Lê's Work on Hypersurface Singularities*

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Abstract

We discuss the foundational work of Lê on the topology and geometry of complex hypersurface singularities.

1 Lecture 1: Introduction and Families of Isolated Hypersurface Singularities

1.1 Introduction

In 1968, Milnor's foundational work, "Singular Points of Complex Hypersurfaces" [23], appeared. In this beautiful monograph, Milnor studies the local topology of complex algebraic hypersurfaces at **singular points**.

Since the appearance of [23], many, many mathematicians have worked to generalize, extend, and/or improve Milnor's work. It is trivial to see that all of Milnor's results can be extended to the case of **analytic** hypersurfaces, but it is not trivial to extend Milnor's work to the case in which the ambient space may itself be singular, nor is it trivial to generalize many of Milnor's results for isolated singular points to the case of non-isolated singular points. Also, "calculating" information about the Milnor monodromy is extremely difficult. In the late 1960's and 1970's, there was a great deal research in these areas by such people as A'Campo, Brieskorn, Buchweitz, Clemens, Gabrielov, Greuel, Hamm, Hironaka, Kato, Landman, Lazzeri, Matsumoto, Oka, Orlik, Perron, Pham, C. P. Ramanujam, K. Saito, Sakamoto, Sebastiani, Teissier, and Thom.

We have, of course, omitted (at least) one name from the above list; many important extensions of Milnor's work have been due to, or are joint work with, one man: Lê Dũng Tráng. In his papers, from 1973 to 1979, [18], [5], [8], [10], [11], [9], [17], [12], [13], [14], [15], and [16], Lê and his coauthors prove: the homotopy-type of the Milnor fiber is an invariant of the local ambient, topological-type of the (reduced) hypersurface, that the constancy of the Milnor number in a family of isolated hypersurface singularities implies that Thom's a_f condition holds and that one has topological constancy (except possibly in the case of single dimension), that one can calculate how many cells of top dimension have to be attached to the Milnor fiber of a hyperplane slice to obtain the original Milnor fiber and that some of these cells must always "cancel out" smaller-dimensional cells, that the Milnor fibration (inside a ball) exists regardless of how singular the domain may be, that the vanishing cycles "cannot split" from one isolated singular point into multiple singular points, and that there is a geometric proof of the Monodromy Theorem.

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Lê's contributions to the field of complex analytic singularities are wide-ranging, and are not limited to the study of hypersurface singularities, and certainly are not limited to work from the 1970's. However, in order to have one coherent theme for these lectures, we will restrict ourselves to the work of Lê which we first encountered in our own research: his work on Milnor fibrations.

In the remainder of this section, we wish to summarize some of the basic results of Milnor from [23]. Some of these results, as presented here, are not actually in [23], but follow quickly; others are proved by Milnor in the case of isolated singularities, but not in the generality that we give. We should mention that this is certainly **not** a complete summary of the results of Milnor; we include only those results which are necessary to understand the results of Lê that we will discuss.

Throughout this paper, we let $\mathcal{U}$ denote a connected, open neighborhood of the origin in $\mathbb{C}^{n+1}$, and use $\mathbf{z} := (z_0, \dots, z_n)$ for the standard coordinate functions. We let $f : (\mathcal{U}, \mathbf{0}) \to (\mathbb{C}, 0)$ denote a non-constant complex analytic function, and let $V(f) := f^{-1}(0)$. Let Σf denote the critical locus of f, and let $s = \dim_{\mathbf{0}} \Sigma f$. (Unless we specifically state otherwise, when we discuss the *dimension* of a complex analytic space, we will mean the complex dimension. For spaces which are not complex analytic, *dimension* will mean the dimension over the real numbers.) By convention, we set $\dim \emptyset$ to be $-\infty$. Using the Curve Selection Lemma (see [23]), it is easy to show that, near $\mathbf{0}$, Σf is contained in $V(f)$. We shall always assume throughout these notes that $\mathcal{U}$ has been chosen so small that $\Sigma f \subseteq V(f)$.

For $\epsilon > 0$, we let S_ϵ (respectively, B_ϵ) denote the sphere (resp., closed ball) of radius ϵ centered at the origin in $\mathbb{C}^{n+1}$. We let $\overset{\circ}{B}_\epsilon$ denote the interior of B_ϵ. In the special case where $n = 0$, we write $\partial\mathbb{D}_\epsilon$, $\mathbb{D}_\epsilon$, and $\overset{\circ}{\mathbb{D}}_\epsilon$ in place of S_ϵ, B_ϵ, and $\overset{\circ}{B}_\epsilon$, respectively. To simplify notation below, if a domain $Y \subseteq \mathcal{U}$ is clearly indicated, we continue to write f in place of $f_{|_Y}$.

The primary goal of Milnor's study in [23] is to investigate the local, ambient topological-type of a complex hypersurface in $\mathcal{U}$, i.e., to investigate, up to homeomorphism, germs of triples $(\mathcal{W}, \mathcal{W} \cap V(f), \{\mathbf{0}\})$, where $\mathcal{W}$ is an open neighborhood of the origin in $\mathcal{U}$. Thus, one wants to describe triples $(\overset{\circ}{B}_\epsilon, \overset{\circ}{B}_\epsilon \cap V(f), \{\mathbf{0}\})$ for sufficiently small $\epsilon > 0$. It would, of course, be sufficient to describe the compact triple $(B_\epsilon, B_\epsilon \cap V(f), \{\mathbf{0}\})$. One has the following theorem, due to Milnor [23] in the case of an isolated singularity, and due to Łojasiewicz [19] and Burghelea and Verona [2] in the general case.

Theorem 1.1. (Milnor, Łojasiewicz, Burghelea and Verona) *There exists $\epsilon_0 > 0$ such, for all ϵ such that $0 < \epsilon \leq \epsilon_0$, the homeomorphism-type of the triple $(B_\epsilon, B_\epsilon \cap V(f), \{\mathbf{0}\})$ is independent of ϵ, and is the same as that of the cone on the pair $(S_\epsilon, S_\epsilon \cap V(f))$ (where this yields a triple by including the cone point subset).*

Definition 1.2. *The space $K := S_\epsilon \cap V(f)$ above (for $0 < \epsilon \leq \epsilon_0$) is called the* **real link of $V(f)$ at the origin**.

Theorem 1.1 reduces the study of the local, ambient topological-type of $V(f)$ at the origin to understanding how the real link embeds inside the sphere. Understanding K and the complement $S_\epsilon - K$ yields a significant partial description of the homeomorphism-type of the pair (S_ϵ, K). Understanding these two spaces is the primary goal of [23].

The most fundamental result of [23] is that $S_\epsilon - K$ is the total space of a specific locally-trivial fibration over a circle.

Theorem 1.3. (Milnor) *There exists $\epsilon_0 > 0$ such that, for all ϵ such that $0 < \epsilon \leq \epsilon_0$, there exists a $\delta_\epsilon > 0$ such that, for all δ such that $0 < \delta \leq \delta_\epsilon$, $f/|f| : S_\epsilon - K \to S^1 \subseteq \mathbb{C}$ is a smooth locally-trivial fibration, which is diffeomorphic to $f : \overset{\circ}{B}_\epsilon \cap f^{-1}(\partial\mathbb{D}_\delta) \to \partial\mathbb{D}_\delta$, and such that this diffeomorphism-type is independent of the choices of such ϵ and δ.*

Definition 1.4. *Either of the fibrations above is referred to as the* **Milnor fibration of f at the origin**. *When it is important to distinguish the two fibrations, they are referred to as the* **Milnor fibration on the sphere** *and the* **Milnor fibration inside the ball**, *respectively.*

The fiber of either fibration is referred to as the **Milnor fiber of f at the origin**, *and is denoted by $F_{f,\mathbf{0}}$.*

An ϵ (resp. (ϵ, δ), resp. B_ϵ, resp. S_ϵ) such as that appearing in Theorem 1.3 is called a **Milnor radius** *(resp.* **Milnor pair**, *resp.* **Milnor ball**, *resp.* **Milnor sphere***) for f at* $\mathbf{0}$.

Associated to the Milnor fibration, one has **characteristic homeomorphisms** *or* **diffeomorphisms** *which describe how $F_{f,\mathbf{0}}$ gets "glued" to itself as one lets the base-point of the fiber travel counterclockwise once around S^1. Such topological or geometric automorphisms are not unique; however, the induced automorphisms on homology or cohomology* **are** *unique. The* **Milnor monodromy automorphism**, *$T^*_{f,\mathbf{0}}$, is the graded automorphism on $H^*(F_{f,\mathbf{0}};\mathbb{Z})$ (or on $H_*(F_{f,\mathbf{0}};\mathbb{Z})$) induced by a characteristic homeomorphism of the Milnor fibration.*

Remark 1.5. There is, of course, no reason to consider the Milnor fibration at only the origin. Later, it will be important for us to look at the Milnor fiber, $F_{f,p}$, at points $p \in V(f)$ other than the origin. One simply replaces the balls and spheres centered at the origin in the definitions above with balls $B_\epsilon(p)$ and spheres $S_\epsilon(p)$ centered at p.

