be of the first kind when the blow down is smooth.

Notice that we can associate a weighted graph $\mathcal{G} = \mathcal{G}(E)$ to a good exceptional divisor $E$ in a complex 2-manifold $X$ as follows: to each irreducible component $E_i$ of $E$ we associate a vertex $v_i$, and if the curves $E_i$ and $E_j$ meet, then we join the vertices $v_i$ and $v_j$ by an edge. Each vertex has two integers attached to it: one is the genus $g_i \geq 0$ of the corresponding Riemann surface $E_i$; the other is the weight $w_i = E_i^2 \in \mathbb{Z}$, which is the self-intersection number of $E_i$ in $X$. This weighted graph is called the dual graph of the exceptional divisor $E$, or the dual graph of the resolution when $E$ is regarded as the exceptional set of a good resolution of a normal singularity.

We observe that every finite graph $\Sigma$ has associated a matrix $I(\Sigma)$ called the matrix of adjacencies of the graph: it has zeroes in the diagonal and if a vertex $v_i$ is joined to $v_j$ by $\delta_{ij}$ edges, then we put this number in the corresponding place of $I(\Sigma)$. It follows that the intersection matrix of the exceptional divisor $E$ is the result of taking the matrix of adjacencies of the dual graph and replacing its diagonal by the vector of weights $w_1, \cdots, w_m$, $w_i = E_i \cdot E_i$.

A beautiful thing of these constructions is that the dual graph of a resolution allows us to re-construct the topology of the resolution, and hence that of the link of the singularity. For this we need to introduce a construction known as plumbing. This was used already by Milnor to construct his first examples of exotic spheres and by Von Randow (1962) in relation with Seifert manifolds, though it was Hirzebruch who made this construction systematic. The plumbing construction is very nicely explained in [25] (see also [18]), we just recall it here briefly.

Let $E$ be a real 2-dimensional oriented vector bundle over a Riemann surface $S$, and denote by $D(E)$ its unit disc bundle for some metric. The total space of $D(E)$, that we denote by the same symbol, is a 4-dimensional smooth manifold with boundary the unit sphere bundle $S(E)$. Notice that restricted to a small disc $D_\varepsilon$ in $S$ the manifold $D(E)$ is a product of the
form $\mathbb{D}^2 \times \mathbb{D}^2$, where the first disc is $\mathbb{D}_e \subset S$ and the second disc is in the fibres of $E$. Now suppose we are given two such bundles $E_i$, $E_j$, over Riemann surfaces $S_i$, $S_j$. To perform plumbing on them we consider the total spaces of the corresponding unit disc bundles $D(E_i)$, $D(E_j)$, we choose small discs $\mathbb{D}_{i,\varepsilon}$, $\mathbb{D}_{j,\varepsilon}$ in $S_i$, $S_j$, and take the restriction of $D(E_i)$, $D(E_j)$ to these discs. Each of them is of the form $\mathbb{D}^2 \times \mathbb{D}^2$ as above. We now identify each point $(x,y) \in \mathbb{D}_{i,\varepsilon} \times \mathbb{D}^2 \subset D(E_i)$ with the corresponding point $(y,x) \in \mathbb{D}_{j,\varepsilon} \times \mathbb{D}^2 \subset D(E_j)$, i.e. interchanging base points in one of them with fibre points in the other. The result is a 4-dimensional, oriented manifold with boundary and with corners, which can be smoothed off in a unique way up to isotopy. We denote this manifold by $P(E_i, E_j)$. One says that $P(E_i, E_j)$ is obtained by plumbing the bundles $E_i$ and $E_j$ over the Riemann surfaces $S_i$ and $S_j$.

![Plumbing line bundles over circles.](image)

The boundary $S(E_i, E_j) = \partial P(E_i, E_j)$ of this 4-manifold is obtained by plumbing the corresponding sphere bundles $S_i(E)$ and $S_j(E)$: we remove from $S_i(E)$ and $S_j(E)$ the interior of the solid tori $\partial D_i \times \mathbb{D}^2$ and similarly for $E_j$. Thus we get two 3-manifolds with boundary a torus $\mathbb{S}^1 \times \mathbb{S}^1$ in each; we then identify these boundaries by glueing the meridians in one torus to the parallels in the other. The result is a 3-manifold with corners, which can be smoothed off in a unique way up to isotopy. The surfaces $S_i, S_j$ are naturally embedded in $P(E_i, E_j)$ as the zero-sections of the corresponding bundles, and they meet transversally in one point.

Notice that the manifolds one gets in this way are entirely described, up to diffeomorphism, by the genera of the Riemann surfaces $S_i$, $S_j$, and by the Euler classes of the corresponding bundles, since these classes determine the isomorphism class of the bundles.

**Definition 2.10** A plumbing graph is a triple $(\Sigma, w, g)$ consisting of a finite
graph $\Sigma$ with vertices $v_1, \ldots, v_r$, $r \geq 1$ and with no loops, a vector $w$ of weights, $w = (w_1, \ldots, w_r)$, $w_i \in \mathbb{Z}$, and a vector $g = (g_1, \ldots, g_r)$ of genera, $g_i \in \mathbb{N}$.

In this definition by a loop we mean an arrow that begins an ends in the same vertex, and we do not allow this (geometrically this means a singular curve in the exceptional divisor that has a double crossing). There can be cycles, i.e. a chain of vertices and edges that returns to itself after a certain time. In [42] Neumann considers a more general situation of plumbing graphs than the one envisaged here, which is important for the plumbing calculus developed there, but it does not really make a difference for the present work.

So the previous results say that dual graph of a good resolution of a normal singularity is a plumbing graph with negative definite intersection matrix. For instance, Figure 5 depicts the resolution corresponding to the surface singularity $z_1^2 + z_2^2 + z_3^r = 0$.

![Dynkin diagrammA_r](image)

Figure 5: Dynkin diagram $A_r$.

More generally, the classical Dynkin diagrams $A_r$, $D_r$, $E_6$, $E_7$ and $E_8$ are the dual graphs of the minimal resolutions of surface singularities of the form $\Gamma/\mathbb{C}^2$, where $\Gamma$ is a finite subgroup of $SU(2)$ (see for instance [58] for details).

