ABOUT THE CLASSIFICATION OF CONFORMALLY INVARIANT OPERATORS

by

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Abstract. — We discuss the construction of jet operators of infinite order on homogeneous spaces and show that Cartan geometries can be defined as first order analogues of these jet operators. We introduce holonomic geometries modelled from homogeneous spaces and construct natural jet operators of infinite order for holonomic geometries modelled on parabolic homogeneous spaces.

***Résumé* (Sur la classification des opérateurs conformement invariants)**
La construction des opérateurs jet de l'ordre infini sur les espaces homogènes sera discutée et ce sera montré que les géometries Cartan peuvent être définites comme analogues de premier ordre des ces opérateurs jet. On introduit des géometries holonomiques modellées en analogie aux espaces homogénes et on construit des opérateurs jet naturels de l'ordre infini pour des géometries holonomiques paraboliques.

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1. Introduction

It is a challenge to provide a suitable definition for the intuitive idea that a homogeneous space G/P can be thought of as model space for a geometric structure. Of course these geometries should be non–trivial and provide a rich class of examples of curved analogues of the homogeneous space G/P. On the other hand a fundamental prerequisite for further investigation is the existence of naturally defined differential operators associated to these geometries.

Besides symplectic and Riemannian geometry the rich conformal geometry has been studied since decades from various points of view. Certainly this geometry can be defined without any reference to a homogeneous space representing the flat model case, in this case the conformal sphere. Nevertheless it is interesting to describe a conformal manifold as a curved analogue of the flat conformal sphere, because all differential operators naturally defined in the class of conformal manifolds must necessarily become operators invariant under the conformal isometry group. In this way the representation theory of the orthogonal groups enters the classification of naturally defined operators.

A very concise definition of what a curved analogue of a homogeneous space should be is given by Cartan geometries. Apriori however there is no geometric object classifying a Cartan geometry, so we cannot tell whether a local diffeomorphism $M \longrightarrow \tilde{M}$ of two manifolds carrying such a geometry is an isometry or not unless we solve a partial differential equation. There is a saying that "a Cartan geometry should be the result of a theorem and not of a definition" reflecting if anything else the need for a classifying geometric object.

Classically a geometric object is associated to the principal bundle $\mathbf{GL}^{\,d}M$ of d–frames of a differentiable manifold M. Reductions of these higher order frame bundles give rise to various geometric structures on M and have been extensively studied ([**K**], [**KMS**]). The examples given by Riemannian or symplectic geometry arising from a reduction of the first order frame bundle are easily generalized to conformal or projective geometry, which are classified by second order geometric objects. However it seems that exterior and symbolic calculus were not really distinguished conceptually, because in the fundamental examples of first order geometries they are almost synonymous. Nevertheless exterior calculus is linked to the concept of translation, whereas symbolic calculus is used in the process of prolongation. Here is a simple example to illustrate the point:

Example 1.1. — *(The Heisenberg Geometry)*
Consider the Heisenberg geometry modelled on the quotient of the simple group $G = \mathbf{SL}_{\,3}\mathbb{R}$ by its Borel subgroup P of upper triangular matrices. In the usual exponential coordinates

$$(x,\,y,\,z) \quad\longmapsto\quad \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z+\frac{xy}{2} & y & 1 \end{pmatrix} P$$

the vector fields generating the left translations by $\mathfrak{sl}_3\mathbb{R}$ are given by

$$\begin{array}{rclcrcl} L_{11} &=& -\,x\,\partial_x \,-\, z\,\partial_z && L_{21} &=& \partial_x \,-\, \frac{y}{2}\,\partial_y \\ L_{22} &=& x\,\partial_x \,-\, y\,\partial_y && L_{32} &=& \partial_y \,+\, \frac{x}{2}\,\partial_x \\ L_{33} &=& y\,\partial_y \,+\, z\,\partial_z && L_{31} &=& \partial_z \end{array}$$

and

$$\begin{array}{rcl} L_{12} &=& -x^2\;\partial_x \,-\, (z-\frac{xy}{2})\,\partial_y \,-\, \frac{x}{2}(z+\frac{xy}{2})\,\partial_z \\ L_{23} &=& (z+\frac{xy}{2})\,\partial_x \,-\, y^2\;\partial_y \,-\, \frac{y}{2}(z-\frac{xy}{2})\,\partial_z \\ L_{13} &=& -x(z+\frac{xy}{2})\,\partial_x \,-\, y(z-\frac{xy}{2})\,\partial_y \,-\, (z^2+\frac{x^2y^2}{4})\,\partial_z \end{array}$$

Note that the Lie algebra of P is not stable under the standard translations ∂_x, ∂_y and ∂_z and is certainly not the maximal prolongation of its projection to first order linear jets in the sense of ([**K**]), because it does not contain the vector fields $z\partial_x$ and $-z\partial_y$. Symbolic calculus however predicts the vector field $L_{13} = -zx\partial_x - zy\partial_y - z^2\partial_z$ modulo higher order terms appearing only at the level of second order jets. In particular we see that the translations form part of the geometry themselves, whereas symbolic calculus is independent of geometry. Thus the classical prolongation procedure for reductions of higher order frame bundles based on the standard translations can only deal properly with Lie algebras generated by homogeneous vector fields.

After a suitable twist modifying the translations according to the model geometry of the homogeneous space the Heisenberg geometry as well as other parabolic geometries can be defined classically as a reduction of the second order frame bundle of a manifold M. We will impose a suitable partial differential equation on the classifying object associated to this reduction to define the notion of a holonomic geometry modelled on a homogeneous space G/P. Moreover if the structure group P is maximally prolonged we can define natural jet operators of arbitrary order for fiber bundles associated to the holonomic geometry. In this way one does not solve the classification problem for natural differential operators, because these jet operators will fail to be equivariant under the full structure group. Nevertheless the construction of the higher order jet operators reduces this problem to a well defined problem in representation theory and abandons the cooking with local coordinates.

The language of jet theory will dominate these notes and we give a self–contained account of this theory as far as we need it in the next section. The geometry of homogeneous spaces will be reviewed in the third section, focussing attention to the construction of jet operators of arbitrary order in these flat model cases. The fourth section gives a brief account of Cartan geometry and a different interpretation in terms of a universal first order jet operator. Finally the last section introduces the concept of holonomic geometries and constructs the higher order jet operators in the special case of holonomic parabolic geometries.

The main ideas of these notes arose from numerous discussions with T. Diemer, who introduced me to the faszinating world of conformal geometry. Presumably I will never understand the huge amount of calculations done by him and others in the quest for natural differential operators, but I got a glimpse of the beauty of the representation theory involved.

2. Jet Theory on Principal Bundles

The language of jet theory will dominate the following sections, most of the ideas and definitions will emerge from this way of expressing calculus. The main object of study in jet theory is of course a jet, which is a generalization of the concept of a Taylor series associated to a smooth map $\mathbb{R} \longrightarrow \mathbb{R}$ to arbitrary smooth maps between manifolds. For the convenience of the reader we will review the basic concepts of jet theory with emphasis on its interplay with the theory of principal bundles.

Let $\mathfrak{u}$ be a fixed real vector space and $\mathcal{F}$ some differentiable manifold. Two smooth maps $f : \mathfrak{u} \longrightarrow \mathcal{F}$ and $\tilde{f} : \mathfrak{u} \longrightarrow \mathcal{F}$ defined in some neighborhood of $0 \in \mathfrak{u}$ are callled equivalent $f \sim \tilde{f}$ up to order $k \geq 0$ if $f(0) = \tilde{f}(0)$ and their partial derivatives up to order k in some and hence every local coordinate system of $\mathcal{F}$ about $f(0) = \tilde{f}(0)$ agree in 0. The equivalence class of a smooth map f up to order k is called the k–th order jet $\mathrm{jet}^k_0 f$ of f and the set of all these equivalence classes is denoted by $\mathrm{Jet}^k_0 \mathcal{F} := \{\mathrm{jet}^k_0 f |\ f : \mathfrak{u} \longrightarrow \mathcal{F}\ \}$. For all $k \geq l \geq 0$ there is a canonical projection

$$\mathrm{pr} : \quad \mathrm{Jet}^k_0 \mathcal{F} \longrightarrow \mathrm{Jet}^l_0 \mathcal{F}, \quad \mathrm{jet}^k_0 f \longmapsto \mathrm{jet}^l_0 f$$

and the evaluation

$$\mathrm{ev} : \quad \mathrm{Jet}^k_0 \mathcal{F} \longrightarrow \mathcal{F}, \quad \mathrm{jet}^k_0 f \longmapsto f(0)$$

which strictly speaking is a special case of the projection since we may identify $\mathrm{Jet}^0_0 \mathcal{F} \cong \mathcal{F}$. We will use a different notation for this special case nevertheless instead of indicating the source and target orders of the projections. If the manifold $\mathcal{F}$ comes along with a distinguished base point $\{*\}$ the jets of pointed smooth maps $f : \mathfrak{u} \longrightarrow \mathcal{F}$ make up the subset of all reduced or pointed jets ${}^*\mathrm{Jet}^k_0\,\mathcal{F} = \{\,\mathrm{jet}^k_0\, f \mid f(0) = *\,\} \subset \mathrm{Jet}^k_0\,\mathcal{F}$, which is just the preimage $\mathrm{ev}^{-1}(*) = {}^*\mathrm{Jet}^k_0\,\mathcal{F}$ of the base point.

A guiding example to picture jets and to set up explicit calculations is given by the jets $\mathrm{Jet}^k_0\, F$ into a vector space F. In this case every equivalence class $\mathrm{jet}^k_0\, f$ has a unique polynomial representative of order $\leq k$ given by the Taylor polynomial of f in $0 \in \mathfrak{u}$, hence there is a canonical identification

$$\mathrm{Sym}^{\leq k}\mathfrak{u}^* \otimes F \stackrel{\cong}{\longrightarrow} \mathrm{Jet}^k_0\, F, \quad \psi \longmapsto \mathrm{jet}^k_0\,[v \longmapsto \psi(v)]$$

which makes $\mathrm{Jet}^k_0\, F$ a vector space itself. For general manifolds $\mathcal{F}$ we are in general very far from having such a simple presentation for $\mathrm{Jet}^k_0\,\mathcal{F}$, however this example serves as a model coordinate patch for a natural smooth manifold structure on the set $\mathrm{Jet}^k_0\,\mathcal{F}$! The total space of its tangent bundle $T\mathrm{Jet}^k_0\,\mathcal{F}$ is canonically diffeomorphic to the manifold $\mathrm{Jet}^k_0\,(T\mathcal{F})$ via:

$$T\mathrm{Jet}^k_0\,\mathcal{F} \stackrel{\cong}{\longrightarrow} \mathrm{Jet}^k_0\,(T\mathcal{F}), \quad \frac{d}{dt}\Big|_0 \mathrm{jet}^k_0\, f_t \longmapsto \mathrm{jet}^k_0\,(\frac{d}{dt}\Big|_0 f_t)$$

The general assertion follows immediately from the special case of vector spaces F, where this map reduces to the linear isomorphism $\mathrm{Sym}^{\leq k}\mathfrak{u}^* \otimes F \oplus \mathrm{Sym}^{\leq k}\mathfrak{u}^* \otimes F \cong \mathrm{Sym}^{\leq k}\mathfrak{u}^* \otimes (F \oplus F)$.

