PHYSICAL REVIEW D **66**, 124001 (2002)

Qualitative and numerical study of the matter-radiation interaction in Kantowski-Sachs cosmologies

Alan A. Coley
Department of Mathematics and Statistics, Dalhousie University, Halifax, Nova Scotia, Canada B3H 3J5

Antonio Sarmiento G.
Instituto de Matemáticas, Unidad Cuernavaca, UNAM, Av. Universidad s/n, Chamilpa, 62200 Morelos, Mexico

Roberto A. Sussman
Instituto de Ciencias Nucleares, Apartado Postal 70543, UNAM, México D. F. 04510, Mexico
(Received 15 May 2002; published 3 December 2002)

We examine, from both a qualitative and a numerical point of view, the evolution of Kantowski-Sachs cosmological models whose source is a mixture of a gas of weakly interacting massive particles (WIMP's) and a radiative gas made up of a ''tightly coupled'' mixture of electrons, baryons and photons. Our analysis is valid from the end of nucleosynthesis up to the duration of radiative interactions (10^6 K$>T>4\times10^3$ K). In this cosmic era annihilation processes are negligible, while the WIMP's only interact gravitationally with the radiative gas and the latter behaves as a single dissipative fluid that can be studied within a hydrodynamical framework. Applying the full transport equations of extended irreversible thermodynamics, coupled with the field and balance equations, we obtain a set of governing equations that becomes an autonomous system of ordinary differential equations once the shear viscosity relaxation time $\tau_{\rm rel}$ is specified. Assuming that $\tau_{\rm rel}$ is proportional to the Hubble time, the qualitative analysis indicates that models begin in the radiation-dominated epoch close to an isotropic equilibrium point (saddle). We show how the form of $\tau_{\rm rel}$ governs the relaxation time scale of the models towards an equilibrium photon entropy, leading to ''near-Eckart'' and transient regimes associated with ''abrupt'' and ''smooth'' relaxation processes, respectively. Assuming the WIMP particle to be a supersymmetric neutralino with a mass $m_{\rm w}\sim100$ GeV, the numerical analysis reveals that a physically plausible evolution, compatible with a stable equilibrium state and with observed bounds on CMB anisotropies and neutralino abundance, is only possible for models characterized by initial conditions associated with nearly zero spatial curvature and total initial energy density close to unity. An expression for the relaxation time, complying with physical requirements, is obtained in terms of the dynamical equations. It is also shown that the ''truncated'' transport equation does not give rise to acceptable physics.

DOI: 10.1103/PhysRevD.66.124001 PACS number(s): 04.40.Nr, 05.70.Ln

I. INTRODUCTION

The radiative era of cosmic evolution extends from the end of cosmological nucleosynthesis to the decoupling of baryonic matter and radiation, covering the temperature range $4\times10^3-10^6$ K (roughly between 1 eV and 1 keV). During this period cosmic matter can be described [1–3] as a mixture of two main non-interacting components: one, a non-relativistic and collisionless gas of weakly interacting massive particles (WIMP's) [cold dark matter (CDM)], the other a tightly coupled mixture of non-relativistic baryons, electrons and ultra-relativistic matter (''radiation,'' i.e., photons and neutrinos). The standard approach to the radiative era consists of using a Friedmann-Lemaître-Robertson-Walker (FLRW) space-time background [1–3] whose sources are described either by equilibrium kinetic theory [4], gauge-invariant perturbations [5], or by hydrodynamical models [6–8], which in general fail to incorporate a physically plausible description of the matter-radiation interaction since they assume a full thermodynamical equilibrium throughout the evolution. Since the tight coupling between electrons, baryonic matter and radiation follows from various processes of radiative interaction [9–11], mostly involving photons and electrons, we can ignore the non-interacting neutrinos and assume the baryon-electron-photon mixture to evolve with a common temperature (local thermal equilibrium) and to behave as a single fluid, the ''radiative fluid.'' This fluid must be dissipative in order to provide an adequate macroscopic model for these interactions [9]. Ideally, all dissipative fluxes (heat flux, bulk and shear viscosities) should be taken into consideration in the study of this tight coupling. However, in order to deal with a mathematically tractable problem, while still aiming at a physically interesting generalization of previous work, we shall study the case in which only shear viscosity is present. Bulk viscosity is not significant in the temperature ranges we are considering [9], and, although neglecting the contribution of heat flux carries physical limitations, this is compensated by the ensuing mathematical simplification of the field and transport equations. This approach has already been tested in various known and new exact solutions [12].