Now, given a plumbing graph we may perform plumbing according to the graph: for each vertex $v_i$ take a Riemann surface $S_i$ of genus $g_i$ and an oriented 2-plane bundle $E_i$ over $S_i$ with Euler class $w_i$. If there is an edge between the vertices $v_i$ and $v_j$, we plumb the corresponding bundles as above. If a vertex $v_i$ is joined with other vertices, we choose pairwise disjoint small discs in each surface, as many as adjacent vertices one has, and perform plumbing by pairs as above. The result is a 4-dimensional manifold $\mathcal{P}(E)$ with boundary $\mathcal{S}(E)$. It follows from the construction that the manifold $\mathcal{P}(E)$ contains the union $E = \bigcup S_i$ as a deformation retract, and these surfaces are contained in $\mathcal{P}(E)$ with self-intersection $w_i$. Hence the homology of $\mathcal{P}(E)$ is that of $E$, and the intersection form on $\mathcal{P}(E)$ is given by the intersection matrix of its corresponding graph.

A manifold obtained in this way is known as a plumbed manifold, and this term may refer either to the 4-manifold $\mathcal{P}(E)$ with boundary, or to
its boundary, which is a 3-manifold. Notice that if the plumbing graph 
\((\Sigma, w, g)\) is the dual graph of a resolution \(\pi : \tilde{V} \to V\), then the manifold 
\(\mathcal{P}(E)\) is diffeomorphic to a regular neighbourhood of the exceptional set \(E\) in 
the resolution, which may be taken to be of the form \(\pi^{-1}(V \cap D_\varepsilon)\), where 
\(D_\varepsilon\) is a small closed disc in the ambient space \(\mathbb{C}^n\) with centre at 0. Since 
the resolution map is a biholomorphism away from \(E\), it follows that the 
boundary \(\mathcal{S}(E)\) is diffeomorphic to the link of \(V\), a fact that we state as a 
theorem:

**Theorem 2.11** Let \(\pi : \tilde{V} \to V\) be a good resolution of \((V,0)\), a normal 
surface singularity. Then the irreducible components \(E_1, \ldots, E_r\) of the 
exceptional divisor \(E\) determine a plumbing graph \(G(E)\), called the dual graph 
of the resolution, and performing plumbing according to this graph we obtain 
a 4-manifold homeomorphic to \(\pi^{-1}(V \cap D_\varepsilon) \subset \tilde{V}\), whose boundary is the link 
\(M\), where \(D_\varepsilon\) is a small closed disc in the ambient space \(\mathbb{C}^n\) with centre at 0.

Hence we know that a plumbing graph is the dual graph of a resolution if 
and only if its intersection matrix is negative definite. This gives a necessary
and sufficient condition for an oriented 3-manifold to be the link of a surface singularity. Let us finish this section by stating a well-known open problem.

**Problem 1.** Which 3-manifolds arise as links of isolated hypersurface singularities in $\mathbb{C}^3$?

In other words, we know from theorems 2.8 and 2.9 that an oriented 3-manifold is a singularity-link if and only if it is a plumbing manifold with negative definite intersection matrix. So the question is, among these manifolds, which ones arise as links of hypersurfaces?

This interesting question was studied by various authors in the 1980s, most notably by A. Durfee and St. Yau, obtaining some partial results. But the problem is, at present, far from being solved.

We notice that every closed oriented 3-manifold embeds in the 5-sphere with trivial normal bundle, and bounding a simply connected parallelizable manifolds. This somehow says that the problem is not only topological, but one needs to look for more subtle (presumably geometric) structures in order to characterize the links of hypersurface singularities.
3 On Milnor’s Fibration Theorem

Milnor’s Fibration Theorem is a result about the topology and geometry of the fibers of a holomorphic function in a neighbourhood of a critical point.

As in Section 1, we begin with the example of the Pham-Brieskorn singularities, 

\[ f(\mathbf{z}) = z_0^{a_0} + \cdots + z_n^{a_n}, \quad a_i > 1, \]

where \( \mathbf{z} = (z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \). The fibres \( V_t = f^{-1}(t) \) are all complex n-manifolds for \( t \neq 0 \) and \( V = f^{-1}(0) \) is a complex hypersurface with an isolated singularity at \( 0 \).

Recall one has a natural \( \mathbb{C}^* \)-action on \( \mathbb{C}^{n+1} \) defined by:

\[ \lambda \cdot (z_0, \cdots, z_n) \mapsto (\lambda^{d/a_0} z_0, \cdots, \lambda^{d/a_n} z_n), \]

where \( d \) is the least common multiple of the \( a_i \), and this action satisfies:

\[ f(\lambda \cdot (z_0, \cdots, z_n)) = \lambda^d \cdot f(z_0, \cdots, z_n). \]

Hence \( V \) is an invariant set of the action. In Section 1 we restricted this action to the real numbers in \( \mathbb{R} \), and we used the corresponding flow on \( V \) to deduce that \( V \) has a conical structure.

Now we look at the restriction of the above \( \mathbb{C}^* \)-action to the unit complex numbers. One has:

**Property 3.1** Restricting the action to the unit circle \( \{e^{i\theta}\} \) we get an \( S^1 \)-action on \( \mathbb{C}^{n+1} \) such that each sphere around \( 0 \) is invariant and:

\[ f(e^{i\theta} \cdot (z_0, \ldots, z_n)) = e^{i\theta d} f(z_0, \cdots, z_n), \]

that is, if we set \( \zeta = f(z_0, \cdots, z_n) \), then multiplication by \( e^{i\theta} \) in \( \mathbb{C}^{n+1} \) transports the fibre \( f^{-1}(\zeta) \) into the fibre over \( e^{i\theta} \cdot \zeta \).

In other words, this action is by isometries of \( \mathbb{C}^{n+1} \), its orbits are transversal to the fibers \( f^{-1}(t) \) for \( t \neq 0 \) and move the points of \( \mathbb{C}^{n+1} \setminus V \) taking fibers diffeomorphically into fibers. Thus, for every disc \( \Delta \) in \( \mathbb{C} \) centered at \( 0 \) one has that,

\[ f|_{f^{-1}(\Delta \setminus \{0\})} : f^{-1}(\Delta \setminus \{0\}) \longrightarrow \Delta \setminus \{0\}, \]

is a \( C^\infty \) fibre bundle. The same statement holds if we consider only the boundary \( \partial \Delta \) of the disc.
Moreover, fix a small ball $\mathbb{B}_\varepsilon$ around 0 of arbitrary positive radius, and let $\delta > 0$ be sufficiently small with respect to $\varepsilon$, so that all the fibers $f^{-1}(t)$ with $\delta \geq |t| > 0$ meet transversally the boundary sphere $S_\varepsilon = \partial \mathbb{B}_\varepsilon$. Let $N(\varepsilon, \delta)$ be defined by:

$$N(\varepsilon, \delta) = \mathbb{B}_\varepsilon \cap (f^{-1}(\partial \Delta_\delta)),$$

where $\Delta_\delta$ is the disc in $\mathbb{C}$ with center at 0 and radius $\delta$. Then

$$f : N(\varepsilon, \delta) \to \partial \Delta_\delta,$$  \hspace{1cm} (3.2)

is a $C^\infty$ fibre bundle.