Consider now the case that P is a Lie group then so are both ${}^*\mathrm{Jet}^k_0\, P$ and $\mathrm{Jet}^k_0\, P$ under pointwise multiplication with Lie algebras ${}^*\mathrm{Jet}^k_0\,\mathfrak{p}$ and $\mathrm{Jet}^k_0\,\mathfrak{p}$ respectively. With the help of the exponential $\exp : \mathfrak{p} \longrightarrow P$ we may identify ${}^*\mathrm{Jet}^k_0\, P$ and ${}^*\mathrm{Jet}^k_0\,\mathfrak{p}$, making the vector space ${}^*\mathrm{Jet}^k_0\,\mathfrak{p}$ an algebraic group with group structure given by the polynomial approximation of the Campbell–Baker–Hausdorff formula. The group $\mathrm{Jet}^k_0\, P$ is then a semidirect product $\mathrm{Jet}^k_0\, P \cong P \ltimes {}^*\mathrm{Jet}^k_0\,\mathfrak{p}$ by the split exact sequence:

$$1 \longrightarrow {}^*\mathrm{Jet}^k_0\, P \longrightarrow \mathrm{Jet}^k_0\, P \stackrel{\mathrm{ev}}{\longrightarrow} P \longrightarrow 1$$

Similarly we could define jets of maps $f : \mathfrak{u} \longrightarrow \mathcal{F}$ at points different from 0 and jets of maps between arbitrary manifolds. However for our purposes it is sufficient to "gauge" the pointwise definitions and constructions given above. Consider therefore the open subset $\mathbf{GL}^k M \subset \mathrm{Jet}^k_0\, M$ of all k–jets of local diffeomorphisms $m : \mathfrak{u} \longrightarrow M$ defined in a neighborhood of $0 \in \mathfrak{u}$ together with the open subset $\mathbf{GL}^k\mathfrak{u} \subset {}^*\mathrm{Jet}^k_0\,\mathfrak{u}$ of all local diffeomorphisms $A : \mathfrak{u} \longrightarrow \mathfrak{u}$ fixing $0 = A(0)$:

$$\begin{aligned} \mathbf{GL}^k M &:= \{\,\mathrm{jet}^k_0\, m \mid m : \mathfrak{u} \longrightarrow M,\ m\,\text{local diffeomorphism}\,\} \\ \mathbf{GL}^k\mathfrak{u} &:= \{\,\mathrm{jet}^k_0\, A \mid A : \mathfrak{u} \longrightarrow \mathfrak{u},\ A(0) = 0\,\text{local diffeomorphism}\,\} \end{aligned}$$

Obviously the set $\mathbf{GL}^k\mathfrak{u}$ is a group under composition acting on $\mathbf{GL}^k M$ again by composition. In this way $\mathbf{GL}^k M$ becomes a principal $\mathbf{GL}^k\mathfrak{u}$–bundle over M with projection $\pi : \mathbf{GL}^k M \longrightarrow M$, $\mathrm{jet}^k_0\, m \longmapsto m(0)$, given by evaluation. Elements of $\mathbf{GL}^k M$ are called k–frames, because in the special case $k = 1$ the principal bundle $\mathbf{GL}^1 M$ is just the usual frame bundle $\mathbf{GL}\, M := \{\,\mathrm{jet}^1_0\, m = m_{*,0} : \mathfrak{u} \stackrel{\cong}{\longrightarrow} T_{m(0)}M\,\}$ on M.

Given now an arbitrary principal bundle $\pi : \mathcal{G} \longrightarrow M$ over M with principal fibre P and some P-representation F there is an associated vector bundle $\mathcal{G} \times_P F$ on M. Historically this construction goes back to Cartan's idea of recovering the tangent bundle from the frame bundle $\mathbf{GL}\, M$ and the left representation of $\mathbf{GL}\,\mathfrak{u}$ on $\mathfrak{u}$. Although left and right P–representations and more generally left and right P–spaces are in bijective correspondence, it is certainly more natural to use right representations instead in order to recover the cotangent bundle. Hence we will always associate fiber bundles by right P–actions

$$\mathcal{G} \times_P \mathcal{F} \;:=\; \mathcal{G} \times \mathcal{F}/_\sim \qquad\qquad (g,\, f) \;\sim\; (g \star p,\, f \star p) \qquad\qquad [g,\, f] \;:=\; (g,\, f)/\sim$$

if not explicitly stated otherwise. The advantage of this choice becomes evident in explicit calculations, because inverting elements of $\mathbf{GL}^k\mathfrak{u}$ is not particularly easy in practice. We hope the reader will not be too disturbed by this choice and will find it appealing in the definitions below. Most notions related to principal bundles are easily adjusted to this convention, only exterior calculus on the principal bundle $\mathcal{G}$ requires some care in signs. By abuse of notation we will identity sections $f \in \Gamma(\mathcal{G} \times_P \mathcal{F})$ of $\mathcal{G} \times_P \mathcal{F}$ and functions $f \in C^\infty(\mathcal{G}, \mathcal{F})^P$ on $\mathcal{G}$ with values in $\mathcal{F}$ satisfying $f(gp) = f(g) \star p$ via $f(\pi\, g) \;=\; [g, f(g)]$.

General jet theory associates to any fiber bundle on M the family of its jet bundles. It is natural to ask how this construction can be formulated naturally in the context of principal bundles and associated fiber bundles such as $\mathcal{G} \times_P \mathcal{F}$. Let

$$\begin{aligned} \mathbf{GL}^k(\mathcal{G}, M) \;&:=\; \{\mathrm{jet}^k_0\, g|\; g:\, \mathfrak{u} \longrightarrow \mathcal{G},\; \pi \circ g\,\text{local diffeomorphism}\} \\ \mathbf{GL}^k(P,\, \mathfrak{u}) \;&:=\; \{\mathrm{jet}^k_0\, A|\; A = (A_\mathfrak{u}, A_P):\, \mathfrak{u} \longrightarrow \mathfrak{u} \times P,\; A_\mathfrak{u}(0) = 0\,\text{local diffeomorphism}\} \end{aligned}$$

The multiplication $\mathrm{jet}^k_0\, A \cdot \mathrm{jet}^k_0\, B \;:=\; \mathrm{jet}^k_0\, (A_\mathfrak{u} \circ B_\mathfrak{u},\, (A_P \circ B_\mathfrak{u}) \cdot B_P)$ makes $\mathbf{GL}^k(P, \mathfrak{u})$ a group acting by

$$\mathrm{jet}^k_0\, g \;\star\; \mathrm{jet}^k_0\, A \quad:=\quad \mathrm{jet}^k_0\, ((g \circ A_\mathfrak{u}) \star A_P)$$

on the principal bundle $\mathbf{GL}^k(\mathcal{G},\, M)$ of k–frames of $\mathcal{G}$ over M. Note that $\mathbf{GL}^k(\mathcal{G},\, M)$ and $\mathbf{GL}^k(P,\, \mathfrak{u})$ project to $\mathbf{GL}^kM$ and $\mathbf{GL}^k\mathfrak{u}$ respectively. The latter projection splits $\mathrm{jet}^k_0\, A \longmapsto \mathrm{jet}^k_0\, (A,\, 1)$, thus $\mathbf{GL}^k(P,\, \mathfrak{u})$ is a semidirect product

$$0 \longrightarrow \mathrm{Jet}^k_0\, P \longrightarrow \mathbf{GL}^k(P,\, \mathfrak{u}) \overset{\longleftarrow}{\longrightarrow} \mathbf{GL}^k\mathfrak{u} \longrightarrow 0$$

of $\mathbf{GL}^k\mathfrak{u}$ with the group $\mathrm{Jet}^k_0\, P$ acting naturally on $\mathrm{Jet}^k_0\, \mathcal{F}$ for any right P–space $\mathcal{F}$ via:

$$\mathrm{jet}^k_0\, f \;\star\; \mathrm{jet}^k_0\, A \quad:=\quad \mathrm{jet}^k_0\, ((f \circ A_\mathfrak{u}) \star A_P)$$

Lemma 2.1. —

$$\mathrm{Jet}^k(\mathcal{G} \times_P \mathcal{F}) \quad\cong\quad \mathbf{GL}^k(\mathcal{G}, M) \times_{\mathbf{GL}^k(P,\mathfrak{u})} \mathrm{Jet}^k_0\mathcal{F}$$

Proof. — A convenient definition of $\mathrm{Jet}^k(\mathcal{G} \times_P \mathcal{F})$ is by its universal property, namely it carries a universal differential operator of order k called the jet operator. It is thus sufficient to write down a differential operator of order k for sections $f \in \Gamma(\mathcal{G} \times_P \mathcal{F}) \;=\; C^\infty(\mathcal{G}, \mathcal{F})^P$, which obviously is a jet operator. In order to do so define the map

$$\begin{array}{cccc} \mathrm{jet}^k \;: & C^\infty(G, \mathcal{F})^P & \longrightarrow & C^\infty(\,\mathbf{GL}^k(G,\, M),\, \mathrm{Jet}^k_0\, \mathcal{F}\,)^{\mathbf{GL}^k(P,\,\mathfrak{u})} \\ & f & \longmapsto & \mathrm{jet}^k f \end{array}$$

by $\mathrm{jet}^k\, f(\,\mathrm{jet}^k_0\, g\,) := \mathrm{jet}^k_0\, (f \circ g)$ and note that $\mathrm{jet}^k f$ is equivariant under $\mathbf{GL}^k(P,\, \mathfrak{u})$. ☐

Definition 2.2. — *A natural object of order $l \geq 0$ on a smooth manifold M is a section of some fiber bundle $\mathbf{GL}^lM \times_{\mathbf{GL}^l\mathfrak{u}} \mathcal{F}$ associated to the principal bundle $\mathbf{GL}^lM$. The k–jet of a natural object of order l is a natural object of order $k + l$, in other words there is a jet operator:*

$$\mathit{jet}^k \;:\quad \Gamma(\,\mathbf{GL}^lM \times_{\mathbf{GL}^l\mathfrak{u}} \mathcal{F}\,) \longrightarrow \Gamma(\,\mathbf{GL}^{k+l}M \times_{\mathbf{GL}^{k+l}\mathfrak{u}} \mathrm{Jet}^k_0\, \mathcal{F}\,)$$

Let us apply Lemma 2.1 to this well known standard situation. For a natural object $f \in \Gamma(\,\mathbf{GL}^{\,l}M \times_{\mathbf{GL}^{\,l}M} \mathcal{F}\,)$ of order l on M we thus get a function $\mathrm{jet}^k f$ on $\mathbf{GL}^{\,k}(\mathbf{GL}^{\,l}M,\, M)$ and the above jet operator defined for natural objects is just the restriction of the jet operator defined in Lemma 2.1 to a natural restriction of the structure group of $\mathbf{GL}^{\,k}(\mathbf{GL}^{\,l}M, M)$. Namely let t_v be the translation $t_v : \mathfrak{u} \longrightarrow \mathfrak{u},\ \tilde{v} \longmapsto \tilde{v} + v$ by v on $\mathfrak{u}$ and set

$$\begin{array}{rccc} \Delta : & \mathbf{GL}^{\,k+l}M & \longrightarrow & \mathbf{GL}^{\,k}(\mathbf{GL}^{\,l}M, M) \\ & \mathrm{jet}^{k+l}_0\, m & \longmapsto & \mathrm{jet}^k_0\,[v \longmapsto \mathrm{jet}^l_v\, m := \mathrm{jet}^l_0\,(m \circ t_v)] \end{array}$$

together with:

$$\begin{array}{rccc} \Delta : & \mathbf{GL}^{\,k+l}\mathfrak{u} & \longrightarrow & \mathbf{GL}^{\,k}(\mathbf{GL}^{\,l}\mathfrak{u}, \mathfrak{u}) \\ & \mathrm{jet}^{k+l}_0\, A & \longmapsto & \mathrm{jet}^k_0\,[v \longmapsto (A(v),\, \mathrm{jet}^l_v\, A := \mathrm{jet}^l_0\,(t_{-A(v)} \circ A \circ t_v))] \end{array}$$

This definition of $\mathrm{jet}^l_v\, A$ satisfies the evident "chain rule" $\mathrm{jet}^l_v\,(A \circ B) = \mathrm{jet}^l_{B(v)}\, A \cdot \mathrm{jet}^l_v\, B$, which makes $\Delta : \mathbf{GL}^{\,k+l}\mathfrak{u} \longrightarrow \mathbf{GL}^{\,k}(\mathbf{GL}^{\,l}\mathfrak{u}, \mathfrak{u})$ an injective group homomorphism. Evidently $\Delta : \mathbf{GL}^{\,k+l}M \longrightarrow \mathbf{GL}^{\,k}(\mathbf{GL}^{\,l}M, M)$ is equivariant over this group homomorphism.