The simplest class of metrics allowing for anisotropic shear viscous stresses are the Kantowski-Sachs cosmologies [13,14], characterized by a 4-dimensional isometric group. As the source of space-time we consider a momentum-energy tensor made of CDM (the WIMP gas) and the dissipative radiative fluid whose anisotropic pressure can be identified with shear viscous pressure. Considering the WIMP particle to be the supersymmetric lightest neutralino, with mass 100 GeV [15–17], we can safely assume that throughout the radiative era these WIMP's are non-relativistic, collisionless and only interact gravitationally with radiation and baryonic matter. It is also reasonable to assume, for the prevailing temperatures of this era, that the pressure of the WIMP gas and the internal energy of the baryons and electrons are negligible in comparison with the radiation equilibrium pressure. Although the radiative era is dominated by radiation, the rest mass-energy density of the WIMP's is not negligible and dominates that of the baryons and electrons, hence the full source, CDM plus radiative fluid, can be well approximated by a momentum energy tensor in which CDM provides the bulk of the rest mass energy ("matter," which ends up dominating the whole dynamics), while the photons ("radiation") provide the bulk of thermal and dissipative effects. The shear viscosity associated with this source must satisfy appropriate constitutive and transport equations from irreversible thermodynamics that comply with causality and stability [18–21]; these thermodynamical theories are known generically as extended irreversible thermodynamics (EIT) [22–24]. The application of such theories to particular physical systems requires phenomenological coefficients, like the coefficient of shear viscosity, to be provided by kinetic theory. In particular, for the tight coupling of electrons, baryonic matter and radiation and its associated photon-electron interaction, the coefficients corresponding to the "radiative gas" model [9,23,25–27] should be employed. The entropy production must be positive definite and the relaxation time of shear viscosity must be a positive and monotonously increasing function, somehow related to the collision times of the radiative processes associated with the radiative era. All these time scales must overtake the Hubble expansion time as baryonic matter and radiation decouple.

The paper is organized as follows. Sections II to IV present and discuss the field equations of Kantowski-Sachs geometry for a mixture of CDM and a radiative fluid, the application of extended irreversible thermodynamics and the appropriate set of equations of state for the models, as well as the evolution equations for the geometric and state variables. The dynamical analysis is carried on in Sec. V by defining a set of normalized variables, which then leads to a self-consistent and well-behaved autonomous system of ordinary differential (governing evolution) equations. From the qualitative analysis, we identify a saddle point associated with a radiation-dominated FLRW cosmology and contained in the invariant set $\chi=0$, associated with the flat Bianchi I model. We argue that initial conditions must be defined near this point. In Sec. VI we discuss various assumptions on the form of the relaxation time for shear viscosity, these assumptions lead to the identification of a "near-Eckart" and transient regimes, respectively, associated with a swift and slow rate of transiency. The effects of using a "truncated" transport equation are discussed qualitatively in Sec. VII, while Sec. VIII deals with the numerical analysis of the models bearing in mind the qualitative results obtained in previous sections. The main result that follows from the qualitative and numerical analysis is the fact that a physically plausible evolution is possible only for (a) initial conditions and evolution close to $\chi=0$, and (b) using the full (not truncated) shear viscosity transport equation of EIT. A detailed discussion and summary of these results is provided in Sec. IX.

II. KANTOWSKI-SACHS COSMOLOGIES WITH ANISOTROPIC STRESSES

The simplest non-FLRW cosmological metric allowing for anisotropic pressure is that of Kantowski-Sachs (KS) models:

$$ds^2=-c^2dt^2+A^2(t)dr^2+B^2(t)[d\theta^2+\sin^2(\theta)d\phi^2]. \tag{1}$$

For a co-moving 4-velocity u^a, the expansion scalar and shear tensor associated with the metric (1) are

$$\Theta=\frac{\dot{A}}{A}+\frac{2\dot{B}}{B},\quad \sigma^a{}_b=\text{diag}[0,-2\sigma,\sigma,\sigma],\quad \sigma\equiv\frac{1}{3}\left(\frac{\dot{B}}{B}-\frac{\dot{A}}{A}\right), \tag{2}$$

where a dot denotes the derivative with respect to proper time of fundamental observers, which for the KS metric (1) in co-moving coordinates is given by t (i.e., $\dot{A}=A_{,t}=u^aA_{,a}$). We consider as the source of (1), the following stress-energy tensor:

$$T^{ab}=\rho u^a u^b+p h^{ab}+\Pi^{ab}, \tag{3}$$

where $h^{ab}=c^{-2}u^a u^b+g^{ab}$ and $\Pi^a{}_b$ is the anisotropic pressure tensor satisfying $\Pi_{ab}u^b=\Pi^a{}_a=0$. The most general form of this tensor for the metric (1) is

$$\Pi^a{}_b=\text{diag}[0,-2P,P,P], \tag{4}$$

where $P=P(t)$ is an arbitrary function to be determined by the field equations and subjected to an evolution law for a given physical model associated with Eqs. (1) and (3). The field equations then become

$$\kappa\rho=-G^t{}_t=\frac{\dot{B}^2}{B^2}+2\frac{\dot{B}}{B}\frac{\dot{A}}{A}+\frac{1}{B^2}, \tag{5}$$

$$3\kappa p=2G^\theta{}_\theta+G^r{}_r=-\frac{\dot{B}^2}{B^2}-\frac{2\ddot{A}}{A}-\frac{4\ddot{B}}{B}-2\frac{\dot{B}}{B}\frac{\dot{A}}{A}-\frac{1}{B^2}, \tag{6}$$

$$3\kappa P=G^\theta{}_\theta-G^r{}_r=\frac{\dot{B}^2}{B^2}-\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}-\frac{\dot{B}}{B}\frac{\dot{A}}{A}+\frac{1}{B^2}, \tag{7}$$

where $\kappa=8\pi G/c^2$, while the energy balance is given by

$$\dot{\rho}+(\rho+p)\Theta+6\sigma P=0. \tag{8}$$

Mixture of cold dark matter and a radiative fluid

We will assume that the stress-energy tensor (3) corresponds to a mixture of a non-relativistic gas of WIMP's and a radiative fluid with shear viscosity corresponding to a "tightly coupled" mixture of photons, electrons and baryons sharing a common temperature T. Hence, ρ and p in Eq. (3) are the total mass-energy density and equilibrium pressure given by