The Manifold $N(\varepsilon, \delta)$ is known as a Milnor tube for $f$. This is the union of the portion of the fibers $f^{-1}(t), |t| = \delta$, contained within the ball $\mathbb{B}_\varepsilon$.

Now observe one also has:

**Property 3.3** The real analytic flow defined by restricting the $C^*$-action to $t \in \mathbb{R}^+$ has the additional property that for points in $\mathbb{C}^{n+1} \setminus V$, the argument of the complex number $f(z)$ is constant on each orbit, i.e., $f(z)/|f(z)| = f(tz)/|f(tz)|$ for $t \in \mathbb{R}^+$, and the norm of $f(z)$ is an strictly increasing function of $t$.

We may thus use this flow to “push” the tube $N(\varepsilon, \delta)$ to the boundary sphere. That is, for each point $x$ in $N(\varepsilon, \delta)$, consider its orbit under the real flow defined by the $C^*$-action (restricted to $\mathbb{R}^+$). The point moves so that its orbit is transversal to all the spheres in $\mathbb{B}_\varepsilon$ centered at 0, and also transversal to all the tubes $f^{-1}(\partial \Delta_\varepsilon)$. Furthermore, at each point of this trajectory one has that the argument of $f(z)$ is constant. Now let $x$ move along this trajectory until it meets the boundary sphere $S_\varepsilon$. This defines a diffeomorphism $\Psi$ between $N(\varepsilon, \delta)$ and the sphere $S_\varepsilon$ minus a tubular neighborhood $N(K)$ of the link $K$. One thus has a fiber bundle,

$$\psi : S_\varepsilon \setminus N(K) \longrightarrow \partial \Delta_\delta,$$

given by $\Psi^{-1}$ followed by $f$. Normalizing, we get a fiber bundle,

$$\phi = \frac{f}{|f|} : S_\varepsilon \setminus N(K) \longrightarrow S^1,$$  \hspace{1cm} (3.4)

With some more work one can show that this fibration actually extends to all of $S_\varepsilon \setminus K$. 

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These two fibrations, 3.2 and 3.4, are the two different versions one usually
has of the Milnor fibration associated to the function \( f \). In the sequel we will
see that one has these two fibrations in general.

In fact the classical Fibration Theorem of Milnor [38] says:

**Theorem 3.5** Let \( f : (\mathbb{C}^{n+1},0) \to (\mathbb{C},0) \) be a holomorphic map-germ, \( \varepsilon > 0 \)
small enough and let \( K = \mathbb{S}_\varepsilon \cap V \) be its link. Then:

\[
\phi = \frac{f}{|f|} : \mathbb{S}_\varepsilon \setminus K \to \mathbb{S}^1,
\]

is a (locally trivial) \( C^\infty \) fiber bundle.

Notice that in 3.5 there are actually two statements melted into a single
one: the first is that \( \mathbb{S}_\varepsilon \setminus K \) is a fibre bundle over \( \mathbb{S}^1 \), the other is that the
projection map can be taken to be the obvious one \( \phi = \frac{f}{|f|} \). One has (see
for instance [58] that the first statement actually extends to a very general
setting, while the second statement is in fact more subtle.

The idea of Milnor’s proof is simple: first show that the map \( \phi \) has
no critical points at all, so the fibres of \( \phi \) are all smooth, codimension-1
submanifolds of \( (\mathbb{S}_\varepsilon \setminus K_\varepsilon) \) where \( K = K_\varepsilon \) is the link. Then construct a
tangent vector field on \( (\mathbb{S}_\varepsilon \setminus K_\varepsilon) \) which is transversal to the fibres of \( \phi \) and the
corresponding flow moves at constant speed with respect to the argument of
the complex number \( \phi(z) \), so it carries fibres of \( \phi \) into fibres of \( \phi \). This proves
one has a product structure around each fibre of \( \phi \). For this, to begin, Milnor
shows that the critical points of \( (\mathbb{S}_\varepsilon \setminus K_\varepsilon) \to \mathbb{S}^1 \), if there were such points,
are exactly the points \( \tilde{z} = (z_0, \ldots, z_n) \) where the vector \( (i \, \text{grad}(\log|f(z)|)) \)
is a real multiple of \( \tilde{z} \). To prove this, set

\[
\phi(z) = \frac{f}{|f|}(\tilde{z}) := e^{i\theta(\tilde{z})},
\]

so one has:

\[
\theta(\tilde{z}) = \text{Re} \left( -i \, \log f(\tilde{z}) \right).
\]

An easy computation shows that given any curve \( \tilde{z} = p(t) \) in \( \mathbb{C}^{n+1} \setminus f^{-1}(0) \),
the chain rule implies:

\[
d\theta(p(t))/dt = \text{Re} \left( \frac{dp}{dt}(t), i \, \text{grad} \log f(\tilde{z}) \right) , \tag{3.6}
\]
where \( \langle \cdot, \cdot \rangle \) denotes the usual Hermitian product in \( \mathbb{C}^{n+1} \). Hence, given a vector \( v(z) \) in \( \mathbb{C}^{n+1} \) based at \( z \), the directional derivative of \( \theta(z) \) in the direction of \( v(z) \) is:

\[
\text{Re} \langle v(z), i \text{grad log } f(z) \rangle.
\]

Since the real part of the hermitian product is the usual inner product in \( \mathbb{R}^{2n} \), it follows that if \( v(z) \) is tangent to the sphere \( S^{2n-1}_0 \), then the corresponding directional derivative vanishes whenever \( i \text{grad log}(f(z)) \) is orthogonal to the sphere, i.e., when it is a real multiple of \( z \); conversely, if this inner product vanishes for all vectors tangent to the sphere, then \( z \) is a critical point of \( f \), and the claim follows.