Although the translations $\{t_v\}$ enter the theory naturally, they are in no way a canonical choice, and other choices should be considered seriously. This ambiguity is just a manifestation of the fact that general relativity is no gauge theory in the strict sense of the word, whatever we try to make it look like one. Eventually the ambiguity in the choice of translations puts a definite limit to this endeavour. En nuce this point can be seen in the subtle differences between a true connection and the so called canonical connection or Spencer form on $\mathbf{GL}^{\,k}M$, unluckily we won't discuss this connection in detail in this article.

We will continue to work with the translations $\{t_v\}$ as before as long as we consider the general theory of jet bundles, but when turning to $(G,\, P)$–geometric structures it is a good idea to insist on the translations $\{t_v\}$ to be elements of the isometry group G. In the almost hermitean geometries like conformal or projective geometry this makes no difference whatsoever, because their isometry group already contains the translations. In the more general parabolic geometries some extra twist in our current concept of "translations" seems appropriate.

At the end of this section we will recall the definition of the Spencer operator, which characterizes genuine jets of sections among sections of jet bundles by a first order differential equation. By the construction given in Lemma 2.1 the k–jet of an arbitrary section of some jet bundle $\mathrm{Jet}^l\,(\mathcal{G} \times_P \mathcal{F}) = \mathbf{GL}^{\,l}(\mathcal{G}, M) \times_{\mathbf{GL}^{\,l}(P,\mathfrak{u})} \mathrm{Jet}^l_0\mathcal{F}$ is a section of the fiber bundle with model fiber $\mathrm{Jet}^k_0\mathrm{Jet}^l_0\mathcal{F}$ associated to the principal bundle $\mathbf{GL}^{\,k}(\mathbf{GL}^{\,l}(\mathcal{G}, M), M)$. However this principal bundle necessarily contains redundant information and consequently its structure group reduces naturally via

$$\begin{array}{rccc} \Delta : & \mathbf{GL}^{\,k+l}(\mathcal{G}, M) & \longrightarrow & \mathbf{GL}^{\,k}(\mathbf{GL}^{\,l}(\mathcal{G}, M), M) \\ & \mathrm{jet}^{k+l}_0 & \longmapsto & \mathrm{jet}^k_0[v \longmapsto \mathrm{jet}^l_v g := \mathrm{jet}^l_0(g \circ t_v)] \end{array}$$

equivariant over:

$$\begin{array}{rccc} \Delta : & \mathbf{GL}^{\,k+l}(P, \mathfrak{u}) & \longrightarrow & \mathbf{GL}^{\,k}(\mathbf{GL}^{\,l}(P, \mathfrak{u}), \mathfrak{u}) \\ & \mathrm{jet}^{k+l}_0(A_\mathfrak{u}, A_P) & \longmapsto & \mathrm{jet}^k_0[v \longmapsto (A_\mathfrak{u}(v),\, \mathrm{jet}^l_v A_\mathfrak{u},\, \mathrm{jet}^l_v A_P)] \end{array}$$

Note that this reduction contains the reduction $\mathbf{GL}^{\,k+l}M \longrightarrow \mathbf{GL}^{\,k}(\mathbf{GL}^{\,l}M, M)$ above as a special case when we consider M as a principal $\{1\}$–bundle over M.

Thus fiber bundles of the form $\mathrm{Jet}^k\,\mathrm{Jet}^l\,(\mathcal{G}\times_P\mathcal{F})$ are associated to the principal bundle $\mathbf{GL}^{\,k+l}(\mathcal{G},M)$ acting by restriction of the action of $\mathbf{GL}^{\,k}(\mathbf{GL}^{\,l}(\mathcal{G},M),M)$ on the model fiber $\mathrm{Jet}^k_0\mathrm{Jet}^l_0\mathcal{F}$. Moreover there is a natural $\mathbf{GL}^{\,k+l}(P,\mathfrak{u})$–equivariant embedding

$$\Delta\ :\quad \mathrm{Jet}^{k+l}_0\,\mathcal{F}\ \longrightarrow\ \mathrm{Jet}^k_0\,\mathrm{Jet}^l_0\,\mathcal{F},\qquad \mathrm{jet}^{k+l}_0\,f\ \longmapsto\ \mathrm{jet}^k_0\,[\upsilon\longmapsto \mathrm{jet}^l_\upsilon\,f := \mathrm{jet}^l_0\,(f\circ t_\upsilon)]$$

Consider now a section $s\in C^\infty(\mathcal{G},\mathcal{F})^P$ of the fiber bundle $\mathcal{G}\times_P\mathcal{F}$. Its iterated jet satisfies

$$\begin{aligned}(\mathrm{jet}^k\mathrm{jet}^l\,s)(\mathrm{jet}^{k+l}_0 g) &= \mathrm{jet}^k_0[\upsilon\longmapsto(\mathrm{jet}^l\,s)(\mathrm{jet}^l_\upsilon g)]\\ &= \mathrm{jet}^k_0[\upsilon\longmapsto \mathrm{jet}^l_0(s\circ g\circ t_\upsilon)]\ =\ \mathrm{jet}^k_0[\upsilon\longmapsto\mathrm{jet}^l_\upsilon(s\circ g)]\\ &= \Delta(\mathrm{jet}^{k+l}s)(\mathrm{jet}^{k+l}_0 g)\end{aligned}$$

for all $\mathrm{jet}^{k+l}_0 g\in\mathbf{GL}^{\,k+l}(\mathcal{G},M)\subset\mathbf{GL}^{\,k}(\mathbf{GL}^{\,l}(\mathcal{G},M),M)$ and thus takes only values in the submanifold $\mathrm{Jet}^{k+l}_0\mathcal{F}\subset\mathrm{Jet}^k_0\mathrm{Jet}^l_0\mathcal{F}$. Conversely it is easy to see say in local coordinates that if the k–jet of a section of the jet bundle $\mathrm{Jet}^l\,(\mathcal{G}\times_P\mathcal{F})$ takes values in $\mathrm{Jet}^{k+l}_0\mathcal{F}\subset\mathrm{Jet}^k_0\mathrm{Jet}^l_0\mathcal{F}$ only then the section is the l–jet of a genuine section of $\mathcal{G}\times_P\mathcal{F}$

This characterization of the sections of $\mathrm{Jet}^l\,(\mathcal{G}\times_P\mathcal{F})$ coming from l–jets of sections of $\mathcal{G}\times_P\mathcal{F}$ can be rewritten in terms of a first order differential equation expressed in terms of the Spencer operator. We will not write down this equation explicitly but remark that $\mathrm{Jet}^{k+l}_0\mathcal{F}\subset\mathrm{Jet}^k_0\mathrm{Jet}^l_0\mathcal{F}$ lies in the double kernel

$$\mathrm{Jet}^{k+l}_0\mathcal{F}\ \longrightarrow\ \mathrm{Jet}^k_0\mathrm{Jet}^l_0\mathcal{F}\ \rightrightarrows\ \mathrm{Jet}^{k-1}_0\mathrm{Jet}^1_0\mathrm{Jet}^{l-1}_0\mathcal{F}\tag{1}$$

of the two maps from $\mathrm{Jet}^k_0\mathrm{Jet}^l_0\mathcal{F}$ to $\mathrm{Jet}^{k-1}_0\mathrm{Jet}^1_0\mathrm{Jet}^{l-1}_0\mathcal{F}$ given by:

$$\begin{aligned}\mathrm{jet}^k_0[\upsilon\longmapsto\mathrm{jet}^l_0 f_\upsilon]\ &\overset{\mathrm{pr}\,\Delta}{\longmapsto}\ \mathrm{jet}^{k-1}_0[\upsilon\longmapsto\mathrm{jet}^1_0[\tilde\upsilon\longmapsto\mathrm{jet}^{l-1}_0(f_\upsilon\circ t_{\tilde\upsilon})]]\\ &\overset{\Delta\mathrm{pr}}{\longmapsto}\ \mathrm{jet}^{k-1}_0[\upsilon\longmapsto\mathrm{jet}^1_0[\tilde\upsilon\longmapsto\mathrm{jet}^{l-1}_0(f_{t_\upsilon\tilde\upsilon})]]\end{aligned}$$

In fact the translations form a group $t_\upsilon\circ t_{\tilde\upsilon}\ =\ t_{t_\upsilon\tilde\upsilon}$ and so $\Delta\mathrm{jet}^{k+l}_0 f\ =\ \mathrm{jet}^k_0[\upsilon\longmapsto\mathrm{jet}^l_0(f\circ t_\upsilon)]$ is mapped to the same image $\mathrm{jet}^{k-1}_0[\upsilon\longmapsto\mathrm{jet}^1_0[\tilde\upsilon\longmapsto\mathrm{jet}^{l-1}_0(f\circ t_\upsilon\circ t_{\tilde\upsilon})]]$ in $\mathrm{Jet}^{k-1}_0\mathrm{Jet}^1_0\mathrm{Jet}^{l-1}_0\mathcal{F}$ under both maps. Note however that the sequence (1) is not exact, i. e. the double kernel is strictly larger than the image of $\mathrm{Jet}^{k+l}_0\mathcal{F}$. This is the mayor drawback of the naive Spencer operator in contrast to the sophisticated Spencer operator, which characterizes the image of $\mathrm{Jet}^{k+l}_0\mathcal{F}$ in $\mathrm{Jet}^k_0\mathrm{Jet}^l_0\mathcal{F}$ completely. However considered as differential equations both operators have the same space of solutions and are in this sense equivalent.

3. Some Remarks on Homogeneous Spaces

Homogeneous spaces have always been an interesting object of study for geometers. However all calculations in analysis on homogeneous spaces G/P can be casted into the language of representation theory, because on these spaces there are naturally defined jet operators of arbitrary order. Nevertheless a homogeneous space G/P can be thought of as model space for a geometric structure related to the pair of groups (G,P). Once it is specified in what sense more general smooth manifolds carry curved analogues of this flat model structure one can proceed to develop this particular geometry in analogy to the analysis on G/P.

In this section we will consider the geometry of homogeneous spaces and define the jet operators for homogeneous vector bundles, postponing the construction of the jet operators for homogeneous fiber bundles to the next section. The question of whether these jet operators are still defined for curved analogues of homogeneous spaces is of genuine importance for the whole topic and serves as a guideline for the correct definition of the geometries modelled on homogeneous spaces.