$$\rho=\rho_{\rm w}+\rho_{\rm b}+\rho_{\rm e}+\rho_{\gamma},\quad p=p_{\rm w}+p_{\rm b}+p_{\rm e}+p_{\gamma},\quad p_{\gamma}=\frac{1}{3}\rho_{\gamma},\tag{9}$$

with the subindices γ, w, b, and e denoting photons, WIMP's, baryons and electrons, respectively. The three latter components satisfy each the equation of state of a non-relativistic ideal gas:

$$\rho_{\rm w}=\left(m_{\rm w}c^2+\frac{3}{2}k_BT_{\rm w}\right)n_{\rm w},\quad p_{\rm w}=n_{\rm w}k_BT_{\rm w},$$
$$\rho_{\rm b}=\left(m_{\rm b}c^2+\frac{3}{2}k_BT\right)n_{\rm b},\quad p_{\rm b}=n_{\rm b}k_BT,$$
$$\rho_{\rm e}=\left(m_{\rm e}c^2+\frac{3}{2}k_BT\right)n_{\rm e},\quad p_{\rm e}=n_{\rm e}k_BT,\tag{10}$$

where $m_{\rm w}, m_{\rm b}, m_{\rm e}$ are the respective particle masses of the WIMP's, the baryons (a proton mass) and the electrons, k_B is Boltzmann's constant, $T_{\rm w}$ is the temperature of the WIMP gas and T is the common temperature of the radiative mixture. During the radiative era creation or annihilation processes cease to be significant and so the particle number densities, $n_{\rm w}, n_{\rm b}, n_{\rm e}$, satisfy conservation laws of the form

$$\dot{n}+n\Theta=0\quad\text{with}\quad n=n_{\rm w},n_{\rm b},n_{\rm e}\tag{11}$$

which can be integrated, leading to

$$n=\frac{N}{AB^2},\quad\text{with}\quad N=N_{\rm w},N_{\rm b},N_{\rm e}\tag{12}$$

where $N_{\rm w}, N_{\rm b}, N_{\rm e}$ are the constant (i.e., conserved) number of WIMP's, baryons and electrons, respectively.

The radiation component of the radiative fluid can be given either in terms of (i) the Stefan-Boltzmann law:

$$\rho_{\gamma}^{sb}=aT^4,\quad p_{\gamma}^{sb}=\frac{1}{3}aT^4,\tag{13}$$

where $a\equiv\pi^2k_B^4/(15\hbar^3c^3)$ is Stefan-Boltzmann constant, or (ii) an ultra-relativistic ideal gas:

$$\rho_{\gamma}^{ig}=3n_{\gamma}k_BT,\qquad p_{\gamma}^{ig}=n_{\gamma}k_BT,\tag{14}$$

where n_{γ} is the number density of ultra-relativistic particles, subjected to a balance law analogous to Eq. (11) and given by an expression similar to Eq. (12) with the conserved photon number N_{γ}.

In order to simplify Eq. (9), we can examine the ratios of particle numbers and rest mass densities of the different particle components. Considering the currently estimated [2] ratio of photons per baryon, we have

$$\nu_{\rm b}\equiv\frac{N_{\rm b}}{N_{\gamma}}=\frac{n_{\rm b}}{n_{\gamma}}\simeq 2.67\times10^{-8}\Omega_{\rm b}h^2,\tag{15}$$

where $\Omega_{\rm b}$ is the baryon abundance today (roughly 0.04 ± 0.01) and $h\simeq0.7$ is the adimensional Hubble factor [28]. Regarding the WIMP gas, if we assume that it is made up of the lightest supersymmetric neutralinos with $m_{\rm w}\sim100$ GeV [15–17], we have that [16]

$$\nu_{\rm w}\equiv\frac{N_{\rm w}}{N_{\gamma}}=\frac{n_{\rm w}}{n_{\gamma}}\simeq 2.82\times10^{-8}\Omega_{\rm w}h^2,\tag{16}$$

where $\Omega_{\rm w}\simeq0.3\pm0.1$ is the neutralino (CDM) abundance today. Using Eqs. (10), (12) and (14), we can rewrite Eq. (9) as

$$\rho=m_{\rm w}c^2n_{\rm w}\left(1+\frac{m_{\rm b}\nu_{\rm b}}{m_{\rm w}\nu_{\rm w}}+\frac{m_{\rm e}\nu_{\rm e}}{m_{\rm w}\nu_{\rm w}}\right)$$
$$+3n_{\gamma}k_BT\left(1+\frac{\nu_{\rm b}}{2}+\frac{\nu_{\rm e}}{2}+\frac{\nu_{\rm w}T_{\rm w}}{2T}\right),$$
$$p=n_{\gamma}k_BT\left(1+\nu_{\rm b}+\nu_{\rm e}+\nu_{\rm w}\frac{T_{\rm w}}{T}\right).\tag{17}$$