Theorem 1.6. (Milnor) *Suppose that $p \in V(f)$. The Milnor fiber has the following properties:*

1. *$F_{f,p}$ is a complex manifold of dimension n;*
2. *$F_{f,p}$ has the homotopy-type of a finite (real) n-dimensional CW-complex;*
3. *if $\dim_p \Sigma f \leq 0$, then $F_{f,p}$ is homotopy-equivalent to the one-point union (wedge) of a finite number of n-spheres;*
4. *if $\dim_p \Sigma f \leq 0$, then the rank of the reduced homology group $\widetilde{H}_n(F_{f,p};\mathbb{Z})$ (i.e., the number of spheres from the item above) is equal to the degree of the normalized Jacobian map*

$$\frac{\epsilon\left(\frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n}\right)}{\left|\left(\frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n}\right)\right|} : S_\epsilon(p) \to S_\epsilon(p)$$

for sufficiently small $\epsilon > 0$; this degree is thus equal to

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathcal{U},p}}{\left\langle \frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n} \right\rangle},$$

where $\mathcal{O}_{\mathcal{U},p} := \mathbb{C}\{z_0 - p_0, \dots, z_n - p_n\}$ is the ring of convergent power series at the origin.

Remark 1.7. One might wonder what happens in the "trivial" case where $n = 0$. In this case, $f(z_0)$ is, up to analytic isomorphism, of the form z_0^d, and the Milnor fiber consists of d points. As a 0-sphere consists of a pair of points, the one-point union of $(d-1)$ 0-spheres is homeomorphic to d points. Thus, even when $n = 0$, all of the numbers described in Items 3 and 4 above are equal; in particular, it is important that we used the **reduced** homology in Item 4.

Definition 1.8. *When $p \in V(f)$ and $\dim_p \Sigma f \leq 0$, the number described in Items 3 and 4 of Theorem 1.6 is called the* **Milnor number of** f **at** p*, and is denoted $\mu_f(p)$.*

More generally, if $\dim_p \Sigma f \leq 0$ (but perhaps $p \notin V(f)$), we define $\mu_f(p) := \mu_{f-f(p)}(p)$.

In general, it is easier to deal with compact spaces with well-behaved boundaries than it is to deal with non-compact spaces with no boundaries. The Milnor fibration on the sphere does not compactify "nicely", since the closure of each fiber of $f/|f| : S_\epsilon - K \to S^1$ contains K. On the other hand, the Milnor fibration inside the ball compactifies very well, yielding a new fibration with a total space and fibers which are smooth manifolds with boundary, which are homotopy-equivalent to the respective spaces without boundary. Thus, we define:

Definition 1.9. *For $0 < \delta \ll \epsilon \ll 1$, the locally trivial fibration $f : B_\epsilon \cap f^{-1}(\partial \mathbb{D}_\delta) \to \partial \mathbb{D}_\delta$ is called the* **compact Milnor fibration of** f **at** $\mathbf{0}$.

Remark 1.10. In practice, all three fibrations from Definition 1.4 and Definition 1.9 are frequently referred to as simply **THE Milnor fibration** of f at $\mathbf{0}$, letting the context clarify which fibration is intended. Of course, as before, we may consider any of our Milnor fibrations at points $p \in V(f)$ other than the origin, simply by using balls or spheres centered at p.

We shall need the main result of A'Campo from [1] (which we state only for the constant sheaf on affine space).

Theorem 1.11. (A'Campo) *Suppose that $\mathbf{0} \in \Sigma f$.*

Then, the Lefschetz number of the Milnor monodromy automorphism at the origin is zero. In particular, $F_{f,\mathbf{0}}$ does not have the homology of a point.

Remark 1.12. Later, it will be important for us to look at the above theorem of A'Campo in the case where $\mathbf{0}$ is an isolated critical point of f. By Item 3 of Theorem 1.6 and Theorem 1.11, it follows that the action induced by the monodromy on the *vanishing homology group*,

$$H_{n+1}(B_\epsilon, F_{f,\mathbf{0}}; \mathbb{Z}) \cong H_{n+1}(B_\epsilon \cap f^{-1}(\mathbb{D}_\delta), B_\epsilon \cap f^{-1}(\delta); \mathbb{Z}) \cong \widetilde{H}_n(F_{f,\mathbf{0}}; \mathbb{Z}),$$

has trace $(-1)^{n+1}$.

As our final result of this introduction, we wish to explain the precise relationship between $\mathbf{0}$ being a critical point of f and $V(f)$ having a singular point at $\mathbf{0}$. While there is no question what one means by a *critical point of* f, there are various concepts of what one might mean by a singular point.

Theorem 1.13. *There are two cases which can occur:*

1. *there exists $g \in \mathcal{O}_{\mathcal{U},\mathbf{0}}$ and $d \in \mathbb{N}$ such that $f = g^d$ in $\mathcal{O}_{\mathcal{U},\mathbf{0}}$, $\mathbf{0} \notin \Sigma g$, and there exists an open neighborhood $\mathcal{W}$ of $\mathbf{0}$ such that $\mathcal{W} \cap V(f) = \mathcal{W} \cap V(g)$ is a complex analytic submanifold of $\mathcal{W}$;*

2. *there does not exist an open neighborhood $\mathcal{W}$ of $\mathbf{0}$ such that $\mathcal{W} \cap V(f)$ is a topological submanifold of $\mathcal{W}$.*

Proof. Case 1 is the case where the reduced form of f, i.e., a generator of the radical of $\langle f \rangle$, has no critical point at the origin; the remainder of Case 1 follows from the Implicit Function Theorem.

It remains to be shown that, if g is reduced and has a critical point at the origin, then there does not exist an open neighborhood $\mathcal{W}$ of $\mathbf{0}$ such that $\mathcal{W} \cap V(g)$ is a topological submanifold of $\mathcal{W}$. By Theorem 1.1, this is equivalent to showing that the real link K is not an unknotted sphere in S_ϵ for small $\epsilon > 0$. If $\dim_{\mathbf{0}} \Sigma g = 0$, then this is proved by Milnor in [23], Corollary 7.3.

Suppose that $\dim_{\mathbf{0}} \Sigma g > 0$, and suppose to the contrary that the real link K **is** an unknotted sphere in S_ϵ. Then, following the proof of Milnor in [23], Corollary 7.3, $F_{g,\mathbf{0}}$ would have to have the homotopy groups of a point. By the Hurewicz Theorem, this would imply that $F_{g,\mathbf{0}}$ has the homology of a point (in fact, combined with Item 2 of Theorem 1.6, this implies that $F_{g,\mathbf{0}}$ must be contractible). However, this is impossible by the main result of A'Campo in [1], which says that the monodromy automorphism has Lefschetz number equal to 0. □

Remark 1.14.The reader should understand the two extremes given in Theorem 1.13: in a neighborhood of the origin in $\mathcal{U}$, either $V(f)$ is a submanifold in the strongest possible sense of the term "submanifold", or $V(f)$ is not a submanifold even using the weakest possible sense of the term "submanifold".

1.2 Topological Invariance

One of the most fundamental properties of the Milnor fiber is often stated without reference: for reduced hypersurfaces, the homotopy-type of the Milnor fiber is an invariant of the local, ambient topological-type of the hypersurface. For an isolated critical point, this statement is equivalent to saying that the Milnor number is an invariant of the local, ambient topological-type of the reduced hypersurface; in this case, this result appears in a remark of Teissier in [25] in 1972 and in [26] in 1973. The general result, with a monodromy statement, is due to Lê in [9] and [8], which both appeared in 1973.

Theorem 1.15.(Teissier, Lê) *Let $f : (\mathcal{U}, \mathbf{0}) \to (\mathbb{C}, 0)$ and $g : (\mathcal{U}, \mathbf{0}) \to (\mathbb{C}, 0)$ be reduced complex analytic functions which define hypersurfaces with the same ambient topological-type at the origin, i.e., such that there exist open neighborhoods $\mathcal{W}$ and $\mathcal{V}$ of the origin and a homeomorphism $h : \mathcal{W} \to \mathcal{V}$ such that $h(\mathbf{0}) = \mathbf{0}$ and $h(\mathcal{W} \cap V(f)) = \mathcal{V} \cap V(g)$.*

Then, there exists a homotopy-equivalence $\alpha : F_{f,\mathbf{0}} \to F_{g,\mathbf{0}}$ such that the induced isomorphism on homology commutes with the respective Milnor monodromy automorphisms.