Let us start with the definition of the Maurer–Cartan form θ^{MC} on an arbitray Lie group G, which expresses the tautological identification of the tangent space $\mathfrak{g} := T_1 G$ of G in the unit element with the space of left–invariant vector fields:

$$\theta^{MC}(\frac{d}{dt}\Big|_0 g_t) \quad := \quad \frac{d}{dt}\Big|_0 g_0^{-1} g_t \ \in \mathfrak{g}$$

The basic properties of the Maurer–Cartan form are summarized in the two formulas:

$$\theta^{MC}(\frac{d}{dt}\Big|_0 g_t\tilde{g}_t) \ = \ \mathrm{Ad}_{\tilde{g}_0^{-1}}\theta^{MC}(\frac{d}{dt}\Big|_0 g_t) + \theta^{MC}(\frac{d}{dt}\Big|_0 \tilde{g}_t) \tag{2}$$

$$\theta^{MC}(\frac{d}{dt}\Big|_0 e^{X_t}) \ = \ \frac{e^{-\mathrm{ad}\,X_0} - 1}{-\mathrm{ad}\,X_0}(\frac{d}{dt}\Big|_0 X_t) \tag{3}$$

The first formula immediately follows from the definition whereas the second gives a well known description of the jacobian of the exponential map $\exp:\ \mathfrak{g} \longrightarrow G,\ X \longmapsto e^X$. The historically important Maurer–Cartan equation

$$d\theta^{MC} + \frac{1}{2}[\,\theta^{MC} \wedge \theta^{MC}\,] \quad = \quad 0 \tag{4}$$

with $\frac{1}{2}[\,\theta^{MC} \wedge \theta^{MC}\,](X,Y) := [\,\theta^{MC}(X), \theta^{MC}(Y)\,]$ becomes trivial if the Lie algebra bracket is defined as the Lie bracket of two left–invariant vector fields.

In order to compare analysis on a homogeneous space with analysis on arbitrary manifolds it will be convenient to choose a linear complement $\mathfrak{u}$ of the subalgebra $\mathfrak{p}$ in $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{u}$, which will play the role of the abstract vector space $\mathfrak{u}$ of the last section. With this complement fixed once and for all we consider the exponential map

$$\exp:\ P \times \mathfrak{u} \longrightarrow G, \quad (p,v) \longmapsto p\,e^v$$

which defines a diffeomorphism of some tubular neighborhood of $P \times \{0\} \subset P \times \mathfrak{u}$ with a neighborhood of P in G. The ordering of the factors p and e^v in this definition is somewhat arbitrary and we could have chosen it the other way round to get a similar diffeomorphism from a neighborhood of P in G to a tubular neighborhood of $\{0\} \times P$ in $\mathfrak{u} \times P$. To account for this ambiguity we introduce the commutator map:

Definition 3.1. — *(The Commutator Map)*
The commutator map $\Phi:\ P \times \mathfrak{u} \longrightarrow \mathfrak{u} \times P,\ (p,v) \longmapsto (\Phi_{\mathfrak{u}}(p,v), \Phi_P(p,v))$ *is uniquely defined in some tubular neighborhood of* $P \times \{0\}$ *in* $P \times \mathfrak{u}$ *by the property:*

$$p\,e^v \quad = \quad e^{\Phi_{\mathfrak{u}}(p,v)}\,\Phi_P(p,v)$$

Whether or not is is defined outside this neighborhood is of no practical importance.

Some special values of the commutator along P and $\mathfrak{u}$ are given by:

$$\begin{array}{lcl} \Phi_{\mathfrak{u}}(1,v) &=& v \\ \Phi_{\mathfrak{u}}(p,0) &=& 0 \end{array} \qquad \begin{array}{lcl} \Phi_P(1,v) &=& 1 \\ \Phi_P(p,0) &=& p \end{array}$$

For more general points (p,v) the commutator is a very important characteristic for the geometry based on the pair (G,P) of groups. Note in particular that $\Phi_{\mathfrak{u}}$ determines a homomorphism from P to the group of germs of local diffeomorphisms of $\mathfrak{u}$ fixing 0 and hence to the groups $\mathbf{GL}^k\mathfrak{u}$ for all $k \geq 1$. Though a general closed formula for the commutator seems quite out of reach one gets a good idea of its behaviour through its infinitesimal version:

Lemma 3.2. — *(Infinitesimal Version of the Commutator)*
For all $\upsilon \in \mathfrak{u}$ and all $X \in \mathfrak{p}$ we have

$$e^{-\mathrm{ad}\,\upsilon}X \;=\; \frac{e^{-\mathrm{ad}\,\upsilon}-1}{-\mathrm{ad}\,\upsilon}\,\frac{d}{dt}\Big|_0 \Phi_{\mathfrak{u}}(\,e^{tX},\,\upsilon\,) \;+\; \frac{d}{dt}\Big|_0 \Phi_P(\,e^{tX},\,\upsilon))$$

Proof. — Let us apply the Maurer–Cartan form θ^{MC} to the curve $t \longmapsto e^{tX}\,e^{\upsilon}$. Using the fundamental properties (2) and (3) of the Maurer–Cartan form we immediately arrive at:

$$\begin{aligned}\theta^{MC}\Big(\frac{d}{dt}\Big|_0 e^{tX}e^{\upsilon}\Big) &= \mathrm{Ad}_{\,e^{-\upsilon}}X \;=\; e^{-\mathrm{ad}\,\upsilon}X\\ &= \frac{e^{-\mathrm{ad}\,\upsilon}-1}{-\mathrm{ad}\,\upsilon}\,\frac{d}{dt}\Big|_0 \Phi_{\mathfrak{u}}(e^{tX},\upsilon) \;+\; \frac{d}{dt}\Big|_0 \Phi_P(e^{tX},\upsilon)\end{aligned}$$

□

By construction this implicit equation for $\frac{d}{dt}\big|_0 \Phi_P(\,e^{tX},\,\upsilon\,)$ and $\frac{d}{dt}\big|_0 \Phi_{\mathfrak{u}}(\,e^{tX},\,\upsilon\,)$ has a unique solution for all υ sufficiently close to $0 \in \mathfrak{u}$. However there are two important cases where this implicit equation can be solved with explicit formulas, namely in the case of parabolic and symmetric geometries. In the first case the complement $\mathfrak{u}$ can be chosen to be a subalgebra of $\mathfrak{g}$ itself and the above equation simplifies to

$$\frac{d}{dt}\Big|_0 \Phi_P(e^{tX},\upsilon) \;=\; \mathrm{pr}_{\,\mathfrak{p}}(e^{-\mathrm{ad}\,\upsilon}X) \qquad\qquad \frac{d}{dt}\Big|_0 \Phi_{\mathfrak{u}}(e^{tX},\upsilon) \;=\; \frac{e^{-\mathrm{ad}\,\upsilon}-1}{-\mathrm{ad}\,\upsilon}\mathrm{pr}_{\,\mathfrak{u}}(e^{-\mathrm{ad}\,\upsilon}X)$$

where $\mathrm{pr}_{\,\mathfrak{p}}$ and $\mathrm{pr}_{\,\mathfrak{u}}$ are the projections associated to the decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{u}$ of the Lie algebra $\mathfrak{g}$. In the second case $\mathfrak{g}$ and $\mathfrak{p}$ is a symmetric pair and $\mathfrak{u}$ can be chosen to be the complement of $\mathfrak{p}$ invariant under the Cartan involution:

$$\frac{d}{dt}\Big|_0 \Phi_P(e^{tX},\upsilon) \;=\; X \qquad\qquad \frac{d}{dt}\Big|_0 \Phi_{\mathfrak{u}}(e^{tX},\upsilon) \;=\; [X,\upsilon]$$

Analysis on homogeneous spaces simplifies dramatically because of the existence of jet operators of infinite order for every homogeneous fiber bundle over G/P. In the rest of this section we describe the construction of these jet operators for homogeneous vector bundles $G \times_P F$ over G/P, which are conceptually easier. The construction of the jet operators for general homogeneous fiber bundles is postponed until the next section, where the necessary definitions appear in the right perspective.

It turns out that there are at least two equivalent definitions of a jet operator of infinite order for a homogeneous vector bundle. The first construction is very simple, consider a section $f \in C^\infty(G,F)^P$ of the homogeneous vector bundle $G \times_P F$ with F being a fixed right P–representation. In exponential coordinates centered at some $\gamma \in G$ the section f becomes the function:

$$f:\; \mathfrak{g} \longrightarrow F, \quad \upsilon \longmapsto f(\gamma e^{\upsilon})$$

Taking the infinite Taylor series of this function at $0 \in \mathfrak{g}$ we get a formal power series $\mathrm{jet}^\infty_0[\upsilon \longmapsto f(\gamma e^{\upsilon})]$ in (a completion of) $\mathrm{Sym}\,\mathfrak{g}^* \otimes F$. In this way we have constructed an infinite order jet operator

$$\mathrm{jet}^\infty:\; C^\infty(G,F)^P \;\longrightarrow\; C^\infty(G,\mathrm{Sym}\,\mathfrak{g}^*\otimes F)^P, \quad f \longmapsto \mathrm{jet}^\infty f$$

with $(\mathrm{jet}^\infty f)(\gamma) := \mathrm{jet}^\infty_0[v \longmapsto f(\gamma e^v)]$. There are two delicate points in this definition. First of all as f is a P–equivariant function on G certainly so is $\mathrm{jet}^\infty f$ for some P–representation on the space $\mathrm{Sym}\,\mathfrak{g}^* \otimes F$. However this P–representation seems very difficult to describe in practice. Moreover it is clear that not all possible Taylor polynomials in $\mathrm{Sym}\,\mathfrak{g}^* \otimes F$ can be realised by P–equivariant functions s on G. In other words the vector space $\mathrm{Sym}\,\mathfrak{g}^* \otimes F$ does not take care of the constraint of P–equivariance and is hence much too large to be a proper target space for a jet operator.

Fortunately these two problems can be solved simultaneously by constructing the same jet operator in a different way. As F is a right P–representation its dual space F^* is naturally a left P–representation, in this way we can even forget about taking the inverse. Now for any left representation of P there is an induced left representation of $\mathfrak{g}$ defined as the quotient of the left $\mathcal{U}\,\mathfrak{g}$–module $\mathcal{U}\,\mathfrak{g} \otimes F^*$

$$\mathcal{U}\,\mathfrak{g} \otimes_{\mathcal{U}\,\mathfrak{p}} F^* \;:=\; \mathcal{U}\,\mathfrak{g} \otimes F^* \big/ \langle\, X \otimes \alpha - 1 \otimes X \star \alpha \mid X \in \mathfrak{p},\ \alpha \in F^* \,\rangle$$

by the $\mathcal{U}\,\mathfrak{g}$–submodule generated by $X \otimes \alpha - 1 \otimes X \star \alpha$ for all $X \in \mathfrak{p}$, $\alpha \in F^*$. In other words the induced representation is the maximal quotient of $\mathcal{U}\,\mathfrak{g} \otimes F^*$, on which the two left $\mathfrak{p}$–representations given by multiplication on $\mathcal{U}\,\mathfrak{g}$ and by the tensor product of the adjoint representation on $\mathcal{U}\,\mathfrak{g}$ and the representation on F^* agree. In case $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$ the induced representations are called parabolic Verma modules ([**V**],[**L**]) and have been extensively studied in representation theory and in differential geometry ([**BE**],[**ES**],[**She**]).