Bearing in mind that for electrons $N_{\rm e}\sim N_{\rm b}$, so that $\nu_{\rm e}\sim\nu_{\rm b}$, while $m_{\rm e}\ll m_{\rm b}\approx1$ GeV, $\nu_{\rm w}\ll1$, and $\nu_{\rm e}\approx\nu_{\rm b}\ll1$, then,

$$\frac{m_{\rm b}\nu_{\rm b}}{m_{\rm w}\nu_{\rm w}}\simeq10^{-2}\frac{\Omega_{\rm b}}{\Omega_{\rm w}}\ll1,\quad\frac{m_{\rm e}\nu_{\rm e}}{m_{\rm w}\nu_{\rm w}}\simeq10^{-5}\frac{\Omega_{\rm b}}{\Omega_{\rm w}}\ll1,$$

while for the temperature range $4\times10^3\ {\rm K}<T<10^6$ K, we have

$$0.013\lesssim\frac{m_{\rm w}c^2n_{\rm w}}{n_{\gamma}k_BT}\lesssim3.25,\tag{18}$$

showing that the radiative era is initially radiation-dominated but rest mass energy density is not negligible and ends up becoming dominant. Therefore, even if $T_{\rm w}/T$ in Eq. (17) is not negligible, we have

$$\rho\simeq m_{\rm w}c^2n_{\rm w}+3n_{\gamma}k_BT,\quad p\simeq n_{\gamma}k_BT,\tag{19}$$

The same type of approximation can be obtained if we use the Stefan-Boltzmann law (13), since the ratio of pressures in Eqs. (10) and (13): $p_{\gamma}^{sb}/p_{\rm b}=aT^3/[3n_{\rm b}k_B]$, is proportional to $\nu_{\rm b}$ (likewise for WIMP's). Therefore, the mixture of a gas of WIMP's and a radiative fluid can be accurately described in the desired temperature range by the approximated equation of state:

$$\rho=mc^2n^{(m)}+\rho^{(r)},\quad p=p^{(r)}=\frac{1}{3}\rho^{(r)}$$
$$\text{with}\quad m\equiv m_{\rm w},\quad n^{(m)}\equiv n_{\rm w},\quad p^{(r)}\equiv p_{\gamma},\tag{20}$$

where $\rho^{(r)}$ follows from either one of Eqs. (13) or (14), hence $n^{(r)}=n_{\gamma}$ and $\rho^{(m)}=m_{\rm w}c^2n_{\rm w}$. For the remaining of this paper, the superindices (r) and (m) will refer to quantities associated with photons (''radiation'') and WIMP's (''matter''), respectively. From Eqs. (5) and (6), the equation of state (20) can be given as the following constraint:

$$\frac{\ddot{A}}{A}+\frac{2\ddot{B}}{B}+\frac{2\dot{A}}{A}\frac{\dot{B}}{B}+\frac{\dot{B}^2}{B^2}+\frac{1}{B^2}-\frac{1}{2}\kappa mc^2 n^{(m)}=0. \quad (21)$$

III. EXTENDED IRREVERSIBLE THERMODYNAMICS

If the source (3) is meant to describe a mixture of nonrelativistic CDM and a radiative fluid (as argued in previous sections), the anisotropic pressure must be identified with a shear viscous stress of the latter fluid and must be compatible with a suitable thermodynamical formalism. We shall consider the so-called ''extended irreversible thermodynamics'' (EIT) [20–23], a theory complying with causality and stability requirements [24] and supported by the kinetic theory of gases, information theory and by the theory of hydrodynamical fluctuations [23]. When shear viscosity is the only dissipative agent, the corresponding generalized entropy current, S^a, obeying the usual balance law with non-negative divergence, and up to second order in Π^{ab}, takes the form

$$S^a=nSu^a, \quad S=S_{(e)}-\frac{\alpha\Pi_{cd}\Pi^{cd}}{2nT}, \quad (22)$$

where S is the entropy per particle, $n=n^{(r)}+n^{(m)}=(1+\nu_{\rm w})n^{(r)}$ is the total particle number density ($\approx n^{(r)}$), α is a phenomenological coefficient to be specified later and $S_{(e)}$ is defined by the equilibrium Gibbs equation:

$$nT\dot{S}_{(e)}=\dot{\rho}-(\rho+p^{(r)})\frac{\dot{n}}{n}=3\dot{p}+4p\Theta=-\sigma_{ab}\Pi^{ab}, \quad (23)$$

where we have used Eqs. (11) and (20) to eliminate $\dot{\rho}$. Fulfillment of the second law of thermodynamics requires $S^a_{;a}\geqslant 0$, which from the definition of S^a and S in Eq. (22) leads to

$$\dot{S}\geqslant 0, \quad (24)$$

together with

$$\alpha=\frac{\tau_{\rm rel}}{2\eta}, \quad (25)$$

and the evolution equation of the viscous pressure, i.e., the transport equation [24,29]:

$$\tau_{\rm rel}\dot{\Pi}_{cd}h^c_a h^d_b+\Pi_{ab}\left[1+\frac{1}{2}\epsilon_0\,\eta T\left(\frac{\tau_{\rm rel}}{T\eta}u^c\right)_{;c}\right]+2\eta\sigma_{ab}$$
$$=\tau_{\rm rel}\dot{\Pi}_{cd}h^c_a h^d_b+\Pi_{ab}\left[1-\frac{1}{2}\epsilon_0\tau_{\rm rel}\left(\frac{\dot{T}}{T}+\frac{\dot{\eta}}{\eta}-\frac{\dot{\tau}_{\rm rel}}{\tau_{\rm rel}}-\Theta\right)\right]$$
$$+2\eta\sigma_{ab}=0, \quad (26)$$