Once we know how to characterize the critical points of \( f \) and how the argument of the complex number \( \phi(z) \) varies as \( z \) moves along paths in \( S \setminus K \), Milnor uses his Curve Selection Lemma (1.3) to conclude that \( f \) has no critical points at all. This part is a little technical and we refer to Milnor’s book (Chapter 4) for details. It follows that all fibres of \( f \) are smooth submanifolds of the sphere \( S \) of real codimension 1. In order to show that \( f \) is actually the projection map of a \( C^\infty \) fibre bundle one must prove that one has a local product structure around each fibre. This is achieved in [38] by constructing a vector field \( w \) on \( S \) satisfying:

i) the real part of the hermitian product \( \langle w(z), i \text{grad log } f(z) \rangle \) is identically equal to 1; recall that this is the directional derivative of the argument of \( \phi \) in the direction of \( w(z) \).

ii) the absolute value of the corresponding imaginary part is less than 1:

\[
|\text{Re} \langle w(z), \text{grad log } f(z) \rangle| < 1.
\]

Consider now the integral curves of this vector field, i.e., the solutions \( p(t) \) of the differential equation \( \frac{d\bar{z}}{dt} = w(t) \). Set \( e^{i\phi(t)} = \phi(z) \) as before. Since the directional derivative of \( \theta(z) \) in the direction \( w(z) \) is identically equal to 1 we have:

\[
\theta(p(t)) = t + \text{constant}.
\]

Therefore the path \( p(t) \) projects to a path which winds around the unit circle in the positive direction with unit velocity. In other words, these paths are transversal to the fibres of \( \phi \) and for each \( t \) they carry a point \( z \in \phi^{-1}(e^{it\omega}) \) into a point in \( \phi^{-1}(e^{it\omega + t}) \). If there is a real number \( t_o > 0 \) so that all these paths are defined for at least a time \( t_o \), then, being solutions of the above differential equation, they will carry each fibre of \( \phi \) diffeomorphically into all
the nearby fibres, proving that one has a local product structure and \( \phi \) is the projection of a locally trivial fibre bundle. Milnor proves this by showing that condition (ii) above implies that all these paths are actually defined for all \( t \in \mathbb{R} \), so we arrive to Theorem 2.1.

The previous formulation of Milnor’s theorem, besides of being the original formulation in [38], provides several geometric insights into the topology of singularities, as for instance showing that the link \( K \) is a fibred knot (or link) in the sphere \( S_\varepsilon \), a fact used by Milnor in his book to study the topology of the fibers and of the link itself. Furthermore, one has a “multi-link” structure (which is a richer structure, see [18], or section 4 below) determined by the map \( f \).

There is, however, another formulation of Milnor’s theorem, which is more famous nowadays, and lends itself more easily to generalizations. This seems to be the most appropriate view-point for algebraic geometry. We call this the Milnor-Lê fibration theorem. For map-germs \( f \) as above, defined on \( \mathbb{C}^{n+1} \), this result is essentially due to Milnor alone, except for a “little” point that we explain below. Lê completed this and extended the result to holomorphic maps defined on arbitrary complex spaces. Let us first state the theorem in \( \mathbb{C}^{n+1} \).

**Theorem 3.7** Let \( S_\varepsilon \) be a sufficiently small sphere in \( \mathbb{C}^{n+1} \) centered at \( 0 \) and choose \( \delta > 0 \) small enough with respect to \( \varepsilon \) so that all the fibres \( f^{-1}(t) \) with \( |t| \leq \delta \) meet \( S_\varepsilon \) transversally. Let \( C_\delta = \partial B_\delta \) be the circle in \( \mathbb{C} \) of radius \( \delta \) and centered at 0, and set \( N(\varepsilon, \delta) = f^{-1}(C_\delta) \cap B_\varepsilon \). Then:

\[
f|_{N(\varepsilon, \delta)} : N(\varepsilon, \delta) \longrightarrow C_\delta \cong \mathbb{S}^1,
\]

is a \( C^\infty \) fibre bundle, equivalent to the fibration in (3.5).

The manifold \( N(\varepsilon, \delta) \) is usually called a Milnor tube for \( f \).

This result is due to Milnor ([37, 38]) when \( \emptyset \in \mathbb{C}^{n+1} \) is an isolated critical point of \( f \). Milnor also proved in [38, §5] that for arbitrary \( f \), the fiber \( f^{-1}(t) \cap B_\varepsilon \) in (3.7) is diffeomorphic to the fiber in (3.5). The missing point is that he did not prove that (3.7) is a fibre bundle (for \( f \) with non-isolated singularity). If \( f \) has an isolated critical point, this is an immediate extension of Ehresmann’s fibration lemma, as noted by Milnor in [37]. However, for general \( f \) one must prove first, for instance, that \( f \) has the so-called Thom \( a_f \)-property. We recall the definition of this property in the simple situation envisaged in this article.
Definition 3.8 Let \((X^{n+k}, p)\) be a real analytic space-germ, and \(h : (X, p) \to (\mathbb{R}^n, 0)\) a real analytic map with an isolated critical value at 0. Then \(f\) has the **Thom property** if there exists a Whitney stratification of \(X\) such that \(V = f^{-1}(0)\) is union of strata, and for each sequence of points \(\{y_m\} \subset X \setminus V\) that converges to a point \(x \in V\), such that the limit \(T\) of the tangent spaces \(T_{y_m}(f^{-1}(f(y_m)))\) exists, one has that \(T\) contains the tangent space \(T_x(X_\alpha)\), where \(X_\alpha\) is the stratum that contains \(\alpha\).

As mentioned before, Hironaka [24] proved that every holomorphic map into \(\mathbb{C}\) has the Thom property. Using this, Lê proved:

**Theorem 3.9** Let \((X, p)\) be a complex analytic germ in \(\mathbb{C}^N\), and \(f : (X, p) \to (\mathbb{C}, 0)\) holomorphic. Let \(\varepsilon > 0\) sufficiently small, so that \(K_X = X \setminus S^{2N-1}_\varepsilon\) is the link of \(X\) and \(L_V = V \cap K_X\) is the link of \(V = f^{-1}(0)\). Choose a disc \(D_\delta\) in \(\mathbb{C}\), \(\varepsilon \gg \delta > 0\), around 0 such that all the fibers \(f^{-1}(t), t \in D_\delta \setminus \{0\}\), meet \(K_X\) transversally, and set \(N(\varepsilon, \delta) = f^{-1}(C_\delta) \cap \mathbb{B}_\varepsilon\), where \(C_\delta = \partial D_\delta\). Then:

\[
f|_{N(\varepsilon, \delta)} : N(\varepsilon, \delta) \longrightarrow C_\delta \cong S^1,
\]

is a continuous fibre bundle.

Furthermore, the tube in this fibration can be “inflated” (see explanation below), so that it takes the form of the classical Milnor fibration (3.5), now defined on \(K_X \setminus L_V\).

We remark that the fact that one can choose \(D_\delta\) so that all the fibers \(f^{-1}(t), t \in D_\delta \setminus \{0\}\), meet \(K_X\) transversally follows easily from the fact that \(f\) has the Thom property, together with the first Thom-Mather isotopy theorem.