The right regular representation of the group G on the functions $G \longrightarrow F$ is the representation of G given by $(R_\gamma f)(\tilde{\gamma}) := f(\tilde{\gamma}\gamma)$. Despite nomenclature the right regular representation is a left representation and thus induces an infinitesimal representation of $\mathcal{U}\,\mathfrak{g}$ on all smooth functions $s \in C^\infty(G, F)$ still denoted by R. With this notation we can give an alternative description of the jet operator defined above

$$\mathrm{jet}^\infty : \quad C^\infty(G, F)^P \;\longrightarrow\; C^\infty(G, (\mathcal{U}\,\mathfrak{g} \otimes_{\mathcal{U}\,\mathfrak{p}} F^*)^*)^P$$

where

$$\langle\, (\mathrm{jet}^\infty f)(\gamma),\ \mathfrak{X} \otimes_{\mathcal{U}\,\mathfrak{p}} \alpha \,\rangle \;\;:=\;\; \langle\, (R_{\mathfrak{X}} f)(\gamma),\ \alpha \,\rangle$$

for all $\mathfrak{X} \in \mathcal{U}\,\mathfrak{g}$ and $\alpha \in F^*$. There are a few things to prove in this definition, namely one has to check that the linear functional $(\mathrm{jet}^\infty f)(\gamma)$ defined on the tensor product $\mathcal{U}\,\mathfrak{g} \otimes F^*$ really factors through to the induced representation $\mathcal{U}\,\mathfrak{g} \otimes_{\mathcal{U}\,\mathfrak{p}} F^*$. Moreover one has to verify that the jet operator maps P–equivariant functions on G again into P–equivariant functions. Neither of these assertions pose any difficulty and so we omit the proof.

It is more important and more elucidating to compare this jet operator with the one previously defined. For this reason we employ the quantization map from the symmetric algebra of $\mathfrak{g}$ to its universal enveloping algebra:

$$q : \;\; \mathrm{Sym}\,\mathfrak{g} \;\longrightarrow\; \mathcal{U}\,\mathfrak{g} \qquad\qquad q(X_1 \cdot \ldots \cdot X_k) \;:=\; \frac{1}{k!} \sum_\sigma X_{\sigma(1)} \,\ldots\, X_{\sigma(k)}$$

The Lemma of Poincaré, Birkhoff and Witt asserts that this map is an isomorphism of vector spaces, in particular we have a surjective map

$$(q \otimes_{\mathcal{U}\,\mathfrak{p}} \mathrm{id}) : \;\; \mathrm{Sym}\,\mathfrak{g} \otimes F^* \;\longrightarrow\; \mathcal{U}\,\mathfrak{g} \otimes_{\mathcal{U}\,\mathfrak{p}} F^*$$

with injective adjoint:

$$(q \otimes_{\mathcal{U}\,\mathfrak{p}} \mathrm{id})^* : \;\; (\mathcal{U}\,\mathfrak{g} \otimes_{\mathcal{U}\,\mathfrak{p}} F^*)^* \;\longrightarrow\; \mathrm{Sym}\,\mathfrak{g}^* \otimes F$$

The crucial point is that this map translates the jet operator taking values in $(\mathcal{U}\,\mathfrak{g}\otimes_{\mathcal{U}\,\mathfrak{p}}F^*)^*$ into the jet operator taking values in $\mathrm{Sym}\,\mathfrak{g}^*\otimes F$. In this way we have determined both the subspace of Taylor series in $\mathrm{Sym}\,\mathfrak{g}^*\otimes F$ arising from P–equivariant functions $s:\ G\longrightarrow F$ and the representation of P on this subspace. In fact the identification of the two jet operators via $(q\otimes_{\mathcal{U}\,\mathfrak{p}}\mathrm{id})^*$ is achieved by the following identity valid for all $\mathfrak{X}\in\mathrm{Sym}\,\mathfrak{g}$ and $\alpha\in F^*$:

Lemma 3.3. —

$$\langle\ jet_0^\infty[v\longmapsto f(\gamma e^v)],\ \mathfrak{X}\otimes\alpha\ \rangle\ =\ \langle\ (R_{q\mathfrak{X}}f)(\gamma),\ \alpha\ \rangle$$

Proof. — Recall that the notation $\mathrm{jet}_0^\infty[v\longmapsto f(\gamma e^v)]\in\mathrm{Sym}\,\mathfrak{g}^*\otimes F$ refers to the infinite Taylor series of the function $v\longmapsto f(\gamma e^v)$ in $0\in\mathfrak{g}$. If we pair off this infinite Taylor series with an element of the form $X^k\otimes\alpha\in\mathrm{Sym}^k\mathfrak{g}\otimes F^*$ with $X\in\mathfrak{g}$ and $\alpha\in F^*$ we get by definition of the pairing:

$$\langle\ \mathrm{jet}_0^\infty[v\longmapsto f(\gamma e^v)],\ X^k\otimes\alpha\ \rangle\ =\ (\frac{d^k}{dt^k}\Big|_0)\,\langle\ f(\gamma e^{tX}),\ \alpha\ \rangle$$

As the elements $X^k\otimes\alpha$ of this form span the vector space $\mathrm{Sym}\,\mathfrak{g}\otimes F^*$ and both sides of Lemma 3.3 are linear in $\mathfrak{X}\otimes\alpha$ it is sufficient to prove

$$(R_{q(X^k)}f)(\gamma)\ =\ (\frac{d^k}{dt^k}\Big|_0)\,f(\gamma e^{tX})$$

for all $\gamma\in G$ and $X\in\mathfrak{g}$, $\alpha\in F^*$. The case $k=0$ is evident from the definiton and the general case follows by induction using the one parameter group $e^{\tau X}e^{tX}=e^{(t+\tau)X}$ in:

$$\begin{aligned}(R_{q(X^{k+1})}f)(\gamma)\ &=\ (R_XR_{q(X^k)}f)(\gamma)\\ &=\ (\frac{d}{d\tau}\Big|_0)\,(\frac{d^k}{dt^k}\Big|_0)\,f(\gamma e^{\tau X}e^{tX})\ =\ (\frac{d^{k+1}}{dt^{k+1}}\Big|_0)\,f(\gamma e^{tX})\end{aligned}$$

□

4. Cartan Geometries and First Order Jet Operators

In the last section we have seen that we have defined jet operators of arbitrary order for homogeneous vector bundles $G\times_P F$ on homogeneous spaces G/P. In this way all questions of analysis on homogeneous vector bundles are reduced to representation theory, say the determination of the spectra of Laplace or Dirac operators as a popular sport. In 1935 Cartan [**C**] introduced Cartan geometries modelled on a homogeneous space G/P to make the idea of a curved analogue of G/P precise. In this section we will recall his definition and show that this concept amounts to postulating the existence of a universal first order jet operator.

Evidently the presence of curvature may be an obstruction against the existence of jet operators of arbitrary order as in the homogeneous case. In order to get a rich geometry which includes many different differentiable manifolds M and a rich local geometry we have to relax the conditions posed on the jet operators somewhat. Taking the existence of the Maurer–Cartan form θ^{MC} on a group G as a guideline, Cartan was led to the following definition:

Definition 4.1. — *(Cartan Geometry and Curvature* [**C**][**Sha**]*)*
A Cartan geometry on a manifold M modelled on the pair of groups (G,P) is a principal P–bundle $\mathcal{G}$ over M endowed with a $\mathfrak{g}$–valued one form $\theta : T\mathcal{G} \longrightarrow \mathfrak{g}$, the Cartan connection, which induces an isomorphism of vector spaces at every point $g \in \mathcal{G}$ and satisfies the equivariance condition

$$\theta(\frac{d}{dt}\Big|_0 g_t \star p_t) \quad = \quad \mathrm{Ad}_{p_0^{-1}}\,\theta(\frac{d}{dt}\Big|_0 g_t) \;+\; \theta^{MC}(\frac{d}{dt}\Big|_0 p_t) \tag{5}$$

Its curvature is then by definition the $\mathfrak{g}$–valued two form κ on $\mathcal{G}$ given by:

$$\kappa \quad := \quad d\theta \;+\; \frac{1}{2}[\theta \wedge \theta] \tag{6}$$

Indeed a Cartan connection θ is a surrogate for the Maurer–Cartan form θ^{MC} on the principal P–bundle $\mathcal{G}$. Its equivariance condition is modelled on the fundamental property (2) of θ^{MC}, whereas the definition of curvature reflects the Maurer–Cartan equation (4). In particular the principal P–bundle G on the homogeneous space $M := G/P$ endowed with the Maurer–Cartan form $\theta := \theta^{MC}$ is a Cartan geometry with vanishing curvature.

Many people would agree that the true geometrical object of a Cartan geometry is not the Cartan connection θ but its curvature κ. Contradictory as this seems at first sight this reflects the idea that a Cartan geometry is a geometric gauge theory in strict analogy to physics. The potential given by the Cartan connection θ has no geometrical meaning in itself but only through the field κ it generates. Just as in physical gauge theories the gauge group of the principal P–bundle $\mathcal{G}$ over M, i. e. the group of all P–equivariant diffeomorphisms of $\mathcal{G}$ covering the identity diffeomorphism of M, acts nontrivially on the set of Cartan connections by pull–back although there is no geometric meaning on M whatsoever associated to this action. In this sense the Cartan geometry on M is not determined by the Cartan connection but only by its gauge equivalence class.

We want to give an equivalent definition for Cartan geometries, which is more in the spirit of these notes. In order to do so we need to introduce P–equivariant versions of the main definitions of Section 2. Consider a P–principal bundle $\mathcal{G}$ over a manifold M and a P–space $\mathcal{F}$ and define:

$$\begin{aligned}
\mathbf{GL}^k_P(\mathcal{G}, M) &:= \{\, \mathrm{jet}^k_1 g \mid g : G \longrightarrow \mathcal{G} \text{ equivariant local diffeomorphism about } P \,\} \\
\mathbf{GL}^k_P G &:= \{\, \mathrm{jet}^k_1 A \mid A : G \longrightarrow G \text{ equivariant local diffeomorphism about } A(P) = P \,\} \\
\mathrm{Jet}^k_P \mathcal{F} &:= \{\, \mathrm{jet}^k_1 f \mid f : G \longrightarrow \mathcal{F} \text{ equivariant } \}
\end{aligned}$$

Actually we haven't defined jets of maps between arbitrary manifolds, however composition with the exponential map $\exp : \mathfrak{g} \longrightarrow G$ allows us in essence to stay with maps whose source is the vector space $\mathfrak{g}$. Indeed the composition with the exponential map has another advantage. Namely we may restrict the composition with exp to the chosen linear complement $\mathfrak{u}$ of $\mathfrak{p}$ in $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{u}$ to get natural diffeomorphisms

$$\mathbf{GL}^k_P(\mathcal{G}, M) \;\cong\; \mathbf{GL}^k(\mathcal{G}, M) \qquad \mathbf{GL}^k_P G \;\cong\; \mathbf{GL}^k(P, \mathfrak{u}) \qquad \mathrm{Jet}^k_P \mathcal{F} \;\cong\; \mathrm{Jet}^k_0 \mathcal{F}$$

which depend explicitly on $\mathfrak{u}$. In this sense the new definitions above associate equivariant versions to the spaces $\mathbf{GL}^k(\mathcal{G}, M)$, $\mathbf{GL}^k(P, \mathfrak{u})$ and $\mathrm{Jet}^k_0 \mathcal{F}$ already defined, which we will employ to make the formulas below manifestly independent of the choice of $\mathfrak{u}$. As an example we can rewrite the jet operator of Lemma 2.1 as follows:

$$\mathrm{jet}^k : \; C^\infty(\mathcal{G}, \mathcal{F})^P \;\longrightarrow\; C^\infty(\mathbf{GL}^k_P(\mathcal{G}, M), \mathrm{Jet}^k_P \mathcal{F})^{\mathbf{GL}^k_P G}, \qquad (\mathrm{jet}^k f)(\mathrm{jet}^k_1 g) \;:=\; \mathrm{jet}^k_1(f \circ g)$$