where $\eta, T, \tau_{\rm rel}, \sigma_{ab}, \Pi_{ab}$ are the coefficients of shear viscosity, the temperature, the relaxation time, and the shear and shear viscosity tensors, respectively. The parameter ϵ_0 can take only two values: $\epsilon_0=1$ (''full'' transport equation) and $\epsilon_0=0$ (''truncated'' transport equation, also known as the Israel-Stewart equation [20,21,23]), while Eckart's noncausal transport equation follows by setting $\tau_{\rm rel}=0$. The coefficient of shear viscosity as well as other related quantities can be obtained by a variety of means [9] including kinetic theory, statistical mechanics or both [25–27,30]. Unless specifically stated otherwise, we shall consider only the full transport equation $\epsilon_0=1$. We will discuss the implications of the truncated equation ($\epsilon_0=0$) in Sec. VII. Evaluating $\dot{S}$ from Eq. (22) and using Eqs. (23), (25), and (26), we obtain

$$\dot{S}=\frac{\Pi_{ab}\Pi^{ab}}{2\eta nT}=\frac{2[S_{(e)}-S]}{\tau_{\rm rel}}\geqslant 0, \quad (27)$$

where we have used Eq. (22) and $\epsilon_0=1$ in Eq. (26). Notice how $\dot{S}$ takes a very simple form, illustrating the role of $\tau_{\rm rel}$ as the reference time scale associated with the entropy change from S to $S_{(e)}>S$. Also, the second law of thermodynamics, [i.e., Eq. (24)] is fulfilled if $\eta\geqslant 0$ or, equivalently, if $\tau_{\rm rel}>0$ and $S_{(e)}\geqslant S$ hold. Another important requirement that follows from the second law of thermodynamics and the stability of equilibrium states is that S be a convex functional, i.e., $\delta^2 S<0$. For the KS models all quantities depend only on time and so a necessary (but not sufficient) condition is given by $\ddot{S}<0$, which leads to

$$\left(1+\frac{1}{2}\dot{\tau}_{\rm rel}\right)\Pi_{ab}\Pi^{ab}+2\eta\sigma_{ab}\Pi^{ab}>0. \quad (28)$$

For the applicability of the general relations, Eqs. (22), (24), (25), (26), (27) and (28), to the models we are concerned with, we must impose the equation of state (20) with either one of the choices Eq. (13) or Eq. (14). As a physical reference to infer the form that the coefficients $\tau_{\rm rel}, \eta, \alpha$ may take for the radiative fluid, consider the''radiative gas'' model associated with the photon-electron interaction [9,22,23,25], characterized by p,ρ complying with the equations of state discussed in Sec. III. For the radiative gas the forms of η,α in terms of the relaxation time of the dissipative process, $\tau_{\rm rel}$, are

$$\eta_{(rg)}=\frac{4}{5}p^{(r)}\tau_{\rm rel}, \qquad \alpha_{(rg)}=\frac{5}{8p^{(r)}}, \quad (29)$$

where $p^{(r)}$ is either $aT^4/3$ or $n^{(r)}k_B T$ [Eq. (13) or (14), respectively] and the subscript ''(rg)'' emphasizes that these quantities are specific to the radiative gas. Applying Eq. (29) into Eqs. (22) and (23), we get for the entropy per particle:

$$S=S_{(e)}-\frac{15P^2}{8p^{(r)}n^{(r)}T}, \quad (30)$$

$$\dot{S}_{(e)}=-\frac{3\sigma P}{n^{(r)}T}, \quad (31)$$

where we have neglected the entropy of the non-relativistic matter, so that $n^{(m)}+n^{(r)}=(1+\nu_w)n^{(r)}\approx n^{(r)}$, and S is approximately the photon entropy (this is justified because the number of photons is so much larger than that of WIMP's, baryons and electrons). The transport equation (26) becomes

$$\dot{P}+\frac{8}{5}p^{(r)}\sigma+\left(\frac{4}{3}\Theta+\frac{1}{\tau_{\rm rel}}\right)P+\lambda\frac{\sigma}{p^{(r)}}P^2=0, \tag{32}$$

where $\lambda\equiv 1+1/(2\lambda_0)$ and $\lambda_0=1/2$ or 2 [for $p^{(r)}$ given by the Stefan-Boltzmann law (13) or the ideal gas law (14), respectively]; also condition (27) takes the form

$$\dot{S}=\frac{15P^2}{4p^{(r)}n^{(r)}T\tau_{\rm rel}}=\frac{2[S_{(e)}-S]}{\tau_{\rm rel}}\geqslant 0. \tag{33}$$

Notice that $S<S_{(e)}$ and $\tau_{\rm rel}>0$ must hold in order for $\dot{S}>0$ to be satisfied, while Eq. (23) implies that we must have $\sigma P<0$ in order that $\dot{S}_{(e)}>0$. Regarding the interpretation of $S_{(e)}$, if we assume[1] $\dot{S}_{(e)}>0$ and the Steffan-Boltzmann law (13), Eq. (23) then yields

$$S_{(e)}=\frac{4aT^3}{3n^{(r)}}, \tag{34}$$

a function that is only constant (equilibrium photon entropy) if $P=0$. This constant is given explicitly by the black body formulas [28]:

$$n^{(r)}|_{P=0}=\frac{30\zeta(3)aT^3}{k_B\pi^4}\Rightarrow S_{(e)}|_{P=0}=\frac{2\pi^4k_B}{45\zeta(3)}\approx 3.60k_B, \tag{35}$$