Sketch of proof. We will sketch the proof of the homotopy-equivalence claim in the special case that the Milnor fibers are simply-connected. The subtlety of the problem lies in the fact that the definition of the Milnor fiber uses actual spheres and balls, not merely neighborhoods which are homeomorphic to spheres and balls.

Let ϵ_0 be a Milnor radius for f at $\mathbf{0}$, and let η_0 be a Milnor radius for g at $\mathbf{0}$. Pick ϵ_1 such that $0 < \epsilon_1 \leq \epsilon_0$ and $B_{\epsilon_1} \subseteq \mathcal{W}$. Pick η_1 such that $0 < \eta_1 \leq \eta_0$ and $B_{\eta_1} \subseteq h(B_{\epsilon_1})$. Pick $\epsilon_2 > 0$ such that $B_{\epsilon_2} \subseteq h^{-1}(B_{\eta_1})$. Finally, pick $\eta_2 > 0$ such that $B_{\eta_2} \subseteq h(B_{\epsilon_2})$.

Then, we have the inclusions

$$h^{-1}(B_{\eta_2} - V(g)) \subseteq B_{\epsilon_2} - V(f) \subseteq h^{-1}(B_{\eta_1} - V(g)) \subseteq B_{\epsilon_1} - V(f),$$

and Theorem 1.1 and Theorem 1.3 imply that the inclusions $h^{-1}(B_{\eta_2} - V(g)) \subseteq h^{-1}(B_{\eta_1} - V(g))$ and $B_{\epsilon_2} - V(f) \subseteq B_{\epsilon_1} - V(f)$ are homotopy-equivalences, and that $h^{-1}(B_{\eta_1} - V(g))$ and $B_{\epsilon_1} - V(f)$ have the homotopy-types of the total spaces of the respective Milnor fibrations of f and g.

It follows that the inclusion $B_{\epsilon_2} - V(f) \subseteq h^{-1}(B_{\eta_1} - V(g))$ is a homotopy-equivalence, i.e., h induces a homotopy-equivalence between the total spaces of the Milnor fibrations of f and g.

Now, consider the cyclic, universal covering map $p : \mathbb{R} \to S^1$. The pull-back of the Milnor fibration of f by p yields a cyclic covering map $\tilde{p}$ from a space E_f to $S_{\epsilon_0} - V(f)$, and a projection map m of a locally-trivial fibration $m : E_f \to \mathbb{R}$, such that the fiber of m is homeomorphic to $F_{f,\mathbf{0}}$. As $\mathbb{R}$ is contractible, this implies that E_f is homeomorphic to $\mathbb{R} \times F_{f,\mathbf{0}}$; in particular, E_f is homotopy-equivalent to $F_{f,\mathbf{0}}$. Thus, there is a cyclic covering map $\tilde{p} : E_f \to S_{\epsilon_0} - V(f)$, where E_f has the homotopy-type of $F_{f,\mathbf{0}}$. Using the same argument, there is a cyclic covering map $\tilde{q} : E_g \to S_{\eta_0} - V(g)$, where E_g has the homotopy-type of $F_{g,\mathbf{0}}$.

If $F_{f,\mathbf{0}}$ and $F_{g,\mathbf{0}}$ are both simply-connected (the *usual case*; see Section 2.3), then E_f and E_g are simply-connected and are thus the universal covers of $S_{\epsilon_0} - V(f)$ and $S_{\eta_0} - V(g)$, respectively. As we know that $S_{\epsilon_0} - V(f)$ and $S_{\eta_0} - V(g)$ have the same homotopy-type, it follows that $F_{f,\mathbf{0}}$ and $F_{g,\mathbf{0}}$ have the same homotopy-type.

If $F_{f,\mathbf{0}}$ and $F_{g,\mathbf{0}}$ are not simply-connected, there is more work to do and, to obtain the full theorem, one must still make the argument about commuting with the Milnor monodromy automorphisms. For these arguments, we direct the reader to [9] and [8]. □

Remark 1.16. It is reasonable to ask if the converse of Theorem 1.15 is true. That is, if $f : (\mathcal{U}, \mathbf{0}) \to (\mathbb{C}, 0)$ and $g : (\mathcal{U}, \mathbf{0}) \to (\mathbb{C}, 0)$ have isolated critical points at the origin, do they define hypersurfaces with the same local ambient topological-type? The answer is: no.

Consider, for instance, $f = y^2 - x^5$ and $g = y^3 - x^3$. We leave it as an exercise for the reader to verify that both of these functions have Milnor number 4 at the origin, but do not define hypersurfaces with the same ambient topological-type at the origin (actually, these hypersurfaces do not have the same topological-tpe at the origin, leaving out the term "ambient").

However, the stunning conclusion of Lê and Ramanujam in [17] is that, if $n \neq 2$, and f and g are part of an analytic family of functions with isolated critical points, all of which have the same Milnor number, then f and g define hypersurfaces with the same local ambient topological-type; see Section 1.7.

1.3 The Relative Polar Curve

The relative polar curve, $\Gamma^1_{f,\mathfrak{l}}$, of f with respect to a linear form $\mathfrak{l} : \mathbb{C}^{n+1} \to \mathbb{C}$ is an important topological device, first used in 1973 by Hamm and Lê in [5], and by Teissier in [26]. As the relative polar curve is used in many of Lê's proofs of results on hypersurface singularities, we need to review the basic definitions and results on this topic.

However, before we define the relative polar curve, we will discuss Thom's a_f condition, for we will need it to prove that two characterizations of the relative polar curve agree, and we shall also need it in Section 1.6, where we discuss the result of Lê and Saito in [18]. We give a definition of the a_f condition in our special setting where the codomain is simply $\mathbb{C}$ and $\Sigma f \subseteq V(f)$.

Definition 1.17. **(version 1)**. *Let M be an analytic submanifold of $\mathcal{U}$ which is contained in $V(f)$.*

We say that **the pair** $(\mathcal{U}-\Sigma f, M)$ **satisfies the** a_f **condition at a point** $p \in M$ *if and only if, for every sequence of points $q_i \in \mathcal{U}-\Sigma f$ such that $q_i \to p$ and such that the limiting tangent planes to level hypersurfaces converge, i.e., such that there exists an n-dimensional linear subspace $\mathcal{T}$ such that $T_{q_i}V(f - f(q_i)) \to \mathcal{T}$, we have that $T_pM \subseteq \mathcal{T}$.*

We say that **the pair** $(\mathcal{U} - \Sigma f, M)$ **satisfies the** a_f **condition** *(without reference to a point) if and only if, for all $p \in M$, $(\mathcal{U} - \Sigma f, M)$ satisfies the a_f condition at p.*

It will be extremely useful for us to have a cotangent definition of the relative polar curve. We let π denote the projection from the cotangent space $T^*\mathcal{U}$ to $\mathcal{U}$. If M be an analytic submanifold of $\mathcal{U}$, then we let $T^*_M\mathcal{U}$ denote the *conormal space to M in $\mathcal{U}$*, i.e., $T^*_M\mathcal{U} := \{(x,\omega) \in T^*\mathcal{U} \mid x \in M, \omega(T_xM) \equiv 0\}$. If $W \subseteq T^*\mathcal{U}$ and $Y \subseteq \mathcal{U}$, we let $W_Y := W \cap \pi^{-1}(Y)$. In particular, if $p \in M$, $(T^*_M\mathcal{U})_p$ is the fiber $T^*_M\mathcal{U} \cap \pi^{-1}(p)$.

The *relative conormal variety of f*, $T^*_f\mathcal{U}$, is defined as a subset of the cotangent space $T^*\mathcal{U}$ by

$$T^*_f\mathcal{U} := \{(x,\omega) \in T^*U \mid \omega(\ker d_xf) \equiv 0\};$$

we will be interested in its closure, $\overline{T^*_f\mathcal{U}}$, which is a closed, irreducible, $(n+2)$-dimensional, analytic subset of $T^*\mathcal{U}$.

Definition 1.17. **(version 2)**. *Let M be an analytic submanifold of $\mathcal{U}$ which is contained in $V(f)$.*

We say that **the pair** $(\mathcal{U} - \Sigma f, M)$ **satisfies the** a_f **condition** *if and only if,* $\left(\overline{T^*_f\mathcal{U}}\right)_p \subseteq (T^*_M\mathcal{U})_p$.