On the homogeneous space G/P itself this construction provides us with jet operators of arbitrary orders. In fact on G/P we have a universal jet operator in the following sense:

Definition 4.2. — *(The Universal Jet Operator for Homogeneous Spaces)*
The left action of P on G defines a homomorphism $P \longrightarrow \mathbf{GL}\,^k_P G$, $p \longmapsto jet^k_1[\tilde\gamma \longmapsto p\tilde\gamma]$ for any $k \geq 0$. The universal jet operator of order k on the homogeneous space G/P is the P–equivariant map:

$$G \;\longrightarrow\; \mathbf{GL}\,^k_P(G, G/P), \quad \gamma \;\longmapsto\; jet^k_1[\tilde\gamma \longmapsto \gamma\tilde\gamma]$$

Associated to the universal jet operator are jet operators for associated fiber bundles via:

$$jet^k:\; C^\infty(G,\mathcal{F})^P \;\longrightarrow\; C^\infty(G,\mathrm{Jet}^k_P\mathcal{F})^P, \qquad (jet^k f)(\gamma) := jet^k_1[\tilde\gamma \longmapsto f(\gamma\tilde\gamma)]$$

In the case of homogeneous vector bundles $G \times_P F$ the exponential map defines an inclusion $\mathrm{Jet}^k_P F \subset \mathrm{Sym}^k\mathfrak{g}^* \otimes F$ and unravelling the definitions we see that the jet operator constructed in the last section for homogeneous vector bundles is actually associated to the universal jet operator. In particular the subspace $\mathrm{Jet}^k_P F$ agrees as a P–representation with the image of $(\mathcal{U}\,\mathfrak{g} \otimes_{\mathcal{U}\,\mathfrak{p}} F^*)^*$ in $\mathrm{Sym}^{\leq k}\mathfrak{g}^* \otimes F$ under $(q \otimes_{\mathcal{U}\,\mathfrak{p}} \mathrm{id})^*$.

Definition 4.3. — *(Cartan Geometries as Universal First Order Jet Operators)*
A Cartan Geometry on a manifold M is a principal P–bundle $\mathcal{G}$ on M endowed with a universal first order jet operator, i. e. a P–equivariant map from $\mathcal{G}$ to the bundle of first order frames $\mathbf{GL}\,^1_P(\mathcal{G}, M)$ of $\mathcal{G}$ over M:

$$\theta^{-1}:\; \mathcal{G} \;\longrightarrow\; \mathbf{GL}\,^1_P(\mathcal{G}, M)$$

In fact the Cartan connection θ is invertible as a linear map $T_g\mathcal{G} \longrightarrow \mathfrak{g}$ in every point $g \in \mathcal{G}$ and its inverse is the differential of a smooth map $G \longrightarrow \mathcal{G}$, $1 \longmapsto g$, which is unique only up to its first order jet in $1 \in G$. In this way a Cartan geometry on M defines first order jet operators for all fiber bundles associated to the principal P–bundle $\mathcal{G}$. Note however that the existence of a first order jet operator does not necessarily imply the existence of a natural connection.

Hence a Cartan connection is perfectly suited for doing first order calculus on a curved analogue of a homogeneous space. In particular it solves the equivalence problem. As a consequence of the sweeping generality of the definition however higher order calculus becomes more and more contrieved. Severe problems arise in this way for the classification of all invariant differential operators present in the geometry modelled on the homogeneous spaces G/P.

A possible way to proceed is to restrict attention to the subclass of invariant differential operators, which can be written as the composition of first order differential operators in an invariant way, see ([**ES**]) for the precise formulation and results of this approach in conformal geometry. On the other hand it is conceivable that the class of Cartan geometries modelled on a homogeneous space G/P contains a subclass of manifolds, which still allow the construction of jet operators of arbitrary order. In any case it seems strange to neglect all holonomy constraints, because in any theory there are always holonomic objects present, if nothing else then at least the curvature will be.

5. Holonomic Geometries on Manifolds

Two of the most fundamental problems in any geometry modelled on a homogeneous space G/P are the description of the set of local covariants and the construction of natural differential operators. The local covariants describe the geometry in question locally and are a direct measure for its richness or rigidity. The existence of natural differential operators on the other hand is a conditio sine qua non for further investigation.

In this section we will describe a different interpretation of the notion of a curved analogue of a homogeneous space G/P, namely holonomic geometries. Moreover we will consider a general strategy to construct universal jet operators of arbitrary order for these geometries. In particular we can construct a universal jet operator for holonomic parabolic geometries and thus make a first step towards the classification of the invariant differential operators.

In the previous sections we have seen various geometrical data of a homogeneous space G/P appearing in our calculations. Since these data describe the local geometry of the homogeneous space G/P in exponential coordinates we want to use them to described a local geometry based on the homogeneous space. Fixing the linear complement $\mathfrak{u}$ of $\mathfrak{p}$ in $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{u}$ once and for all we have at our disposal the commutator map and the translations defined in some neighborhood of $0 \in \mathfrak{u}$ by:

$$t_v : \quad \mathfrak{u} \longrightarrow \mathfrak{u} \qquad\qquad e^{t_v \tilde{v}} P \;=\; e^v e^{\tilde{v}} P$$

Evident properties are:

$$t_0 \;=\; \mathrm{id} \qquad\qquad t_v^{-1} \;=\; t_{-v} \qquad\qquad t_v 0 \;=\; v \qquad\qquad t_v \tilde{v} \;=\; v \,+\, \tilde{v} \,+\, O(v\tilde{v})$$

Loosely speaking we will say that the translations form a group if in addition $t_v \circ t_{\tilde{v}} \;=\; t_{t_v \tilde{v}}$ is satisfied equivalently if we can choose the linear complement $\mathfrak{u}$ of $\mathfrak{p}$ in $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{u}$ to be a subalgebra itself. The property $t_v(\tilde{v}) \;=\; v + \tilde{v} + O(v\tilde{y})$ is particularly important, because in any case the symbolic calculus must be independent of the choice of the translations. Some statements of Section 2 depend implicitly on the choice to the $\{t_v\}$, but most of them can be proved by induction using symbolic calculations only. However for the construction of the naive Spencer operator in the way we have sketched the translations must necessarily form a group.

Besides the translations the commutator map will enter the geometry modelled on the homogeneous space G/P. First of all we have group homomorphisms for all $d \geq 0$

$$P \;\longrightarrow\; \mathbf{GL}^{\,d}\mathfrak{u}, \quad p \;\longmapsto\; \mathrm{jet}_0^d[v \longmapsto \Phi_{\mathfrak{u}}(p, v)]$$

or more succintly into the inverse limit:

$$P \;\longrightarrow\; \mathbf{GL}^{\,\infty}\mathfrak{u} \;:=\; \varprojlim \mathbf{GL}^{\,d}\mathfrak{u}$$

We will assume throughout that the latter homomorphism is injective, i. e. P embeds into $\mathbf{GL}^{\,d}\mathfrak{u}$ for sufficiently large $d \geq 0$. More generally we could allow a discrete kernel to get analogues of spin structures for the geometry modelled on G/P. The image of P in $\mathbf{GL}^{\,d}\mathfrak{u}$ will be denoted by P^d.

Recall that the diagonal group homomorphism $\Delta : \mathbf{GL}^{\,k+d}\mathfrak{u} \longrightarrow \mathbf{GL}^{\,k}(\mathbf{GL}^{\,d}\mathfrak{u}, \mathfrak{u})$ embeds $\mathbf{GL}^{\,k+d}\mathfrak{u}$ into $\mathbf{GL}^{\,k}(\mathbf{GL}^{\,l}\mathfrak{u}, \mathfrak{u})$. Evidently the latter group is a semidirect product $\mathbf{GL}^{\,k}(\mathbf{GL}^{\,d}\mathfrak{u}, \mathfrak{u}) \cong \mathbf{GL}^{\,k}\mathfrak{u} \ltimes \mathrm{Jet}_0^k \mathbf{GL}^{\,d}\mathfrak{u}$, so that we have a well defined projection though not a group homomorphism:

$$\mathcal{J} : \;\; \mathbf{GL}^{\,k+d}\mathfrak{u} \;\longrightarrow\; \mathrm{Jet}_0^k \mathbf{GL}^{\,d}\mathfrak{u}, \qquad \mathrm{jet}_0^{k+d} A \;\longmapsto\; \mathrm{jet}_0^k[v \longmapsto \mathrm{jet}_v^d A]$$

The definition of the commutator map implies

$$e^{-\Phi_{\mathfrak{u}}(p,v)}\, p\, e^{v}\, e^{\tilde{v}}\, P \;\; = \;\; \Phi_P(p,v)\, e^{\tilde{v}}\, P$$

so that $\mathcal{J}$ is compatible with the embedding of P into $\mathbf{GL}^{\,k+d}\mathfrak{u}$, $k \geq 0$, in the sense:

$$(7) \qquad \mathcal{J}: \;\; P^{k+d} \longrightarrow \mathrm{Jet}^k_0 P^d, \qquad \mathcal{J}(\,\mathrm{jet}^{k+d}_0\Phi_{\mathfrak{u}}(p,\cdot)\,) \;=\; \mathrm{jet}^k_0[v \longmapsto \mathrm{jet}^d_0\Phi_{\mathfrak{u}}(\Phi_P(p,v),\cdot)]$$

In essence a holonomic geometry on a manifold M modelled on the homogeneous space G/P will be a reduction $\mathcal{G}$ of the bundle $\mathbf{GL}^{\,d}M$ of d–frames on M to the structure group $P \cong P^d$. This definition is absolutely classical ([**K**]) but we will impose a particular first order partial differential equation on this reduction. Any reduction $\mathcal{G}$ of $\mathbf{GL}^{\,d}M$ comes along with a classifying section

$$\Omega \;\in\; C^\infty(\mathbf{GL}^{\,d}M,\; {}_{P^d}\backslash\mathbf{GL}^{\,d}\mathfrak{u})^{\mathbf{GL}^{\,d}\mathfrak{u}}$$

defined in such a way as to make the following sequence of maps exact:

$$\mathcal{G} \;\longrightarrow\; \mathbf{GL}^{\,d}M \;\stackrel{\Omega}{\longrightarrow}\; {}_{P^d}\backslash\mathbf{GL}^{\,d}\mathfrak{u}$$

Equivalently Ω is characterized by $\mathrm{jet}^d_0 m \star \Omega^{-1}(\mathrm{jet}^d_0 m) \in \mathcal{G}$ for all $\mathrm{jet}^d_0 m \in \mathbf{GL}^{\,d}M$.