If \(X\) has an isolated singularity at \(p\) the proof of (3.9) is rather simple (and this was proved by H. Hamm [22]). The idea is that since \(0 \in \mathbb{C}\) is an isolated critical value, the map \(f\) in (3.9) is a submersion and we can lift a vector field on the circle \(C_\delta\) to a vector field on the tube \(N(\varepsilon, \delta)\), transversal to the fibres of \(f\). We may further choose this vector field to be a \(C^\infty\) and integrable, and also tangent to \(N(\varepsilon, \delta) \cap S_\varepsilon\) (since the fibers are transversal to the sphere). Since the fibres are compact, we can assume that for each fibre \(F_0\) there is a time \(t_0 > 0\) so that all solutions passing by \(F_0\) are defined for at least time \(t_0\). Then the corresponding local flow identifies \(F_0\) with all nearby fibres, which are thus diffeomorphic to \(F_0\) and one has a local product structure, hence a fibre bundle.
The proof in general follows the same ideas, but it is technically more difficult. The key point is noticing that also in this situation, the (proof of the) first Thom-Mather isotopy theorem gives that one can lift a smooth vector field on $C_\delta$ to a stratified, integrable vector field on the tube $N(\varepsilon, \delta)$ (smooth on each stratum) and transversal to the fibres of $f$. After that, the proof is exactly as in the isolated singularity case.

Once we have (3.9), in order to show that this fibration is equivalent to a fibration of the type of (3.5), one follows exactly the same steps followed by Milnor in [38, Ch. 5]. Up to this point, essentially everything works in the real analytic category, but from now on the holomorphic structure is used thoroughly. The idea is to show that there exists a vector field on $(\mathbb{B}_\varepsilon \cap X) \setminus f^{-1}(0)$ whose solutions move away from the origin being transversal to all the spheres around 0, transversal to all the tubes $f^{-1}(C'_\delta)$ and the argument of the complex number $f(z)$ is constant along the path of each solution. This allows us to “inflate” the Milnor tube $f^{-1}(\partial D_\delta) \cap \mathbb{B}_\varepsilon$ in (3.7) to become the complement of a neighbourhood $T(\varepsilon, \delta)$ of the link $K$ in the sphere $S_\varepsilon$, taking the fibres of (2.3) into the fibres of the map $\phi$ in Theorem 2.1 (just as we did for the Pham-Brieskorn singularities at the beginning of this section). Then one must show that this fibration extends to $T(\varepsilon, \delta) \setminus K_\varepsilon$.

**Problem 2.** Let $(V, p)$ be a two-dimensional complex analytic non-isolated singularity in $\mathbb{C}^3$, say defined by a map-germ $f$ with $f(p) = 0$, and equip $V$ with a Whitney stratification. Let $r > 0$ be sufficiently small, so that every sphere around $p \in \mathbb{C}^3$ meets every Whitney stratum transversally, and let $K = V \setminus S_r(p)$ be the link of $p$. Now choose $\delta > 0$ sufficiently small, so that for every $t$ with $0 < |t| < \delta$, the complex manifold $f^{-1}(t)$ meets $S_r(p)$ transversally, and set $\partial F = f^{-1}(t) \cap S_r(p)$. What type of 3-manifold is $\partial F$?, can we describe the topology of the link $K$ from that of $\partial F$?

We remark that the existence of $r > 0$ as above comes from the fact that every complex analytic map-germ $\mathbb{C}^n \rightarrow \mathbb{C}$ has the Thom $a_f$-property, by a theorem of Hironaka (improved by Parusinsky few years ago).

Of course the same questions are valid in higher dimensions. Notice $F = f^{-1}(t) \cap \mathbb{B}_r(p)$ is the Milnor fibre of $f$. If $p$ were an isolated singularity, then the link $K$ is smooth and $\partial F$ is diffeomorphic to $K$. In general, there is a collapsing map $F \rightarrow V$, defined by Lê Dũng Tráng in [28] and others, which can be used to study the above problem.

Let us now say a few words about the topology of the fibers in Milnor’s
fibration. This will be explained in much more detail, and in a more general setting, in Lê Dũng Tráng’s course next week.

First notice that the description of this fibration as in Theorem 3.9 implies that the fibers, \( F_t = f^{-1}(t) \cap B_r(p) \), are Stein spaces. Therefore the theorem of Andreotti-Frankel [2] says that they have the homotopy type of CW-complexes of real dimension \( n \), where \( n \) is the complex dimension of \( F_t \). This holds for all holomorphic map-germs, regardless of whether or not the critical of \( f \) at 0 is isolated. This same statement was proved directly by Milnor using the real-valued function \( |f| \), restricted to \( F_t \). The first observation for this is that Theorem 3.9 implies that \( F_t \) can be considered as a compact manifold with boundary the link \( K \). Now, although \( |f| \) may not be itself a Morse function on \( F_t \), its Morse index is well defined at each critical point, and Milnor shows that the fact that \( f \) is holomorphic implies that all its Morse indices are necessarily more than \( n \). Then, one can always approximate \( |f| \) by a Morse function \( \hat{f} \), which agrees with \( |f| \) away from a neighborhood of its critical set on \( F_t \). Then the Morse indices of \( \hat{f} \) must be also more than \( n \), thus implying that \( F_t \) has the homotopy of a CW-complex of dimension \( n \).

In the special case when \( f \) has an isolated critical point, the previous statements can be made stronger, as proved by Milnor. In fact, using now the fibration as in Theorem 3.5, one has the the compact manifold with boundary \( F_t \) is embedded in \( S_r \) in such a way that it has the same homotopy type as its complement (since \( S_r \setminus F_t \) is a fibre bundle over \( R \) with fiber diffeomorphic to \( F_t \)). Then, the Alexander duality theorem implies that the reduced homology of \( F_t \) vanishes in dimensions less than \( n \). With a little extra work one can show that for \( n > 1 \) the fiber \( F_t \) is simply connected, which implies, together with the previous statement, that if is \((n-1)\)-connected. Milnor further shows that the homology group \( H_n(F_t) \) (with integer coefficients) is free abelian, and the Hurewicz theorem implies that for \( n > 1 \) it is isomorphic to the homotopy group \( \pi_n(F_t) \). From this one gets:

**Theorem 3.10** The fiber \( F_t \) has the homotopy type of a CW-complex of middle dimension \( n \). Furthermore, if \( f \) has an isolated critical point, then \( F_t \) actually has the homotopy type of a bouquet of spheres of dimension \( n \),

\[
F_t \cong \bigvee_{\mu} S^n,
\]

the number \( \mu \) of spheres in this wedge being some positive integer (unless \( f \) is regular at 0, then \( \mu = 0 \)).
For instance, for a Morse function $z_0^2 + \cdots + z_n^2$ one has $\mu = 1$. More generally, as proved by Pham in (and later by Milnor), for the polynomial map $f(z) = z_0^{a_0} + \cdots + z_n^{a_n}$ one has:

$$
\mu = (a_0 - 1) \cdots (a_n - 1).
$$

**Definition 3.11** The above number $\mu$ is called the Milnor number of $f$.

This number is an important invariant of isolated hypersurface singularities. By definition, it measures the rank of the middle homology of the Milnor fibre $F_t$, which has homology only in middle-dimension. This invariant has been generalized in various ways for non-isolated singularities, giving rise to the so-called $L^e$-numbers and classes, as well as to the so-called Milnor classes. We refer to [36] for the former, and to [8] for the latter.
4 Additional geometric structures on surface singularities

We now restrict the discussion to the case when $V$ is a complex analytic surface in some $\mathbb{C}^N$ with an isolated singularity at 0, though much of what we will say holds for higher dimensional complex analytic varieties.

Thus $V^* = V \setminus \{0\}$ is a complex 2-dimensional manifold and the link $K = V \cap S_r$ is an oriented 3-manifold. We know from the previous section that the diffeomorphism type of $K$ does not depend on the choice of the sphere $S_r$ used to define it. In fact, slightly more refined arguments, due independently to Durfee [17] and Le-Teissier [33] show that this statement can be made much stronger: the diffeomorphism type of the link $K$ depends only on the analytic structure of $V$ at 0 and not on other choices, such as the equations used to define the embedding of $V$ in $\mathbb{C}^N$, the metric, etc.

This means that whatever invariant of 3-manifolds one has, is naturally an invariant of surface singularities. This has been used for various authors, such as Laufer, Durfee and several others including myself, to obtain information about surface singularities (see for instance Chapter IV in [58] for more on the subject). More recently, Nemethi and others have been getting remarkable results for surface singularities using some of the recent 3-manifolds invariants coming from gauge theory, such as the Seiberg-Witten invariants, Floer homology, etc.

In this section we look briefly at some finer structures one has on the manifolds $V^*$ and $K$, that allow us to use important ideas and tools of other areas of geometry and topology to study surface singularities. These structures are somehow the basic ground for the use of the recent new manifolds-invariants.

4.1 Contact structures

A contact structure on an odd dimensional manifold $M$ is a field of tangent hyperplanes $\mathcal{H}$, called contact hyperplanes, satisfying a certain maximal non-integrability condition, that we now explain. At each point of $M$ the field $\mathcal{H}$ of hyperplanes is defined locally by a (non-unique) 1-form $\alpha$, whose kernel is, at each point $x$, the given hyperplane $\mathcal{H}_x$. The 1-form $\alpha$ is called a contact form if the 2-form $d\alpha$ is non degenerate on each $\mathcal{H}_x$. This means that at each point $x$ the alternate bilinear form $d\alpha_x : \mathcal{H}_x \times \mathcal{H}_x \to \mathbb{R}$ satisfies that if $d\alpha(x, y) = 0$ for all $x \in V$, then $y = 0$. 

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Notice that the classical Frobenius integrability theorem says that a field of hyperplanes is integrable at a point \( x \) if the corresponding 1-form \( \omega \) satisfies \( \omega \wedge d\omega = 0 \). Thus a contact form is in some sense “as far as possible” from being integrable. The form that defines the hyperplanes is locally unique up to multiplication by a non-vanishing function, and this does not change the fact of being a contact form. Thus the condition depends only on the choice of hyperplanes.

The simplest example of a contact structure is given on \( \mathbb{R}^{2n+1} \) by equipping it with coordinates \( (q, p, y) = (q_1, \cdots, q_2, p_1, \cdots, p_n, y) \) and its usual contact form \( \alpha = dy - pdq \). An important theorem of Darboux actually says that this example is “universal”, in the sense that every contact form is equivalent to this one, up to a smooth change of coordinates.

Now let \( V \subset \mathbb{C}^N \) be an affine complex analytic variety of dimension \( n > 1 \), with an isolated singularity at 0, and let \( K \) be its link. Then \( K \) is a codimension 1 oriented submanifold of \( V^* = V \setminus \{0\} \), and therefore its normal bundle \( \nu(K) \) is trivial. At each \( x \in K \), one has a real line \( \nu_x(K) \); multiplying each vector in this line by the complex number \( i \) we obtain a real line \( i\nu_x(K) \) in the tangent space \( T_xK \). If we endow \( V^* \) with the hermitian metric inherited from that in \( \mathbb{C}^N \), one can consider the orthogonal complement \( \mathcal{H}_x \) of \( i\nu_x(K) \) in \( T_xK \). An important theorem of Varchenko says:

**Theorem 4.1** The field of hyperplanes that one gets in this way satisfies the maximal non-integrability condition, and therefore defines a contact structure on \( K \). Moreover, this contact structure is independent of all choices, up to contact diffeomorphism.

We follow [13] and call this the *canonical contact structure* on \( K \).

### 4.2 Spin and Spin\(^c\) structures

We recall that given a smooth n-manifold \( M \), one has its tangent bundle \( TM \) with structure group \( (Gl(n), \mathbb{R}) \). This means one has a smooth atlas for \( M \), such that on each coordinate chart \( TM \) is trivial, and the transition functions \( U_i \cap U_j \) have natural liftings to \( (U_i \cap U_j) \times \mathbb{R}^n \), where \( \mathbb{R}^n \) is being regarded as the corresponding tangent space and the maps on these tangent fibers are linear isomorphisms, that is elements in \( (Gl(n), \mathbb{R}) \). These *clutching functions* on the charts \( (U_i \cap U_j) \times \mathbb{R}^n \) allow us to reconstruct not only \( M \), but also its tangent bundle \( TM \).
In other words, we have associated to $TM$ a principal $(Gl(n), \mathbb{R})$-bundle $P(M; Gl(n))$ over $M$, whose fiber over each point $x \in M$ is the group of linear isomorphisms in the tangent space $T_x M$. We can always equip $M$ with a Riemannian metric. This means that on each tangent space we have introduced a non-degenerate bilinear form, that varies smoothly over $M$. Since $(Gl(n), \mathbb{R})$ has the orthogonal group $O(n)$ as a deformation retract, having a Riemannian metric on $M$ allows us to reduce the structure group of $TM$ from $(Gl(n), \mathbb{R})$ to $O(n)$, that is, we can take all the clutching functions of $TM$ to be elements in $O(n)$.