We say that **the pair** $(\mathcal{U} - \Sigma f, M)$ **satisfies the** a_f **condition** *(without reference to a point) if and only if, for all $p \in M$, $(\mathcal{U} - \Sigma f, M)$ satisfies the a_f condition at p, i.e.,* $\left(\overline{T^*_f\mathcal{U}}\right)_M \subseteq T^*_M\mathcal{U}$.

It is a good exercise to verify that the two versions of Definition 1.17 are, in fact, equivalent.

Definition 1.18. *A complex analytic stratification, $\mathfrak{S}$, of $V(f)$ such that $V(f) - \Sigma f$ is in $\mathfrak{S}$ and such that, for all $S \in \mathfrak{S}$, $(\mathcal{U} - \Sigma f, S)$ satisfies the a_f condition is called a* **good stratification** *for f. Given a good stratification for f, we refer to the strata contained in Σf as the* **critical strata**.

Thus, $\mathfrak{S}$ is a good stratification for f if and only if $V(f) - \Sigma f \in \mathfrak{S}$ and $\left(\overline{T^*_f\mathcal{U}}\right)_{V(f)} \subseteq \bigcup_{S\in\mathfrak{S}} T^*_S\mathcal{U}$.

The following theorem appears in [5], where it is attributed to F. Pham. See also [6].

Theorem 1.19. *For all complex analytic $f : \mathcal{U} \to \mathbb{C}$, there exists a good stratification for f.*

Definition 1.20. *Let $p \in V(f)$ and suppose that $\mathfrak{S}$ is a good stratification for f in a neighborhood of p. Then, a non-zero linear form $\mathfrak{l}$ is a* **prepolar form for** f **at** p **with respect to** $\mathfrak{S}$ *if and only if there exists a neighborhood $\mathcal{W}$ of p in $\mathcal{U}$ such that, for all $S \in \mathfrak{S}$, $V(\mathfrak{l} - \mathfrak{l}(p))$ transversely intersects $S - \{\mathbf{0}\}$ in $\mathcal{W}$.*

A non-zero linear form $\mathfrak{l}$ is a **prepolar form for** f **at** p *if and only if there exists a good stratification $\mathfrak{S}$ of $V(f)$ in a neighborhood of p such that $\mathfrak{l}$ is a prepolar form for f at p with respect to $\mathfrak{S}$.*

From Theorem 1.19, we immediately conclude:

Proposition 1.21. *The set of prepolar linear forms for f at a point $p \in V(f)$ is generic.*

Remark 1.22. Suppose that $\dim_{\mathbf{0}} \Sigma f \leq 1$, and $\mathfrak{l}$ is a linear form such that $\dim_{\mathbf{0}} \Sigma(f_{|_{V(\mathfrak{l})}}) \leq 0$. Then, it is a trivial exercise to conclude that $\mathfrak{l}$ is a prepolar form for f at $\mathbf{0}$.

At last, we are ready to proceed with our discussion of the relative polar curve, $\Gamma^1_{f,\mathfrak{l}}$, of f with respect to a linear form $\mathfrak{l} : \mathbb{C}^{n+1} \to \mathbb{C}$.

One may view $\Gamma^1_{f,\mathfrak{l}}$ as a set, a scheme, or as an analytic cycle. In fact, the scheme structure will not be used in this paper. Thus, we will let $\Gamma^1_{f,\mathfrak{l}}$ denote the relative polar curve as a cycle, and use $\left|\Gamma^1_{f,\mathfrak{l}}\right|$ for the underlying set. In general, if V denotes a scheme, we write $[V]$ for the associated analytic cycle; frequently, however, the context will make it clear that we are considering only the cycle structure, and we will omit the square brackets.

We will use the intersection theory of proper intersections inside analytic manifolds. This theory gives one well-defined intersection cycles, not intersection classes. The reader is directed to Appendix A of [20] for this easy case, or to [3] for a complete treatment of intersection theory.

It will be helpful for us to have notation for the sum (respectively, union) of the components of an analytic cycle $C := \sum_i m_i[V_i]$ (respectively, analytic set $X := \bigcup_i V_i$) which are not contained in an analytic set W. We let $C \neg W := \displaystyle\sum_{V_i \not\subseteq W} m_i[V_i]$ and $X \neg W := \displaystyle\bigcup_{V_i \not\subseteq W} V_i$. Thus, $|C \neg W| = |C| \neg W$, and $X \neg W = \overline{X - W}$.

Definition 1.23. *Let $\mathfrak{l}$ be a non-zero linear form, and let $\Sigma(f, \mathfrak{l})$ denote the critical locus of the map $(f, \mathfrak{l})$. Let $\mathrm{Jac}(f, \mathfrak{l})$ denote the ideal generated by 2×2 minors of the Jacobian matrix of $(f, \mathfrak{l})$.*

Then, the **polar curve**, *$\Gamma^1_{f,\mathfrak{l}}$, is defined (as a cycle) to be $\left[V(\mathrm{Jac}(f, \mathfrak{l})\right] \neg \Sigma f$. Thus, the set $\left|\Gamma^1_{f,\mathfrak{l}}\right|$ is $\Sigma(f, \mathfrak{l}) \neg \Sigma f$.*

In particular, if $\mathfrak{l} = z_0$, our first coordinate function, then we have

$$\Gamma^1_{f,z_0} = \left[V\left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)\right] \neg \Sigma f.$$

Note that we refer to $\Gamma^1_{f,\mathfrak{l}}$ as a *curve*, despite the fact that, for non-generic $\mathfrak{l}$, $\Gamma^1_{f,\mathfrak{l}}$ may not be 1-dimensional. It is immediate that $\Gamma^1_{f,\mathfrak{l}}$ has no 0-dimensional components, since $V\left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)$ is defined by n equations in an open subset of $\mathbb{C}^{n+1}$. As we shall see below, for generic $\mathfrak{l}$, either $\mathbf{0} \notin \Gamma^1_{f,\mathfrak{l}}$ or $\dim_{\mathbf{0}} \Gamma^1_{f,\mathfrak{l}} = 1$.

Example 1.24. Let $f = y^2 - x^a - tx^b$ where $a > b > 1$, and let $\mathfrak{l} = t$. Then, $\Sigma f = V(x, y)$, which is the t-axis, and

$$\Gamma^1_{f,t} = \left[V\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\right] \neg \Sigma f.$$

Now, the cycle $\left[V\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\right]$ is equal to $[V(ax^{a-1} + btx^{b-1}, y)] = [V(x^{b-1}(ax^{a-b} + bt), y)]$, which equals $(b-1)[V(x, y)] + [V(ax^{a-b} + bt, y)]$. Disposing of the component(s) whose underlying sets are contained in $\Sigma f = V(x, y)$, we find that $\Gamma^1_{f,t} = [V(ax^{a-b} + bt, y)]$.

One can also give a cotangent description of $\Gamma^1_{f,\mathfrak{l}}$, which agrees with Definition 1.23 when $\mathfrak{l}$ is generically chosen. Most of the theorem below is from 2.1.3.1 of [5], though our presentation is slightly different; however, Items 3 and 4 use an argument of Teissier from [26] (which was stated for isolated critical points, but the argument works more generally). We use the proper push-forward of cycles, and again direct the reader to Appendix A of [20] or to [3]. The notation $\operatorname{im} d\mathfrak{l}$ is used for the image of section of the cotangent bundle given by $d\mathfrak{l}$, i.e., $\operatorname{im} d\mathfrak{l} = \{(x, d_x\mathfrak{l}) \in T^*\mathcal{U} \mid x \in \mathcal{U}\}$.

Theorem 1.25. (2.1.3.1 of [5], and [26]) *Let $\mathfrak{l}$ be a prepolar linear form for f at $\mathbf{0}$. Then, there exists a neighborhood $\mathcal{W}$ of $\mathbf{0}$ in $\mathcal{U}$ such that*

1. $|\Gamma^1_{f,\mathfrak{l}}| \cap \mathcal{W}$ *is purely 1-dimensional (which includes the possibility of being empty);*
2. $\left|\Gamma^1_{f,\mathfrak{l}}\right| \cap V(f) \cap \mathcal{W} \subseteq \{\mathbf{0}\}$;
3. $\left|\Gamma^1_{f,\mathfrak{l}}\right| \cap V(\mathfrak{l}) \cap \mathcal{W} \subseteq \{\mathbf{0}\}$;
4. $\left(\Gamma^1_{f,\mathfrak{l}} \cdot V(f)\right)_{\mathbf{0}} = \left(\Gamma^1_{f,\mathfrak{l}} \cdot V\left(\frac{\partial f}{\partial \mathfrak{l}}\right)\right)_{\mathbf{0}} + \left(\Gamma^1_{f,\mathfrak{l}} \cdot V(\mathfrak{l})\right)_{\mathbf{0}}$;
5. $(\overline{T^*_f\mathcal{U}})_{\mathcal{W}} \cap \operatorname{im} d\mathfrak{l}$ *is purely 1-dimensional and, inside $\mathcal{W}$, the cycle $\Gamma^1_{f,\mathfrak{l}}$ equals the proper push-forward* $\pi_*\left((\overline{T^*_f\mathcal{U}})_{\mathcal{W}} \cdot \operatorname{im} d\mathfrak{l}\right)$.