Definition 5.1. — *(Holonomic Geometries)*
Let G/P be a homogeneous space such that the associated translations form a group and suppose $d \geq 1$ is chosen large enough to embed P into $\mathbf{GL}^{\,d}\mathfrak{u}$. A holonomic geometry of order d on a manifold M is a reduction $\mathcal{G}$ of the bundle $\mathbf{GL}^{\,d}M$ of d–frames on M to the structure group $P^d \cong P$, such that the first order jet of the classifying section

$$jet^1\Omega \;\in\; C^\infty(\,\mathbf{GL}^{\,d+1}M,\; \mathrm{Jet}^1_0(\,{}_{P^d}\backslash\mathbf{GL}^{\,d}\mathfrak{u})\,)^{\mathbf{GL}^{\,d+1}\mathfrak{u}}$$

takes values only in the double kernel

$$\mathbf{GL}^{\,d+1}M \;\stackrel{jet^1\Omega}{\longrightarrow}\; \mathrm{Jet}^1_0(\,{}_{P^d}\backslash\mathbf{GL}^{\,d}\mathfrak{u}) \;\cong\; {}_{\mathrm{Jet}^1_0 P^d}\backslash\mathrm{Jet}^1_0\mathbf{GL}^{\,d}\mathfrak{u} \;\rightrightarrows\; {}_{\mathrm{Jet}^1_0 P^{d-1}}\backslash\mathrm{Jet}^1_0\mathbf{GL}^{\,d-1}\mathfrak{u}$$

of the following two maps descending to the respective quotients according to (7):

$$\begin{array}{ccc} \mathrm{Jet}^1_0\mathbf{GL}^{\,d}\mathfrak{u} & \stackrel{\mathcal{J}\circ\mathrm{ev}}{\longrightarrow} & \mathrm{Jet}^1_0\mathbf{GL}^{\,d-1}\mathfrak{u} \\ & \stackrel{\mathrm{Jet}^1_0\mathrm{pr}}{\longrightarrow} & \mathrm{Jet}^1_0\mathbf{GL}^{\,d-1}\mathfrak{u} \end{array}$$

The partial differential equation imposed on holonomic geometries is modelled on the naive Spencer operator discussed in Section 2 and is far less stringent than the integrability of the subbundle $\mathcal{G}$. In fact a holonomic geometry of order $d = 2$ modelled on the homogeneous space $P \ltimes \mathfrak{u}/P$ with $P \subset \mathbf{GL}^{\,1}\mathfrak{u}$ is the same as a torsion free but not necessarily flat connection tangent to the reduction $\mathrm{pr}\,\mathcal{G} \subset \mathbf{GL}^{\,1}M$ of $\mathbf{GL}^{\,1}M$. Moreover in Riemannian and conformal geometry we have a natural bijection

$$\mathcal{J}: \;\; {}_{P^2}\backslash\mathbf{GL}^{\,2}\mathfrak{u} \;\stackrel{\cong}{\longrightarrow}\; \mathrm{Jet}^1_0(\,{}_{P^1}\backslash\mathbf{GL}^{\,1}\mathfrak{u}\,)$$

and so the holonomy constraint on the geometry of order $d = 2$ is the holonomy constraint on the first order jet of the Riemannian or conformal structure in disguise. Intuitvely the notion of holonomic geometries is closely linked to the vanishing of torsion, i. e. to the frame bundle $\mathbf{GL}^{\,d}M$, although the reader should observe that the notion of torsion itself depends on the choice of translations. Nevertheless the partial differential equation has a generalization to the bundle of semiholonomic frames $\mathbf{GL}^{\,d}_{\,\mathrm{semi}}M$ of order d on M allowing torsion as well.

Definition 5.2. — *(Prolongation)*
Let $\mathcal{G}$ be a reduction of the bundle $\mathbf{GL}^{\,d}M$ of d–frames on M to the structure group P^d. Its k–th prolongation is the principal bundle $\mathcal{G}^{(k)} \subset \mathbf{GL}^{\,k}(\mathcal{G}, M) \times \mathbf{GL}^{\,k+d}M$ defined by:

$$\begin{aligned} \mathcal{G}^{(k)} &:= \{\, (jet^k_0 g,\, jet^{k+d}_0 m) \mid g(0) = jet^d_0 m \text{ and } jet^d_0(\mathrm{ev} \circ g) = jet^d_0 m \,\} \\ P^{(k)} &:= \{\, (jet^k_0 A_P,\, jet^{k+d}_0 A_{\mathfrak{u}}) \mid A_P(0) = jet^d_0 A_{\mathfrak{u}} \,\} \end{aligned}$$

Note that the group $P^{(k)}$ acts simply transitively on the fibers of $\mathcal{G}^{(k)}$ via:

$$(\, jet^k_0 g,\, j0^{k+d} m) \,\star\, (jet^k_0 A_P,\, jet^{k+d}_0 A_{\mathfrak{u}}\,) \;:=\; (\, jet^k_0(\, (g \circ A_{\mathfrak{u}}) \star A_P\,),\, jet^{k+d}_0(m \circ A_{\mathfrak{u}})\,)$$

The condition $g(0) = \mathrm{jet}^d_0 m$ for a tuple $(\mathrm{jet}^k_0 g,\, j0^{k+d}m) \in \mathcal{G}^{(k)} \subset \mathbf{GL}^{\,k}(\mathcal{G}, M) \times \mathbf{GL}^{\,k+d}M$ is very natural for it ensures that we have a well defined projection $\mathrm{pr} : \mathcal{G}^{(k)} \longrightarrow \mathcal{G}$. Similarly the other condition allows a well defined projection $\mathrm{pr} : \mathcal{G}^{(k)} \longrightarrow \mathbf{GL}^{\,k}M$. In general the philosophy is to impose as many natural conditions as possible to get a simple description of the situation. The two factors $\mathbf{GL}^{\,k}(\mathbf{GL}^{\,d}M, M)$ and $\mathbf{GL}^{\,k+d}M$ in $\mathcal{G}^{(k)}$ serve entirely different purposes. The first one defines a jet operator for all fiber bundles associated to $\mathcal{G}$, whereas the second detects curvature. Consider now the two maps from $\mathcal{G}^{(k)}$ to $\mathbf{GL}^{\,k}(\mathbf{GL}^{\,d}M, M)$:

$$\begin{tikzcd} & \mathbf{GL}^{\,k+d}M \arrow[dr, "\Delta"] & \\ \mathcal{G}^{(k)} \arrow[ur] \arrow[dr] & & \mathbf{GL}^{\,k}(\mathbf{GL}^{\,d}M, M) \\ & \mathbf{GL}^{\,k}(\mathcal{G}, M) \arrow[ur, "\mathbf{GL}^{\,k}\iota"'] & \end{tikzcd}$$

As $\mathbf{GL}^{\,k}(\mathbf{GL}^{\,d}M, M)$ is a principal bundle the difference between them defines a function from $\mathcal{G}^{(k)}$ to the structure group $\mathbf{GL}^{\,k}(\mathbf{GL}^{\,d}\mathfrak{u}, \mathfrak{u})$. However the constraints in the definition of $\mathcal{G}^{(k)}$ allow only values in ${}^*\mathrm{Jet}^k_0\mathbf{GL}^{\,d}\mathfrak{u} \subset \mathbf{GL}^{\,k}(\mathbf{GL}^{\,d}\mathfrak{u}, \mathfrak{u})$. In fact the constraint $\mathrm{jet}^k_0(\mathrm{ev} \circ g) = \mathrm{jet}^k_0 m$ discards $\mathbf{GL}^{\,k}\mathfrak{u} \subset \mathbf{GL}^{\,k}(\mathbf{GL}^{\,d}\mathfrak{u}, \mathfrak{u})$ and $g(0) = \mathrm{jet}^d_0 m$ show that we get only pointed jets in $\mathrm{Jet}^k_0\mathbf{GL}^{\,d}\mathfrak{u}$. The natural difference function on $\mathcal{G}^{(k)}$ is thus well defined by:

$$\text{Д} :\; \mathcal{G}^{(k)} \longmapsto {}^*\mathrm{Jet}^k_0\mathbf{GL}^{\,d}\mathfrak{u}, \qquad\qquad \text{Д}(\,\mathrm{jet}^k_0 g,\, \mathrm{jet}^{k+d}_0 m\,) \;:=\; (\mathrm{jet}^k_0 g)^{-1}\Delta(\mathrm{jet}^{k+d}_0 m)$$

Since Д measures the difference of the two embeddings of $\mathcal{G}^{(k)}$ into $\mathbf{GL}^{\,k}(\mathbf{GL}^{\,d}M, M)$ it is $P^{(k)}$–equivariant for the restriction $\star :\; {}^*\mathrm{Jet}^k_0\mathbf{GL}^{\,d}\mathfrak{u} \times P^{(k)} \longrightarrow {}^*\mathrm{Jet}^k_0\mathbf{GL}^{\,d}\mathfrak{u}$ of the adjoint action given explicitly by:

$$\begin{aligned} &\mathrm{jet}^k_0[\upsilon \longmapsto \mathrm{jet}^d_0 B_\upsilon] \star (\,\mathrm{jet}^k_0 A_P,\, \mathrm{jet}^{k+d}_0 A_{\mathfrak{u}}\,) \\ &\quad:=\; \mathrm{jet}^k_0(A_P, A_{\mathfrak{u}})^{-1} \cdot \mathrm{jet}^k_0[\upsilon \longmapsto \mathrm{jet}^d_0 B_\upsilon] \cdot \mathrm{jet}^k_0[\upsilon \longmapsto (\mathrm{jet}^d_\upsilon A_{\mathfrak{u}}, A_{\mathfrak{u}}(\upsilon))] \\ &\quad=\; \mathrm{jet}^k_0[\upsilon \longmapsto A_P^{-1}(A_{\mathfrak{u}}^{-1}(\upsilon)) \cdot \mathrm{jet}^d_0 B_{A_{\mathfrak{u}}(\upsilon)} \cdot \mathrm{jet}^d_\upsilon A_{\mathfrak{u}}] \end{aligned}$$

The local covariants of holonomic geometries show up in the values of the difference function Д. This is somewhat unsettling at first sight because holonomic geometries are defined using the geometric object Ω and more or less by definition the local covariants should appear in its jet. However all information about the jet of Ω is encoded in the difference function, because for any tuple $(\mathrm{jet}^k_0 g, \mathrm{jet}^{k+d}_0 m) \in \mathcal{G}^{(k)}$ we have:

$$\begin{aligned}
(\mathrm{jet}^k\Omega)(\Delta(\mathrm{jet}^{k+d}_0 m)) &= (\mathrm{jet}^k\Omega)(\,\mathrm{jet}^k_0 g \star \varPi(\mathrm{jet}^k_0 g, \mathrm{jet}^{k+d}_0 m)\,) \\
&= (\mathrm{jet}^k\Omega)(\mathrm{jet}^k_0 g) \star \varPi(\mathrm{jet}^k_0 g, \mathrm{jet}^{k+d}_0 m) \\
&= \mathrm{Jet}^k_0 P^d \cdot \varPi(\mathrm{jet}^k_0 g, \mathrm{jet}^{k+d}_0 m)
\end{aligned}$$

In this way the difference function determines the jet of Ω and conversely. The basic strategy is now to define a map $\mathcal{G} \longrightarrow \mathcal{G}^{(k)}$, which gives rise to both a universal jet operator of order k via the projection to $\mathbf{GL}^{\,k}(\mathcal{G}, M)$ and to generalized normal coordinates via the projection to $\mathbf{GL}^{\,k+d}M$:

$$\begin{tikzcd}
 & & \mathbf{GL}^{\,k+d}M \arrow[dr, "\Delta"] & \\
\mathcal{G} \arrow[r] \arrow[urr] \arrow[drr] & \mathcal{G}^{(k)} \arrow[ur] \arrow[dr] & & \mathbf{GL}^{\,k}(\mathbf{GL}^{\,d}M, M) \\
 & & \mathbf{GL}^{\,k}(\mathcal{G}, M) \arrow[ur, "\mathbf{GL}^{\,k}\iota"'] &
\end{tikzcd}$$