In other words, equipping $TM$ with a Riemannian metric means that we have retracted the bundle $P(M; Gl(n))$ over $M$ to the principal bundle $P(M; O(n))$, whose fiber over each point is the group $O(n)$ of orthogonal transformations in the tangent space. This group can also be regarded as the space of orthonormal frames at the tangent space, and therefore $P(M; O(n))$ is often referred to as the bundle of orthonormal frames over $M$.

If we now suppose that the manifold $M$ is orientable, then to actually equip $M$ with an orientation means that we are choosing all the clutching functions of $TM$ to be linear maps with the same determinant, that is, elements in $SO(n)$. This means we have reduced the structure group of $TM$ further to $SO(n)$. We thus have a corresponding principal $SO(n)$-bundle $P(M; SO(n))$ over $M$.

We recall that $SO(2)$ is the group of rotations of $\mathbb{R}^2$ and therefore can be identified with the circle $S^1$. For $n \geq 3$ one has that the fundamental group $\pi_1(SO(n))$ is cyclic of order 2. This means that its universal cover $\widetilde{SO}(n)$ is a two-fold cover $p : \widetilde{SO}(n) \rightarrow SO(n)$.

**Definition 4.2** For $n \geq 3$ the Spin$(n)$ group is defined to be the universal covering group $\widetilde{SO}(n)$ of $SO(n)$.

For $n = 2$, Spin$(2)$ can be defined to be $S^1$ regarded as the double cover of $SO(2) \equiv S^1$ via the map $z \mapsto z^2$.

We remark that this geometric definition of the spin groups is certainly correct and useful for many purposes, but it hides away many important additional properties. In fact the Spin$(n)$ group is naturally a subgroup of the group of units of the Clifford algebra of $\mathbb{R}^n$ and inherits from it its group structure. We refer to the literature for more on this topic.

For us, in this lecture it is enough to think of the spin groups as defined above. In fact we are mostly interested in the cases $n = 3, 4$, and one knows
very well these groups: $SO(3)$ is diffeomorphic to the real projective space $\mathbb{R}P^3$ and therefore

$$Spin(3) \cong S^3 \cong SU(2),$$

where the 3-sphere $S^3$ is being regarded as the group $SU(2)$ of complex matrices of the form,

$$
\begin{pmatrix}
  z_1 & z_2 \\
  -\bar{z}_2 & \bar{z}_1
\end{pmatrix},
$$

with determinant 1.

For $n = 4$ one knows that $SO(4) \cong (SO(3) \times S^3)$ and therefore one has:

$$Spin(4) \cong S^3 \times S^3 \cong SU(2) \times SU(2).$$

**Definition 4.3** A spin structure on an oriented manifold $M$ means a lifting of its structure group from $SO(n)$ to $Spin(n)$.

In other words, one has a principal $Spin(n)$-bundle $P(M; Spin(n))$ over $M$, together with a projection map $P(M; Spin(n)) \to P(M; SO(n))$ which on each fiber restricts to the double cover $Spin(n) \to SO(n)$.

Just as not every manifold admits an orientation, so too, not every manifold admits a spin structure. This can be easily said in terms of characteristic classes (we refer to the literature for a proof of the following statements): $M$ compact admits an orientation iff its first Stiefel-Whitney class $w_1(M)$ vanishes, and it admits a spin structure iff one further has $w_2(M) = 0$.

If $M$ admits a spin structure, then it may admit several such structures, and any two of them are regarded as equivalent (or being lousy, as being "equal") if the corresponding principal bundles are isomorphic. It can be proved that for $M$ compact, the different spin structures on $M$, a manifold with $w_1(M) = w_2(M) = 0$, are classified by the cohomology group $H_1(M)$ with $\mathbb{Z}_2$ coefficients.

**Definition 4.4** A spin manifold means a (usually smooth, but this concept can be adapted to topological manifolds) manifold $M$, equipped with an orientation and a compatible spin structure.

Thus, for instance, every smooth manifold which is a $\mathbb{Z}_2$-homotopy sphere admits a unique spin structure.

Spin manifolds have remarkable geometric and topological properties, and we refer for instance to the beautiful book of Lawson and Michelson for an account on the subject.
There are however very interesting manifolds which are not spin, as for instance the complex projective space $\mathbb{CP}^2$. Thus, we now introduce a larger class of manifolds, the $\text{Spin}^c$ manifolds, which also have remarkable properties and have the additional feature that every Spin manifold and every complex manifold is canonically $\text{Spin}^c$.

These structures are the starting point for the theory of Seiberg-Witten invariants, discussed in the courses of Ron Stern and A. Nemethi.

For this we must first define the $\text{Spin}^c$ groups.

**Definition 4.5** Consider the subgroup $SO(n) \times SO(2)$ of $SO(n + 2)$, and the projection map,

$$p : \text{Spin}(n + 2) \to SO(n + 2).$$

The group $\text{Spin}^c(n)$ is by definition $p^{-1}(SO(n) \times SO(2))$.

**Definition 4.6** A $\text{Spin}^c$ structure on an oriented manifold $M$ means that we have equipped $M$ with an oriented 2-plane bundle $\mathcal{D}$ with structure group $SO(2)$ such that the principal $SO(n) \times SO(2)$-bundle corresponding to $TM \times \mathcal{D}$ lifts to a principal $\text{Spin}^c$-bundle.

The bundle $\mathcal{D}$ is called the determinant bundle of the $\text{Spin}^c$ structure.

Thus, for instance, if $M$ has a spin structure, then we can take $\mathcal{D}$ to be the trivial bundle $M \times \mathbb{R}^2$, and one has an induced $\text{Spin}^c$ structure. That is, every Spin manifold is canonically $\text{Spin}^c$, with trivial determinant bundle.

On the other hand, if $M$ is a complex $n$-manifold, then $TM$ is a complex bundle, then it is automatically oriented. In this case the structure group of $TM$ is $GL(n, \mathbb{C})$ and by choosing a hermitian metric we can reduce it further to $U(n)$, which is a subgroup of $SO(2n)$. Let $\mathcal{K}^* = \Lambda^n TM$ be the corresponding anti-canonical bundle of $M$, dual of the bundle of holomorphic $n$-forms on $M$. The complex bundle $\mathcal{K}^*$ is 1-dimensional and so its structure group is $U(1) \equiv SO(2)$. Then the structure group of $TM \times \mathcal{K}^*$ is canonically a subgroup of $SO(2n) \times SO(2)$ and it has a canonical lifting to $\text{Spin}^c$. Hence every complex manifold is canonically $\text{Spin}^c$, with determinant bundle $\mathcal{K}^* = \Lambda^n TM$.