   Furthermore, if $\mathbf{0} \notin |\Gamma^1_{f,\mathfrak{l}}|$ (i.e., the relative polar curve is empty at $\mathbf{0}$), then $\mathbf{0} \notin |\Gamma^1_{f,\mathfrak{l}'}|$ for generic $\mathfrak{l}'$.

Sketch of proof. Define the conormal relative polar curve, $|{}^c\Gamma^1_{f,\mathfrak{l}}|$, as a set, to be $\pi\left(\overline{T^*_f\mathcal{U}} \cap \operatorname{im} d\mathfrak{l}\right)$. It is trivial to show that $|\Gamma^1_{f,\mathfrak{l}}| - \Sigma f = |{}^c\Gamma^1_{f,\mathfrak{l}}| - \Sigma f$.

Projectivize $T^*\mathcal{U}$ to obtain $\mathbb{P}(T^*\mathcal{U}) \cong \mathcal{U} \times \mathbb{P}^n$ and, using this identification, let π_1 denote the projection to $\mathcal{U}$ and let π_2 denote the projection onto $\mathbb{P}^n$. Note that π_1 is a proper map and, hence, is closed. As $\overline{T^*_f\mathcal{U}}$ is closed and conic (i.e., the fibers via π are closed under scalar multiplication), $\mathbb{P}\left(\overline{T^*_f\mathcal{U}}\right)$ is a closed subset of $\mathbb{P}(T^*\mathcal{U})$. If $\mathfrak{l} = a_0z_0 + \dots a_nz_n$, then $|{}^c\Gamma^1_{f,\mathfrak{l}}| = \pi_1(\pi_2^{-1}([a_0 : \dots : a_n]))$. Thus, $|{}^c\Gamma^1_{f,\mathfrak{l}}|$ is closed. Then, it follows immediately that $|\Gamma^1_{f,\mathfrak{l}}| = \overline{|\Gamma^1_{f,\mathfrak{l}}| - \Sigma f} \subseteq |{}^c\Gamma^1_{f,\mathfrak{l}}|$.

Let $\mathfrak{l}$ be a prepolar linear form for f at $\mathbf{0}$ with respect to a good stratification $\mathfrak{S}$ for f. Recall that this implies that $\left(\overline{T^*_f\mathcal{U}}\right)_{V(f)} \subseteq \bigcup_{S\in\mathfrak{S}} T^*_S\mathcal{U}$.

If we show that $\dim_0 |{}^c\Gamma^1_{f,\mathfrak{l}}| \cap V(f) = 0$, then Items 1 and 2 follow and, furthermore, $|\Gamma^1_{f,\mathfrak{l}}| = |{}^c\Gamma^1_{f,\mathfrak{l}}|$.

It is well-known, and follows from the Curve Selection Lemma (see [23], though one can actually use a complex analytic version here), that there is an open neighborhood $\mathcal{W}$ of $\mathbf{0}$ in $\mathcal{U}$ such that, for all $S \in \mathfrak{S}$, $\mathcal{W} \cap \Sigma(\mathfrak{l}_{|_S}) \subseteq V(\mathfrak{l})$ (one says that, *locally, the stratified critical values are isolated*). As we are assuming that $\mathfrak{l}$ is prepolar with respect to $\mathfrak{S}$ at $\mathbf{0}$, we conclude that one can choose $\mathcal{W}$ so that, for all $S \in \mathfrak{S}$, $\mathcal{W} \cap \Sigma(\mathfrak{l}_{|_S}) \subseteq \{\mathbf{0}\}$, i.e., for all $S \in \mathfrak{S}$, for all $x \in S - \{\mathbf{0}\}$, $d_x\mathfrak{l} \notin (T^*_S\mathcal{U})_x$. This is equivalent to $\pi\left((\bigcup_{S\in\mathfrak{S}} T^*_S\mathcal{U})_{\mathcal{W}} \cap \operatorname{im} d\mathfrak{l}\right) \subseteq \{\mathbf{0}\}$. Therefore, we have

$$\mathcal{W} \cap |{}^c\Gamma^1_{f,\mathfrak{l}}| \cap V(f) = \pi\left(\left(\overline{T^*_f\mathcal{U}}\right)_{\mathcal{W}\cap V(f)} \cap \operatorname{im} d\mathfrak{l}\right) \subseteq \pi\Big(\big(\bigcup_{S\in\mathfrak{S}} T^*_S\mathcal{U}\big)_{\mathcal{W}} \cap \operatorname{im} d\mathfrak{l}\Big) \subseteq \{\mathbf{0}\}.$$

Items 1 and 2 follow, as does the fact that $|\Gamma^1_{f,\mathfrak{l}}| = |{}^c\Gamma^1_{f,\mathfrak{l}}|$.

Item 5 is the cycle version of the equality of sets $|\Gamma^1_{f,\mathfrak{l}}| = |^c\Gamma^1_{f,\mathfrak{l}}|$. It is now easy to verify by using coordinates and letting $\mathfrak{l} = z_0$. We leave this as an exercise.

Items 3 and 4 follows from Item 2, together with an argument of Teissier from [26]. After a change of coordinates, assume that $\mathfrak{l} = z_0$. Suppose that $\left|\Gamma^1_{f,z_0}\right| \cap V(z_0)$ contains a complex analytic curve C which contains the origin and is irreducible there, i.e., $V(z_0)$ contains an irreducible component of $\left|\Gamma^1_{f,z_0}\right|$ which contains the origin. Take a complex analytic parameterization of C near $\mathbf{0}$; that is, let $\alpha : \overset{\circ}{\mathbb{D}} \to C$ be a complex analytic map from an open disk into C such that $\alpha^{-1}(\mathbf{0}) = 0$. Using the chain rule, we find

$$(\dagger) \qquad \big(f(\alpha(t))\big)' = \left(\frac{\partial f}{\partial z_0}\right)_{|_{\alpha(t)}} \cdot \big(z_0(\alpha(t))\big)' + \cdots + \left(\frac{\partial f}{\partial z_n}\right)_{|_{\alpha(t)}} \cdot \big(z_n(\alpha(t))\big)'.$$

As C is a component of $\left|\Gamma^1_{f,z_0}\right|$, all of the terms in the summation above are 0, except possibly the first term. However, even the first term of the sum in (†) is 0, since we are assuming that $C \subseteq V(z_0)$. Therefore, $\big(f(\alpha(t))\big)' = 0$, and so $f(\alpha(t)) \equiv 0$, which contradicts Item 2. Thus, we have proved Item 3. Now, the formula in (†) immediately yields Item 4.

Finally, we need to prove the "Furthermore". Consider $\pi_1^{-1}(\mathbf{0}) \cap \mathbb{P}\big(\overline{T^*_f\mathcal{U}}\big) \subseteq \{\mathbf{0}\} \times \mathbb{P}^n$. Suppose that $\mathbf{0} \notin |\Gamma^1_{f,\mathfrak{l}}| = |^c\Gamma^1_{f,\mathfrak{l}}|$. Then, $\pi_1^{-1}(\mathbf{0}) \cap \mathbb{P}\big(\overline{T^*_f\mathcal{U}}\big)$ is a proper analytic subset of $\{\mathbf{0}\} \times \mathbb{P}^n$ and, hence, for generic $\mathfrak{l}'$, $\mathbf{0} \notin |^c\Gamma^1_{f,\mathfrak{l}'}|$. Since prepolar forms are generic, the desired conclusion follows. □

In fact, in 2.2.1 of [5], Hamm and Lê prove that if we assume that $\mathfrak{l}$ is generic enough, then each component of $\Gamma^1_{f,\mathfrak{l}}$ appears in the cycle with multiplicity 1. We shall not need this extra genericity.