The guiding example for this procedure is the homogeneous space itself with the reduction $G \longrightarrow \mathbf{GL}^{\,d}G/P$, $\gamma \longmapsto \mathrm{jet}^d_0[\upsilon \longmapsto \gamma e^\upsilon P]$. In this "flat" model case we get both the universal jet operator of Section 4 and the usual exponential coordinates from:

$$\Psi: \quad G \longrightarrow G^{(k)}, \qquad \gamma \longmapsto (\,\mathrm{jet}^k_0[\upsilon \longmapsto \mathrm{jet}^d_0[\tilde{\upsilon} \longmapsto \gamma e^\upsilon e^{\tilde{\upsilon}} P]], \mathrm{jet}^{k+d}_0[\upsilon \longmapsto \gamma e^\upsilon P]\,)$$

Since $\Delta(\mathrm{jet}^{k+d}_0[\upsilon \longmapsto \gamma e^\upsilon P]) = \mathrm{jet}^k_0[\upsilon \longmapsto \mathrm{jet}^d_0[\tilde{\upsilon} \longmapsto \gamma e^{t_\upsilon \tilde{\upsilon}} P]]$ the difference $\varPi$ vanishes on the image $\Psi(G)$ of G in $G^{(k)}$, although $\Psi(G) \subset \{\varPi = *\} \subset G^{(k)}$ is in general a strict subset. The problem is that the Lie group P may fail to be "maximally prolonged" at order d:

Definition 5.3. — *(Maximally Prolonged Groups)*
A subgroup $P \subset \mathbf{GL}^{\,\infty}\mathfrak{u}$ is said to be maximally prolonged at order $d \geq 1$ if

$$\mathcal{J}(\,P^{d+1}\,) \;=\; \mathrm{Jet}^1_0 P^d \,\cap\, \mathcal{J}(\,\mathbf{GL}^{\,d+1}\mathfrak{u}\,)$$

It has been conceived long time ago that this property is part of symbolic calculus and thus independent of the choice of translations as soon as the compatiblity condition (7) is met. In this case $\mathcal{J}(P^{d+1}) \subset \mathrm{Jet}^1_0 P^d \cap \mathcal{J}(\mathbf{GL}^{\,d+1}\mathfrak{u})$ and so the evalutation map

$$\mathrm{ev}\;:\;\; \mathrm{Jet}^1_0 P^d \,\cap\, \mathcal{J}(\,\mathbf{GL}^{\,d+1}\mathfrak{u}\,) \;\longrightarrow\; P^d$$

is surjective. Since $P \cong P^d \cong P^{d+1}$ the group P^d is maximally prolonged if and only if the evaluation map is injective, too, equivalently $\mathfrak{u}^* \otimes \mathfrak{p}^d \cap \mathrm{Sym}^{\,d+1}\mathfrak{u}^* \otimes \mathfrak{u} = \{0\}$ in the vector group ${}^*\mathrm{Jet}^1_0\mathbf{GL}^{\,d}\mathfrak{u} \cong \mathfrak{u}^* \otimes \mathfrak{gl}^d\mathfrak{u}$. This latter condition immediately implies

$$\mathrm{Sym}^{\,k}\mathfrak{u}^* \otimes \mathfrak{p}^d \,\cap\, \mathrm{Sym}^{\,k+d}\mathfrak{u}^* \otimes \mathfrak{u} \;=\; \{0\}$$

for all $k \geq 0$ at the symbolic level for every maximally prolonged group P and thus:

$$\mathcal{J}(\,P^{k+d}\,) \;=\; \mathrm{Jet}^k_0 P^d \,\cap\, \mathcal{J}(\,\mathbf{GL}^{\,k+d}\mathfrak{u}\,)$$

At the end of this section we want to illustrate these rather general remarks by a class of examples, namely the holonomic parabolic geometries of order $d = 2$. These geometries are modelled on a homogeneous space G/P with G semisimple and P a parabolic subgroup.

The adjoint representation $\mathfrak{g}$ splits under the reductive component $L \subset P$ of P into a direct sum of subalgebras:

$$\mathfrak{g} \;:=\; \mathfrak{u} \oplus \mathfrak{p} \;=\; \mathfrak{u} \oplus (\mathfrak{l} \oplus \mathfrak{u}^*)$$

In particular $le^v = e^{\mathrm{Ad}_l v} l$ for all $l \in L$ and $v \in \mathfrak{u}$ or $\Phi_{\mathfrak{u}}(l, v) = \mathrm{Ad}_l v$ and $\Phi_P(l, v) = l$ for the commutator map. Hence the embedding of P into $\mathbf{GL}^{\,\infty}\mathfrak{u}$ defined using $\Phi_{\mathfrak{u}}$ maps L into the subgroup $\mathbf{GL}^{\,1}\mathfrak{u} \subset \mathbf{GL}^{\,\infty}\mathfrak{u}$ of rigid transformations. Consequently the right action of L on ${}^*\mathrm{Jet}^k_0\mathbf{GL}^{\,d}\mathfrak{u} \cong \mathrm{Sym}^{\,1\leq k}\mathfrak{u}^* \otimes \mathfrak{gl}^d\mathfrak{u}$ via $\star$ is just the usual vector representation. The set of local covariants is an L–invariant vector subspace of ${}^*\mathrm{Jet}^k_0\mathbf{GL}^{\,d}\mathfrak{u}$ transversal to the action of $P^{(k)}$ in the following sense:

Definition 5.4. — *(Local Covariants of Parabolic Geometries)*
Consider for $l \geq 0$ the kernel of the projection $\mathrm{pr} : P^{(l+1)} \longrightarrow P^{(l)}$*, which is simply the vector group* $\mathrm{Sym}^{\,l+1}\mathfrak{u}^* \otimes \mathfrak{p}^d \oplus \mathrm{Sym}^{\,l+d+1}\mathfrak{u}^* \otimes \mathfrak{u}$*. The action $\star$ of this group on ${}^*\mathrm{Jet}^{l+1}_0\mathbf{GL}^{\,d}\mathfrak{u}$ reduces to addition on the top order term* $\mathrm{Sym}^{\,l+1}\mathfrak{u}^* \otimes \mathfrak{gl}^d\mathfrak{u}$ *so that it is possible to find an L–invariant direct sum decomposition:*

$$\mathrm{Sym}^{\,l+1}\mathfrak{u}^* \otimes \mathfrak{gl}^d\mathfrak{u} \;=\; \mathrm{Sym}^{\,l+1}\mathfrak{u}^* \otimes \mathfrak{p}^d \,\oplus\, \mathrm{Sym}^{\,l+d+1}\mathfrak{u}^* \otimes \mathfrak{u} \,\oplus\, \mathcal{R}^{l+1}$$

The vector spaces $\mathcal{R}^{l+1}$, $l \geq 0$, are the spaces of local covariants of strict order $l+1$ and their direct sums up to order k are the spaces of local covariants of order k:

$$\mathcal{R}^{(k)} \;:=\; \mathcal{R}^1 \oplus \ldots \oplus \mathcal{R}^k \;\subset\; {}^*\mathrm{Jet}^k_0\mathbf{GL}^{\,d}\mathfrak{u}$$

Note that a priori there may be different choices for the L–invariant vector subspace $\mathcal{R}^{(k)} \subset {}^*\mathrm{Jet}^k_0\mathbf{GL}^{\,d}\mathfrak{u}$. The situation is different for the holonomic parabolic geometries, because the holonomy constraint on the classifying geometric object Ω restricts the possible values assumed by the difference function considerably. For general reasons the local covariants of holonomic parabolic geometries will define an L–invariant subspace $\mathcal{R}^{(k)}_{\mathrm{holonom}} \subset \mathcal{R}^{(k)}$.

Unluckily at the time of writing we haven't succeeded in describing this subspace in complete generality. In particular we don't know whether general holonomic parabolic geometries are non–rigid in the sense that they may have a non–trivial space of local covariants. The situation is different in the case of conformal and projective geometry, where we get the well known space of local covariants, namely the jet of the Weyl curvature tensor modulo the second Bianchi identity. Moreover for a holonomic almost hermitean geometry the subspace $\mathcal{R}^{(1)}_{\mathrm{holonom}}$ is not only L– but already P–invariant simply by reasons of degree. In this case the holonomic parabolic geometries are special geometrically defined Cartan geometries due to the following more general theorem:

Theorem 5.5. — *(Universal Jet Operator for Holonomic Parabolic Geometries)*
Let $\mathcal{G}$ be a holonomic parabolic geometry of order $d = 2$. The group $P \subset \mathbf{GL}^{\,\infty}\mathfrak{u}$ is maximally prolonged at order $d = 2$ and for all $k \geq 0$ there exist natural L–equivariant maps

$$\Psi :\; \mathcal{G} \;\longrightarrow\; \mathcal{G}^{(k)}$$

defined by sending each point in $\mathcal{G}$ to the unique point in $\mathcal{G}^{(k)}$ in which the difference function Д assumes a value in the space $\mathcal{R}^{(k)}_{holonom}$ of local covariants. For homogeneous spaces G this map agrees with the map Ψ already defined above, because the difference function assumes the value $* \in \mathcal{R}_{holonom}$ on the image $\Psi(G)$.

Proof. — As the space of local covariants $\mathcal{R}^{(k)}_{\text{holonom}}$ is an L–invariant subspace of $^{*}\mathrm{Jet}^k_0 \mathbf{GL}^{\,d}\mathfrak{u}$ the map Ψ will automatically be L–equivariant as soon as we can show that Ψ is well defined. The case $k = 0$ is of course trivial as $\mathcal{G} = \mathcal{G}^{(0)}$ by definition. For the inductive step we assume that we have constructed $\Psi : \mathcal{G} \longrightarrow \mathcal{G}^{(k)}$ by sending $\mathrm{jet}^d_0 g$ to the unique point of $\mathcal{G}^{(k)}$ over $\mathrm{jet}^d_0 g$, where the difference function assumes a value in $\mathcal{R}^{(k)}_{\text{holonom}}$. The kernel of the projection $\mathrm{pr} : P^{(k+1)} \longrightarrow P^{(k)}$ acts simply transitively on the preimage of $\Psi(\mathrm{jet}^d_0 g)$ in $\mathcal{G}^{(k+1)}$. Since the difference function is equivariant the direct sum decomposition of Lemma 5.4

$$\mathrm{Sym}^{\,k+1}\mathfrak{u}^* \otimes \mathfrak{gl}^d\mathfrak{u} \;=\; \mathrm{Sym}^{\,k+1}\mathfrak{u}^* \otimes \mathfrak{p}^d \,\oplus\, \mathrm{Sym}^{\,k+d+1}\mathfrak{u}^* \otimes \mathfrak{u} \,\oplus\, \mathcal{R}^{k+1}$$

defining the space $\mathcal{R}^{k+1}$ of local covariants shows that there is a unique point in this fiber on which the difference function assumes a value in $\mathcal{R}^{(k+1)}_{\text{holonom}}$. □

Corollary 5.6. — *(Universal Jet Operators)*
Let $\mathcal{G}$ be a holonomic parabolic geometry of order $d = 2$ on M modelled on the parabolic homogeneous space G/P. For all $k \geq 0$ there is an L–equivariant universal jet operator of order k:

$$\mathcal{G} \;\longrightarrow\; \mathbf{GL}^{\,k}(\mathcal{G}, M)$$

Associated to the universal jet operator are jet operators of order k for all associated fiber bundles $\mathcal{G} \times_P \mathcal{F}$:

$$\mathit{jet}^k : \;\; C^\infty(\mathcal{G}, \mathcal{F})^P \;\longmapsto\; C^\infty(\mathcal{G}, \mathrm{Jet}^k_0\mathcal{F})^L$$

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