Now consider the case when $M = V^* = V \setminus 0$, where $V$ is the above complex surface with an isolated singularity. This is a complex manifold, and therefore it has a canonical $\text{Spin}^c$ structure. Since the link $K$ is a codimension 1 oriented submanifold of $V^*$, it follows that its normal bundle in $V^*$ is trivial,
and it has a canonical trivialization by the unit outwards normal vector field. Hence the canonical $\text{Spin}^c$ structure on $V^*$ defines one on $K$.

That is, one has the following result, first due to Nemethi and Nicolaescu in [41], where they study the Seiberg-Witten invariant of the link $K$ with this $\text{Spin}^c$ structure.

**Proposition 4.7** The link $K$ of an isolated complex surface singularity has a canonical $\text{Spin}^c$ structure.

### 4.3 The hypersurface case

Now assume the surface $V$ is defined by a single equation $f : \mathbb{C}^3 \to \mathbb{C}$; $V = f^{-1}(0)$. Notice that on the regular points of each surface $V_t = f^{-1}(t)$ one has a canonical, nowhere-vanishing holomorphic 2-form $\omega$ obtained by contracting the 3-form $dz_1 \wedge dz_2 \wedge dz_3$ with respect to the vector field $\nabla f = \left( \frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \frac{\partial f}{\partial z_3} \right)$. In local coordinates $\omega$ is:

$$\omega = \frac{dz_1 \wedge dz_2}{\partial f/\partial z_3} = \frac{dz_2 \wedge dz_3}{\partial f/\partial z_1} = \frac{dz_3 \wedge dz_1}{\partial f/\partial z_2}. \quad (4.8)$$

It is easy to see that the 2-forms on the right hand side of this equation, which are defined where the denominator is not zero, coincide when any two of them are defined and so they give a global 2-form on the regular part of each $V_t$, which coincides with the one given by contracting $\bar{\Omega}$ with $\nabla f$.

Assume for simplicity that 0 is the only critical point of $f$, and consider again the complex manifold $V^* = V \setminus \{0\}$. The structure group of its tangent bundle $TV^*$ is $GL(2, \mathbb{C})$, and we can always reduce it to $U(2)$ by endowing $V^*$ with some Hermitian metric. If we try to reduce the structure group of $TV^*$ further to $SU(2)$ we may run into problems, since this is not always possible. The obstruction for doing so is the first Chern class of $TV^*$ (c.f. IV.1 in [58]). To see this we notice that $U(2)$, being the structure group of $TV^*$, acts naturally on the space of differential forms (of all degrees) on $V^*$. In particular it acts on the forms of type $(2,0)$, i.e. on the sections of the bundle associated to $TV^*$ whose fibre at $x \in V^*$ is $\Lambda^{(2,0)} T_x V^*$; this is isomorphic to the bundle of holomorphic 2-forms on $V^*$, which is called the canonical bundle $\mathcal{K}$ of $V^*$. The action of $U(2)$ on $\Lambda^{(2,0)} T_x V^*$ is by the determinant, and $SU(2)$ is precisely the subgroup of $U(2)$ which acts trivially on $\mathcal{K}$. Hence we can reduce the structure group of $TV^*$ from $U(2)$ further to
SU(2) if and only if the bundle $K$ is trivial, and the specific reductions of the structure group of $TV^*$ to $SU(2)$ correspond to the specific trivializations of $K$. Now, the canonical bundle $K$ is 1-dimensional, so it is classified (up to a differentiable isomorphism) by its Chern class, which is the negative of the first Chern class of $V^*$. Hence $c_1(V^*)$ vanishes iff $K$ is trivial and this happens iff we can reduce the structure group of $TV^*$ to $SU(2)$.

Now equip $V^*$ with the above holomorphic 2-form $\omega$. Then this form defines a specific reduction to $SU(2)$ of the structure group of $TV^*$. Since $SU(2)$ is isomorphic to the group of unit quaternions $Sp(1)$, this means that we have at each point of $V^*$ multiplication of tangent vectors by the quaternions $i, j, k$.

Now choose, at each point $x \in K$, a normal vector field $\tau(x)$ of unit length and pointing outwards. Multiplying $\tau(x)$ at each point of $K$ by the quaternions $i, j, k$, we get a trivialization $\rho$ of $TK$, the tangent bundle of the link. This trivialization $\rho$ of $TK$ was called in [55] the canonical framing of the link $K$. This determines a canonical spin structure on $K$, and therefore defines a $Spin^c$ structure too, which coincides with that of [41] explained in the previous section.

We remark that the previous discussion generalizes immediately to all germs of complex surfaces with an isolated complete intersection germ, and to some extent to all normal Gorenstein surface singularities (see [58]). Notice that the above discussion also implies that the holomorphic 2-form $\omega$ is well-defined and never-vanishing on the Milnor fibers $F_t$ of $f$. Hence it defines a reduction to $SU(2)$ of the structure group of $Tf$. Since $SU(2) \equiv Spin(3)$ one also has a Spin structure on all of $F_t$. If one considers the fibre $F_t$ as a compact manifold with boundary $K$, as in the previous section, then one gets that the spin structure on $K$ extends to $F_t$.

One gets:

**Theorem 4.9** Let $(V, 0)$ be an isolated 2-dimensional complete intersection singularity. Let $V^* = V \setminus \{0\}$ and let $K$ be its link. Then:

i) The structure group of the tangent bundle $TV^*$ has a canonical reduction to $SU(2) \equiv Sp(1)$.

ii) This reduction to $Sp(1)$ of the structure group of $TV^*$ defines a canonical trivialization $\rho$ of the tangent bundle $TK$, determined by the three tangent vector fields $i\tau, j\tau, k\tau$, where $\tau$ is the unit normal outwards vector field of $K$ in $V^*$.
iii) The trivialization $\rho$ of $TK$ determines a canonical spin structure on the link, which in this case coincides with the canonical $\text{Spin}^c$ structure of Nemethi and Nicolaescu. With this spin structure, the link $K$ is the spin-boundary of the Milnor fibre $F_t$.

iv) The 2-plane field on $K$ spanned by the vector fields $j\tau$, $k\tau$ determines the canonical contact structure on the link $K$, and the vector field $i\tau$ is (up to scaling) the Reeb vector field of the contact structure.

References


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J. Seade, Instituto de Matemáticas, Unidad Cuernavaca, Universidad Nacional Autónoma de México, Av. Universidad s/n, Lomas de Chamilpa, 62210 Cuernavaca, Morelos, A. P. 273-3, México
jseade@matem.unam.mx