where ζ is the Riemann zeta function. However, if we characterize the evolution to equilibrium as $P\to 0$, then, as this evolution proceeds, $S_{(e)}$ in Eq. (34) must tend to the constant entropy given in Eq. (35). Hence, we can identify Eq. (35) as the equilibrium state associated with Eq. (27) and Eq. (33), attained as $P\to 0$ and $S\to S_{(e)}$ (i.e. as the radiation relaxes) in the time scale provided by the relaxation time $\tau_{\rm rel}$. For the ideal gas law, Eq. (23) does not yield Eq. (34), but $S_{(e)}\propto k_B\ln(T^3/n^{(r)})$, an expression that coincides with Eqs. (34) and (35) only in equilibrium (if $\dot{S}_{(e)}=0$). However, since both EIT and Eckart's theory assume near equilibrium states, quantities like P^2 and σP, appearing in Eqs. (30) to (33) must be small, hence the ratio $T^3/n^{(r)}$ is nearly constant and so we can also assume that $S_{(e)}$ given by Eq. (34) is approximately valid for the ideal gas law.

Conditions (33) and $\delta^2 S<0$ associated with Eq. (28) must be satisfied by any self-consistent thermodynamical system. The importance of these conditions will become evident when discussing the numerical integration of the evolution equations. The relaxation time, $\tau_{\rm rel}$, is qualitatively analogous to and larger than the mean collision time between particles and it may, in principle, be estimated by collision integrals provided the interaction potential is known. Since we are concerned with mixtures of baryons, electrons and photons evolving in the temperature range 4×10^3 K$<T<10^6$ K (from the end of cosmic nucleosynthesis to decoupling), convenient references for comparing $\tau_{\rm rel}$ are the collision times associated with Compton and Thompson scatterings [31]:

$$t_c=\frac{m_ec^2}{k_BT}t_\gamma, \tag{36}$$

$$t_\gamma=\frac{1}{2c\sigma_Tn_b}\left[1+\left(1+\frac{4h^3n_be^{B_0/k_BT}}{(2\pi m_ek_BT)^{3/2}}\right)^{1/2}\right], \tag{37}$$

where σ_T, B_0, m_e, and h are the Thompson scattering cross section (6.65×10^{-25} cm^2), the hydrogen atom binding energy (13.6 eV), and the electron mass and Planck's constant, respectively. Equation (37) is obtained from the number density of free electrons provided by the Saha equation. Notice that we are using the baryon number density, n_b, and not the number density of WIMPs, $n^{(m)}$. For higher temperatures in the range of interest, Compton scattering is the most efficient radiative process keeping baryonic matter and radiation tightly coupled, though it is no longer effective in lower and intermediate temperature ranges ($T<10^4$ K). The photon-electron interaction of the radiative era requires that microscopic collision times t_γ, t_c, as well as $\tau_{\rm rel}$, be much smaller than the time scale of cosmic expansion given by the Hubble time, approximately $t_H\equiv 3/\Theta$. For the lower temperature range of the radiative era, just before recombination, Thomson scattering becomes the dominant radiative process, so that the decoupling of baryonic matter and radiation can be associated with the condition $t_\gamma=t_H$, which should be approximately equivalent to $\tau_{\rm rel}=t_H$.

IV. EVOLUTION EQUATIONS

Since we need to determine a self-consistent set of ordinary differential equations governing the KS models, it is convenient to express the field equations and Eq. (21) in terms of Θ and σ by eliminating $\dot{A},\dot{B}$ from Eq. (2):

$$\frac{\dot{A}}{A}=\frac{\Theta}{3}-2\sigma, \tag{38a}$$

$$\frac{\dot{B}}{B}=\frac{\Theta}{3}+\sigma, \tag{38b}$$

which leads to

$$\kappa\rho=\frac{\Theta^2}{3}-3\sigma^2+\frac{1}{B^2}, \tag{39}$$

[1]Notice that $\dot{S}_{(e)}>0$ is a sufficient but not necessary condition for $\dot{S}>0$. Under the framework of extended irreversible thermodynamics, the latter is the physicaly relevant condition.

$$\kappa p^{(r)} = -\frac{\Theta^2}{3} - \frac{2\dot{\Theta}}{3} - 3\sigma^2 - \frac{1}{3B^2}, \tag{40}$$

$$\kappa P = \dot{\sigma} + \sigma\Theta + \frac{1}{3B^2}, \tag{41}$$

while, using Eqs. (39) and (40), the equation of state (21) becomes

$$\dot{\Theta} + \frac{2}{3}\Theta^2 + 3\sigma^2 + \frac{1}{B^2} - \frac{1}{2}\kappa m c^2 n^{(m)} = 0. \tag{42}$$