1.4 Families of Isolated Hypersurface Singularities

In the remainder of this first lecture, we shall discuss three results of Lê and others on families of isolated hypersurface singularities. We continue to let $\mathcal{U}$ denote a connected, open neighborhood of the origin in $\mathbb{C}^{n+1}$, and we also fix an open complex disk $\overset{\circ}{\mathbb{D}}_\tau$ centered at the origin. We continue to use $\mathbf{z} := (z_0, \dots, z_n)$ as coordinates on $\mathcal{U}$, and let t denote the coordinate on $\overset{\circ}{\mathbb{D}}_\tau$.

We let $g : (\mathcal{U} \times \overset{\circ}{\mathbb{D}}_\tau, \{\mathbf{0}\} \times \overset{\circ}{\mathbb{D}}_\tau) \to (\mathbb{C}, 0)$ be an analytic function. For all $a \in \overset{\circ}{\mathbb{D}}_\tau$, we let $g_a : (\mathcal{U}, \mathbf{0}) \to (\mathbb{C}, 0)$ be defined by $g_a(\mathbf{z}) := g(\mathbf{z}, a)$, and we assume that $g_a \not\equiv 0$. One says that g **(or the collection of g_a) defines a (1-parameter, analytic) family of hypersurfaces at the origin.**

We suppose that $\dim_{\mathbf{0}} \Sigma g_0 = 0$ (in particular, we are assuming that $\mathbf{0}$ is, in fact, a critical point of g_0). It is trivial to show that this implies that, if we re-choose $\mathcal{U}$ and τ small enough, then, for all $a \in \overset{\circ}{\mathbb{D}}_\tau$, $\dim \Sigma g_a \leq 0$; we assume that $\mathcal{U}$ and τ are re-chosen this small. We also assume that $\mathcal{U}$ and τ are small enough so that $\Sigma g \subseteq V(g)$ and $\Sigma g_0 \subseteq \{\mathbf{0}\}$. One says that g **(or the collection of g_a) defines a family of isolated hypersurface singularities at the origin.**

The fact that $\Sigma g_a = V\left(\frac{\partial g}{\partial z_0}, \dots, \frac{\partial g}{\partial z_n}, t - a\right)$ is 0-dimensional (or empty) at each point in $\mathcal{U} \times \overset{\circ}{\mathbb{D}}_\tau$ implies that $\dim V\left(\frac{\partial g}{\partial z_0}, \dots, \frac{\partial g}{\partial z_n}\right) \leq 1$, which certainly implies that $\Sigma g = V\left(\frac{\partial g}{\partial z_0}, \dots, \frac{\partial g}{\partial z_n}, \frac{\partial g}{\partial t}\right)$ has dimension at most 1. Thus, $\dim \Sigma g \leq 1$ and, by assumption, $\dim_{\mathbf{0}} \Sigma g_0 = 0$. As we pointed out in Remark 1.22, this implies that t is a prepolar linear form for g at $\mathbf{0}$. We re-choose $\mathcal{U}$ and τ small enough one last time; we assume that $\mathcal{W} := \mathcal{U} \times \overset{\circ}{\mathbb{D}}_\tau$ can be used in place of $\mathcal{W}$ in Theorem 1.25, and that, inside $\mathcal{W}$, every component of the relative polar curve contains the origin.

Therefore, we have that the cycle $\Gamma^1_{g,t}$ is purely 1-dimensional, $|\Gamma^1_{g,t}| \cap V(g) \subseteq \{\mathbf{0}\}$, $|\Gamma^1_{g,t}| \cap V(t) \subseteq \{\mathbf{0}\}$, and $\Gamma^1_{g,t} = \pi_*\left(\overline{T^*_g\mathcal{W}} \cdot \operatorname{im} dt\right)$.

1.5 Non-splitting of the Vanishing Cycles

In this section and the following two, we will be concerned with the Milnor numbers (recall Definition 1.8) of level hypersurfaces defined by g_a at various points. We will once again use some basic intersection theory, in place of the more classical degrees of maps on spheres, but, aside from this change of language, the arguments are all the same. We will use all of our notation from the previous sections.

The result of this section proved independently by Lê [10] in 1973, Gabrielov [4] in 1974, and Lazzeri [7] in 1974. The method of Lê's proof is not similar to the other two.

Before we can discuss this non-splitting result, we need a trivial lemma:

Lemma 1.26. *For all $a \in \mathbb{C}$ such that $|a|$ is sufficiently close to 0,*

$$\mu_{g_0}(\mathbf{0}) = \left(\Gamma^1_{g,t} \cdot V(t)\right)_{\mathbf{0}} \;+ \sum_{p \in V(t-a)\cap\Sigma g} \mu_{g_a}(p).$$

In particular, $\mu_{g_0}(\mathbf{0}) = \sum_{p \in V(t-a)\cap\Sigma g} \mu_{g_a}(p)$ if and only if $|\Gamma^1_{g,t}|$ is empty.

Proof. In terms of analytic cycles and intersection theory, $\mu_{g_0}(\mathbf{0})$ equals the multiplicity of the origin in the cycle $\left[V\left(\frac{\partial g}{\partial z_0}, \dots, \frac{\partial g}{\partial z_n}, t\right)\right]$; this is denoted by $\left[V\left(\frac{\partial g}{\partial z_0}, \dots, \frac{\partial g}{\partial z_n}, t\right)\right]_{\mathbf{0}}$.

Consider the purely 1-dimensional cycle $\left[V\left(\frac{\partial g}{\partial z_0}, \dots, \frac{\partial g}{\partial z_n}\right)\right]$. The sum of the components of this cycle whose underlying sets are not contained in Σg is precisely $\Gamma^1_{g,t}$. Let us denote the sum of the remaining components, those whose underlying sets are contained in Σg, by $\Sigma_{g,t}$.

With this notation, we have

$$\left[V\left(\frac{\partial g}{\partial z_0}, \dots, \frac{\partial g}{\partial z_n}, t\right)\right]_{\mathbf{0}} = \left(V\left(\frac{\partial g}{\partial z_0}, \dots, \frac{\partial g}{\partial z_n}\right) \cdot V(t)\right)_{\mathbf{0}} = \left(\left(\Gamma^1_{g,t} + \Sigma_{g,t}\right) \cdot V(t)\right)_{\mathbf{0}} =$$

$$\left(\Gamma^1_{g,t} \cdot V(t)\right)_{\mathbf{0}} \;+\; \left(\Sigma_{g,t} \cdot V(t)\right)_{\mathbf{0}}.$$

Now,

$$\left(\Sigma_{g,t} \cdot V(t)\right)_{\mathbf{0}} \;=\; \sum_{p \in V(t-a)\cap\Sigma g} \left(\Sigma_{g,t} \cdot V(t-a)\right)_p \;=\; \sum_{p \in V(t-a)\cap\Sigma g} \mu_{g_a}(p),$$

and we are finished. □

So, if there is no polar curve, then $\mu_{g_0}(\mathbf{0}) = \sum_{p \in V(t-a)\cap\Sigma g} \mu_{g_a}(p)$ for small a. A priori, there is no reason to expect that the lack of a polar curve implies anything stronger. However, the non-splitting result of Lê, Gabrielov, and Lazzeri says precisely that the Milnor numbers cannot split among various critical points as one moves from $t = 0$ to $t = a$.

Theorem 1.27. *Suppose that* $\mu_{g_0}(\mathbf{0}) = \sum_{p\in V(t-a)\cap\Sigma g} \mu_{g_a}(p)$ *for small* a. *Then, there is only one point in* $V(t-a)\cap\Sigma g$ *and, hence,* Σg *is smooth and transversely intersected by* $V(t)$ *at* $\mathbf{0}$, *and the Milnor number of* g_a *along* Σg *is constant for small* a.

Sketch of proof. The proof of Lê in [10] can be simplified quite a bit by using Remark 1.12. Nonetheless, the heart of the argument is the same.