We can eliminate B from the equations above with the help of Eq. (39) (which becomes a constraint) and Eq. (38b) (the evolution equation for B). Using Eq. (20), Eqs. (41) and (42) then become

$$\dot{\sigma} + \sigma^2 + \sigma\Theta - \frac{\Theta^2}{9} + \frac{\kappa}{3} m c^2 n^{(m)} + \kappa p^{(r)} - \kappa P = 0, \tag{43}$$

$$\dot{\Theta} + \frac{\Theta^2}{3} + 6\sigma^2 + \frac{\kappa}{2} m c^2 n^{(m)} + 3\kappa p^{(r)} = 0, \tag{44}$$

which are the evolution equations for σ and Θ. Since we are assuming particle number conservation, an evolution equation for non-relativistic particle number density (the WIMP's) follows from Eq. (11):

$$\dot{n}^{(m)} + n^{(m)}\Theta = 0, \tag{45}$$

a conservation law satisfied also by $n^{(r)} = n^{(m)}/\nu_{\rm w}$ [if using the ideal gas law (14)]. Another evolution equation is provided by Eq. (8), which applied to the equation of state (20) and using $\sigma_{ab}\Pi^{ab} = 6\sigma P$ yields

$$\dot{p}^{(r)} + \frac{4}{3} p^{(r)}\Theta + 2\sigma P = 0, \tag{46}$$

becoming the evolution equation for $p^{(r)}$. The transport equation (32) derived in the previous section, namely,

$$\dot{P} + \frac{8}{5} p^{(r)}\sigma + \left(\frac{4}{3}\Theta + \frac{1}{\tau_{\rm rel}}\right) P + \lambda \frac{\sigma}{p^{(r)}} P^2 = 0, \tag{47}$$

is the evolution equation for the shear stress P. In addition to these evolution laws, Eq. (33) can be thought of as an evolution equation for S, while we can transform Eq. (46) into an evolution equation for T by using either Eq. (13) or (14), leading to

$$\frac{\dot{T}}{T} + \frac{\Theta}{3} + \frac{\sigma}{\lambda_0 p^{(r)}} P = 0. \tag{48}$$

Equations (43), (44), (45), (46), and (47) represent a self-consistent and closed system of first order ODE's for $n^{(m)}, P, p^{(r)}, \sigma, \Theta$. Notice that this set of evolution equations is fully determined if $\tau_{\rm rel}$ is known. In the cosmological context $\tau_{\rm rel}$ might be proportional to the time scale defined by the expansion scalar (approximately the Hubble time):

$$\tau_{\rm rel} \propto t_H \equiv 3/\Theta. \tag{49}$$

Alternatively, and depending on the temperature and energy range one is considering, $\tau_{\rm rel}$ could be identified as proportional to a collision time (say, Thomson or Compton scattering) given by Eq. (36) or (37), i.e.,

$$\tau_{\rm rel} \propto t_\gamma(n,T), \quad \tau_{\rm rel} \propto t_c(n,T). \tag{50}$$

Since we can assume in Eq. (47) two different equations of state for the radiation component, strictly speaking, Eqs. (43)–(47) constitute two different systems of evolution equations parametrized by the two possible values of λ_0 and λ, the Stefan-Boltzmann law:

$$3p^{(r)} = aT^4 \quad \text{with} \quad \lambda_0 = 2 \quad \text{and} \quad \lambda = 5/4, \tag{51a}$$

and the ideal gas law:

$$p^{(r)} = n^{(r)} k_B T, \quad \text{with} \quad \lambda_0 = 1/2 \quad \text{and} \quad \lambda = 2. \tag{51b}$$

V. DYNAMICAL ANALYSIS

A. The governing equations

Let us define

$$Q \equiv \frac{P}{p^{(r)}}, \tag{52}$$

a ratio that is related [from Eq. (30)] to the deviation from equilibrium:

$$S_{(e)} - S = \frac{15}{8}\lambda_0 Q^2, \quad \text{where}$$

$$\lambda_0 = \frac{p^{(r)}}{n^{(r)}T} = \begin{cases} \frac{3}{4} S_{(e)}, & \text{Stefan-Boltzmann law} \\ k_B, & \text{ideal gas law} \end{cases} \tag{53}$$

with $S_{(e)}$ given by Eq. (34). Therefore, Q must be a small quantity for the cosmic times we are interested in. Note that when P (and hence Q) is zero, the shear vanishes and the Kantowski-Sachs model reduces to an isotropic FLRW model. In principle, P (and hence Q) can be positive or negative, but since P represents a viscous pressure it should be negative in the expanding regime. We shall focus henceforth on the case in which Q is negative. The energy conditions imply that $-1 \leqslant Q \leqslant 1/2$. In addition, on physical grounds we expect the second term in Eq. (8) to dominate the third term in this equation, which is satisfied whenever $\sigma^2 Q^2/(\Theta/3)^2 < 4$. However, since physically we expect Q^2 to be small, these constraints are satisfied handily.

We introduce now the new normalized variables:

$$\Sigma \equiv \frac{\sigma}{\Theta/3}, \qquad \Omega_{(m)} \equiv \frac{\kappa m c^2 n^{(m)}}{\Theta^2/3}, \qquad \Omega_{(r)} \equiv \frac{\kappa p^{(r)}}{(\Theta/3)^2}, \tag{54}$$

a definition that is motivated by the fact that $\Theta/3$ is approximately the Hubble expansion factor $H=\Theta/3+\sigma_{ab}n^a n^b$ for unit space-like vectors defined by $n^a n_a=1, u^a n_a=0$, hence $\Omega_{(m)}$ and $\Omega_{(r)}$ are approximately equivalent to the observational parameters (the Ω's) for the CDM and radiation contents of the mixture. We also define (for $\Theta>0$) the new independent variable:

$$\frac{d}{dt}=\frac{\Theta}{3}\frac{d}{d\tau} \Rightarrow \tau=\int \frac{\Theta}{3} dt, \tag{55}$$