First note that Lemma 1.26 implies that $|\Gamma^1_{g,t}|$ is empty. Now, pick an ϵ and δ_0 such that $0<\delta_0\ll\epsilon\ll 1$ and such that, for all δ such that $0<\delta\leq\delta_0$, $B_\epsilon\cap g_0^{-1}(\partial\mathbb{D}_\delta)$ is the total space of the Milnor fibration of g_0. Let $a\in\mathbb{C}$ be such that $0<|a|\ll\delta_0$. For each point $p_i\in B_\epsilon\cap V(t-a)\cap\Sigma g$, let $B_{\epsilon_i}(p_i)$ be a small enough ball in which to define the Milnor fibration of g_a at p_i. Fix a $\delta>0$ such that $\delta\leq\delta_0$ and such that $B_{\epsilon_i}(p_i)\cap g_a^{-1}(\partial\mathbb{D}_\delta)$ yields the Milnor fibration of g_a at each p_i. We also choose δ so small that

$$(\dagger)\qquad H_{n+1}(B_\epsilon\cap g_a^{-1}(\mathbb{D}_\delta), B_\epsilon\cap g_a^{-1}(\delta))\cong\bigoplus_i H_{n+1}(B_\epsilon(p_i)\cap g_a^{-1}(\mathbb{D}_\delta), B_\epsilon(p_i)\cap g_a^{-1}(\delta)).$$

As the relative polar curve is empty, the pair $(B_\epsilon\cap g_0^{-1}(\mathbb{D}_\delta), F_{g_0,\mathbf{0}})$ lifts, in a monodromy-preserving way, to $(B_\epsilon\cap g_a^{-1}(\mathbb{D}_\delta), B_\epsilon\cap g_a^{-1}(\delta))$. Combining this with (†), we conclude that the trace of the monodromy action of g_0 on $H_{n+1}(B_\epsilon\cap g_0^{-1}(\mathbb{D}_\delta), F_{g_0,\mathbf{0}})$ is the sum of the traces of the monodromy actions of g_a on each of the $H_{n+1}(B_\epsilon(p_i)\cap g_a^{-1}(\mathbb{D}_\delta), F_{g_a,p_i})$. Therefore, by Remark 1.12, we conclude that $(-1)^{n+1}=\sum_{p\in V(t-a)\cap\Sigma g}(-1)^{n+1}$. The desired conclusions follow. □

1.6 The Result of Lê and Saito

In this section, we will continue with all of our assumptions and notations involving the family g of isolated hypersurface singularities.

The a_f condition (or , a_g condition, in our current setting) of Definition 1.17 is important for a number of reasons. In Section 1.3, we discussed how Hamm and Lê used good (or a_f) stratifications to define prepolar linear forms (the term "prepolar" is ours, but this type of form was used in [5]), which allowed them to prove Theorem 1.25. The a_f condition is also an hypothesis of Thom's Second Isotopy Lemma (see [21]), a lemma whose conclusion if that a function trivializes along strata. In Section 2.1, we shall see how Lê used the existence of an a_f stratification to prove the existence of a Milnor fibration for f, even when the domain of f is singular.

Finally, in our setting of a family of isolated hypersurface singularities, if one wants to consider the vector field obtained by projecting $\frac{\partial}{\partial t}$ into the level hypersurfaces of g and have this vector field extend continuously to $\frac{\partial}{\partial t}$ along the t-axis, then one needs the a_g condition to be satisfied along the t-axis. It is of some importance to have such a vector field to integrate; we shall discuss this further in the next section.

Having explained why the a_g condition is of so much interest, we will now give the result of Lê and Saito in [18] from 1973.

Theorem 1.28.(Lê and Saito) *Suppose that the Milnor number* $\mu_{g_a}(\mathbf{0})$ *is independent of* a. *Then,* $\Sigma g = \{\mathbf{0}\}\times\overset{\circ}{\mathbb{D}}_\tau$, *and* $(\mathcal{W}-\{\mathbf{0}\}\times\overset{\circ}{\mathbb{D}}_\tau, \{\mathbf{0}\}\times\overset{\circ}{\mathbb{D}}_\tau)$ *satisfies the* a_g *condition.*

Proof. From Lemma 1.26, we know that $|\Gamma^1_{g,t}|$ is empty and that $\Sigma g=\{\mathbf{0}\}\times\overset{\circ}{\mathbb{D}}_\tau$. Let $T:=\Sigma g$.

By the existence of a good stratification for g (Theorem 1.19), the a_g condition is satisfied everywhere along T, except perhaps at isolated points. It suffices, then, to assume that $(\mathcal{W} - \Sigma g, T)$ satisfies the a_g condition everywhere along T except at $\mathbf{0}$, and then show that it also holds at $\mathbf{0}$.

As in the proof of Theorem 1.25, we consider the closed subset $\mathbb{P}\big(\overline{T^*_g\mathcal{W}}\big) \subseteq \mathbb{P}(T^*\mathcal{W}) \cong \mathcal{W} \times \mathbb{P}^{n+1}$, and let π_1 denote the projection to $\mathcal{W}$ and let π_2 denote the projection onto $\mathbb{P}^{n+1}$. Our assumption in the paragraph above tells us that

$$(\dagger) \qquad \mathbb{P}\big(\overline{T^*_g\mathcal{W}}\big)_T \subseteq \big(T \times (\{0\} \times \mathbb{P}^n)\big) \cup \big(\{\mathbf{0}\} \times \mathbb{P}^{n+1}\big).$$

To show that $(\mathcal{W} - T, T)$ satisfies the a_g condition at $\mathbf{0}$, we must show that $\mathbb{P}\big(\overline{T^*_g\mathcal{W}}\big)_T \subseteq \big(T \times (\{0\} \times \mathbb{P}^n)\big)$.

Consider the blow-up $\mathrm{Bl}_{J(f)}\mathcal{W}$ of $\mathcal{W}$ along the Jacobian ideal $J(g) := \langle \frac{\partial g}{\partial z_0}, \dots, \frac{\partial g}{\partial z_n}, \frac{\partial g}{\partial t}\rangle$ (more precisely, we blow up $\mathcal{W}$ along the ordered set of functions which we have given as generators of $J(f)$). This blow-up is a subset of $\mathcal{W} \times \mathbb{P}^{n+1}$. Let us denote the exceptional divisor of this blow-up by E.

It is trivial to see that $E = \mathbb{P}\big(\overline{T^*_g\mathcal{W}}\big)_T$, and so we conclude that $\mathbb{P}\big(\overline{T^*_g\mathcal{W}}\big)_T$ is purely $(n+1)$-dimensional. Combining this with ($\dagger$), we see that in order to show that $\mathbb{P}\big(\overline{T^*_g\mathcal{W}}\big)_T \subseteq \big(T \times (\{0\} \times \mathbb{P}^n)\big)$, we have only to show that $\mathbb{P}\big(\overline{T^*_g\mathcal{W}}\big)$ does not contain $\{\mathbf{0}\} \times \mathbb{P}^{n+1}$, i.e., that $\big(\overline{T^*_g\mathcal{W}}\big)_{\mathbf{0}} \neq \{\mathbf{0}\} \times \mathbb{C}^{n+2}$.

However, this is trivial, for t is a prepolar form for g at $\mathbf{0}$ and $|\Gamma^1_{g,t}|$ is empty. By Item 5 of Theorem 1.25, this implies that $(\mathbf{0}, d_{\mathbf{0}}t) \notin \big(\overline{T^*_g\mathcal{W}}\big)_{\mathbf{0}}$. □

1.7 The Result of Lê and Ramanujam

We continue with all of our previous notation.

The final result that we shall discuss in this lecture is the famous result of Lê and Ramanujam from [17] that the constancy of the Milnor number $\mu_{g_a}(\mathbf{0})$ implies the constancy of the local, ambient topological-type of the hypersurface defined by g_a at the origin, provided that $n \neq 2$. This "dimensional gap" is caused by the (at the time) surprising use of Smale's h-cobordism Theorem (see [24], [22]) in the proof.

The reader should understand how surprising this result is. Recall Remark 1.16, in which we gave an example which showed that two isolated hypersurface singularities having the same Milnor number can be far from having the same topological-type. However, the theorem of Lê and Ramanujam tells one that if the isolated hypersurface singularities occur in an analytic family of isolated hypersurface singularities with constant Milnor number, then the local, ambient topological-type is also constant (except, possibly, in one dimension).

We begin with a lemma.

Lemma 1.29. *Suppose that $f : (\mathcal{U}, \mathbf{0}) \to (\mathbb{C}, 0)$ has an isolated critical point at the origin. Then, the topological-type of the compact Milnor fibration $f : B_\epsilon \cap f^{-1}(\partial\mathbb{D}_\delta) \to \partial\mathbb{D}_\delta$ determines the local, ambient topological-type of $V(f)$ at $\mathcal{U}$ at $\mathbf{0}$.*

Sketch of proof. As Milnor shows in [23], the triple $(B_\epsilon, B_\epsilon \cap V(f), \{\mathbf{0}\})$ is homeomorphic to the triple $(B_\epsilon \cap f^{-1}(\mathbb{D}_\delta), B_\epsilon \cap V(f), \{\mathbf{0}\})$, and so the local, ambient topological-type of $V(f)$ at $\mathbf{0}$ is the cone on the pair $\big((B_\epsilon \cap f^{-1}(\partial\mathbb{D}_\delta)) \cup (S_\epsilon \cap f^{-1}(\mathbb{D}_\delta))\ ,\ S_\epsilon \cap V(f)\big)$.

However, it is easy to show that $f : S_\epsilon \cap f^{-1}(\mathbb{D}_\delta) \to \mathbb{D}_\delta$ is a locally-trivial and, hence, trivial fibration, whose fiber is precisely the boundary, $\partial F_{f,\mathbf{0}}$, of the compact Milnor fiber.

Therefore, given $f : B_\epsilon \cap f^{-1}(\partial\mathbb{D}_\delta) \to \partial\mathbb{D}_\delta$, one may "glue" $\partial F_{f,\mathbf{0}} \times \mathbb{D}_\delta$ onto $B_\epsilon \cap f^{-1}(\partial\mathbb{D}_\delta)$ along their common boundary $\partial F_{f,\mathbf{0}} \times \partial\mathbb{D}_\delta$. The pair

$$\Big(\big(B_\epsilon \cap f^{-1}(\partial\mathbb{D}_\delta)\big) \cup_{\partial F_{f,\mathbf{0}}\times\partial\mathbb{D}_\delta} \big(\partial F_{f,\mathbf{0}} \times \mathbb{D}_\delta\big)\ ,\ \partial F_{f,\mathbf{0}} \times \{0\}\big)\Big)$$

determines the local, ambient topological-type of $V(f)$ at $\mathbf{0}$. □

Now we can prove:

Theorem 1.30. (Lê and Ramanujam) *Suppose that $n \neq 2$ and that the Milnor number $\mu_{g_a}(\mathbf{0})$ is independent of a Then, the local, ambient topological-type of $V(g_a)$ at $\mathbf{0}$ is independent of a.*

Sketch of proof. It suffices to prove the theorem for all a close to 0.

As have seen above, the assumption that $\mu_{g_a}(\mathbf{0})$ is constant implies that $|\Gamma^1_{g,t}| = \emptyset$.

Pick a Milnor pair (ϵ_0, δ_0) for g_0 at the origin, and pick ϵ_0 so small that S_{ϵ_0} transversely intersects $V(g_0)$. It follows that there exists a δ' and τ' such that $0 < \delta' < \delta_0$ and $0 < \tau' < \tau$ and such that, for all $b \in \mathring{\mathbb{D}}_{\delta'}$ and $a \in \mathring{\mathbb{D}}_{\tau'}$, S_{ϵ_0} transversely intersects $g_a^{-1}(b)$.

Consider the map $G := (g, t) : \big(B_{\epsilon_0} \times \mathring{\mathbb{D}}_{\tau'}\big) \cap g^{-1}\big(\mathring{\mathbb{D}}_{\delta'} - \{\mathbf{0}\}\big) \to \big(\mathring{\mathbb{D}}_{\delta'} - \{\mathbf{0}\}\big) \times \mathring{\mathbb{D}}_{\tau'}$. Then, by our choices in the previous paragraph, G has no critical points on the boundary of the domain. As $|\Gamma^1_{g,t}| = \emptyset$, G has no critical points in the interior of the domain. Thus, G is a proper stratified submersion and, hence, is a locally-trivial fibration.

Now, fix $a \in \mathring{\mathbb{D}}_{\tau'}$. Let ϵ_a be a Milnor radius for g_a at $\mathbf{0}$ such that $\epsilon_a < \epsilon_0$. Re-choose δ' (smaller) so that $0 < \delta' \leq \delta_0$ and such that (ϵ_0, δ') and (ϵ_a, δ') are Milnor pairs for g_0 and g_a, respectively, at the origin. Let δ be such that $0 < \delta < \delta'$. We conclude from the previous paragraph that the map (which we still denote by G) $G := (g, t) : \big(B_{\epsilon_0} \times \mathring{\mathbb{D}}_{\tau'}\big) \cap g^{-1}\big(\partial\mathbb{D}_\delta\big) \to \big(\mathring{\mathbb{D}}_{\delta'} - \{\mathbf{0}\}\big) \times \mathring{\mathbb{D}}_{\tau'}$.

As $\mathring{\mathbb{D}}_{\tau'}$ is contractible, it follows that the compact Milnor fibration $g_0 : B_{\epsilon_0} \cap g_0^{-1}(\partial\mathbb{D}_\delta) \to \partial\mathbb{D}_\delta$ is diffeomorphic to the smooth fibration $g_a : B_{\epsilon_0} \cap g_a^{-1}(\partial\mathbb{D}_\delta) \to \partial\mathbb{D}_\delta$.

Given Lemma 1.29, the reader may believe that we have completed the proof. This is far from true. The problem at this point is that B_{ϵ_0} may be too big to use for a Milnor ball for g_a at $\mathbf{0}$.

So, we have the following situation:

The total space of the Milnor fibration of g_a at the origin, $B_{\epsilon_a} \cap g_a^{-1}(\partial\mathbb{D}_\delta)$, includes into $B_{\epsilon_0} \cap g_a^{-1}(\partial\mathbb{D}_\delta)$, which is diffeomorphic to the total space of the Milnor fibration of g_0 at the origin, in a fiber-preserving manner. In addition,the fibers $F_{g_a,\mathbf{0}}$ and $F'_{g_0,\mathbf{0}} := B_{\epsilon_0} \cap g_a^{-1}(\delta)$ have the same homotopy-type, since the Milnor number μ of g_a and g_0 at the origin are the same. Moreover, the map $g_a : \big(B_{\epsilon_0} - \mathring{B}_{\epsilon_a}\big) \cap g_a^{-1}(\mathbb{D}_\delta) \to \mathbb{D}_\delta$ is locally trivial and, hence, trivial.

If we can show that the fiber of this last fibration is a cross-product $E := \big(B_{\epsilon_0} - \mathring{B}_{\epsilon_a}\big) \cap g_a^{-1}(\delta) \cong \partial F_{g_a,\mathbf{0}} \times [0, 1]$, then, by taking a collared, fibered neighborhood of the boundary of $B_{\epsilon_a} \cap g_a^{-1}(\partial\mathbb{D}_\delta)$, we could conclude that the Milnor fibration of g_a at $\mathbf{0}$ is homeomorphic to the Milnor fibration of of g_0 at $\mathbf{0}$ (actually, we will show be able to conclude that the fibrations are diffeomorphic).

The case where $n = 0$ is trivial. The case where $n = 1$ will follow from the argument below, combined with the classification of surfaces. So, assume that $n \geq 3$. Thus, E is a smooth manifold with boundary, of dimension at least 6. The boundary of E, ∂E, is a disjoint union of $\partial F_{g_a,\mathbf{0}}$ and $\partial F'_{g_0,\mathbf{0}} \cong \partial F_{g_0,\mathbf{0}}$, and each of these two boundary components are simply-connected by [23], since $n \geq 3$. We claim that both of the inclusions $\partial F_{g_a,\mathbf{0}} \hookrightarrow E$ and $\partial F'_{g_0,\mathbf{0}} \hookrightarrow E$ are homotopy-equivalences. Then, the h-cobordism Theorem [24] implies that E is diffeomorphic to $\partial F_{g_a,\mathbf{0}} \times [0,1]$, and we are finished.

By excision, $H_*(E, \partial F_{g_a,\mathbf{0}}) \cong H_*(F'_{g_0,\mathbf{0}}, F_{g_a,\mathbf{0}})$. In addition, by perturbing slightly the function given by distance from the origin, one finds - via Morse Theory - that $F'_{g_0,\mathbf{0}}$ is obtained from $F_{g_a,\mathbf{0}}$ by attaching cells of dimension at most n. Combining this with the fact that $F'_{g_0,\mathbf{0}}$ and $F_{g_a,\mathbf{0}}$ both have the homotopy-type of a wedge of μ n-spheres, the homology long exact sequence for the pair $(F'_{g_0,\mathbf{0}}, F_{g_a,\mathbf{0}})$ yields that $H_*(E, \partial F_{g_a,\mathbf{0}}) \cong H_*(F'_{g_0,\mathbf{0}}, F_{g_a,\mathbf{0}}) = 0$. By a version of Poincaré duality, it follows that $H_*(E, \partial F'_{g_0,\mathbf{0}}) = 0$. By the Whitehead and Hurewicz Theorems, if we can show that E is simply-connected, we can conclude that the inclusions $\partial F_{g_a,\mathbf{0}} \hookrightarrow E$ and $\partial F'_{g_0,\mathbf{0}} \hookrightarrow E$ are homotopy-equivalences.

By perturbing slightly the function given by negative the distance from the origin, one finds that E is obtained from $\partial F'_{g_0,\mathbf{0}}$ by attaching cells of dimension at least n; as $n \geq 3$ and $\partial F'_{g_0,\mathbf{0}}$ is simply-connected, E is also simply-connected, and we are finished. $\square$

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