the evolution equations (43) to (47) become

$$\Omega'_{(m)} = -\Omega_{(m)}[1-4\Sigma^2-\Omega_{(m)}-2\Omega_{(r)}], \tag{56}$$

$$Q' = -\frac{8}{5}\Sigma - \frac{1}{\tau_{\rm rel}\Theta/3}Q-(\lambda-2)\Sigma Q^2. \tag{57}$$

$$\Omega'_{(r)} = -\Omega_{(r)}[2+2\Sigma(Q-2\Sigma)-\Omega_{(m)}-2\Omega_{(r)}], \tag{58}$$

$$\Sigma' = (1-2\Sigma)(1-\Sigma^2) - \frac{\Omega_{(m)}}{2}(2-\Sigma) - \Omega_{(r)}(1-Q-\Sigma), \tag{59}$$

where we have used

$$\frac{3\dot{\Theta}}{\Theta^2}=\frac{\Theta'}{\Theta}=-1-2\Sigma^2-\frac{\Omega_{(m)}}{2}-\Omega_{(r)}, \tag{60}$$

and a prime denotes differentiation with respect to τ. Note that this last equation implies that Θ is monotonically decreasing. The constraint (42) becomes

$$\chi \equiv 1-\Sigma^2-\Omega_{(m)}-\Omega_{(r)} = -\frac{3}{B^2\Theta^2}. \tag{61}$$

We note that

$$\chi' = \chi[2\Sigma(2\Sigma-1)+\Omega_{(m)}+2\Omega_{(r)}].$$

Clearly $\chi=0$ is an invariant set of the above differential equations, which corresponds to the Bianchi I (zero curvature) sub-case.

Eventually, the models re-collapse and Θ changes sign. At the point of maximum expansion (when $\Theta=0$) the variables above diverge and the normalized equations are no longer valid. However, for the times we are interested in, in the expanding phase far from re-collapse, the above variables and equations are valid. Indeed, in principle we can use the above system to follow the evolution of the models all the way back to the big bang. From Eq. (44) we can see that in this regime the curvature is small and that the variables $\Sigma^2, \Omega_{(m)}, \Omega_{(r)}$ are well behaved. Compact variables can be defined by normalizing with Θ^2+B^{-2} (instead of Θ^2) that are valid for all times [32]; however, the physical assumptions used here are not valid at later times.

In addition to Eqs. (56) to (59), evolution equations for $n, T, S_{(e)}$ and S follow by using the variables defined in Eqs. (54) and (55) in Eqs. (45), (48), (31) and (33):

$$n' = -3n, \Rightarrow n(\tau)=n_i e^{-3\tau} \quad \text{where} \quad n=n^{(m)} \quad \text{or} \quad n^{(r)}, \tag{62}$$

$$T' = -T\left[1+\frac{\Sigma Q}{\lambda_0}\right], \tag{63}$$

$$S'_{(e)} = -3\lambda_0 \Sigma Q, \tag{64}$$

$$S' = \frac{2t_H}{\tau_{\rm rel}}[S_{(e)}-S] = \frac{15 t_H}{4\tau_{\rm rel}}\lambda_0 Q^2, \tag{65}$$

where λ_0 is given by Eq. (53), the subindex i denotes evaluation at an initial time $\tau=\tau_i$, and $t_H=3/\Theta$ is approximately the Hubble time which follows from Eq. (60) as

$$t_H = \frac{3}{\Theta_i}\exp\left\{\int\left[1+2\Sigma^2+\frac{\Omega_{(m)}}{2}+\Omega_{(r)}\right]d\tau\right\}, \tag{66}$$

while the relation between physical time t and τ follows from Eqs. (55) and (60):

$$t=\int t_H d\tau, \tag{67}$$

where t_H is given by Eq. (66) above.

B. Qualitative properties

Consider the dynamical implications of assuming that $\tau_{\rm rel}$ is given by Eq. (49), namely,

$$\tau_{\rm rel} = \frac{\gamma_0}{\Theta/3} = \gamma_0 t_H, \tag{68}$$

where $\gamma_0>0$ is a constant. For the range we are interested in, $\gamma_0 \leqslant 1$. Equation (57) becomes

$$Q' = -\frac{8}{5}\Sigma - \frac{1}{\gamma_0}Q-(\lambda-2)\Sigma Q^2, \tag{69}$$

and so Eqs. (56), (58), (59) and (69) now constitute a closed four-dimensional system of first order autonomous differential equations for $(\Omega_{(r)}, \Omega_{(m)}, \Sigma, Q)$, a system that depends on the value of the constant parameter γ_0. Moreover, from Eq. (61) and the above discussion, in the regime we are interested in $\Omega_{(m)}, \Omega_{(r)}$, and Σ are bounded and physical conditions imply that $-1 \leqslant Q \leqslant 1/2$. Consequently a local analysis of the stability of the equilibrium points of this system will provide useful dynamical information.

Setting the right-hand side of Eq. (69) to zero we obtain

$$(\lambda-2)\Sigma Q^2 + \frac{1}{\gamma_0}Q + \frac{8}{5}\Sigma = 0, \tag{70}$$

which is a quadratic equation for Q if Σ is given and $\lambda \neq 2$.

Stefan-Boltzman law ($\lambda = 5/4$). The equilibrium points at finite values are [note that all such points are given below; however, $(\Omega_{(m)}, \Omega_{(r)}, \Sigma^2)$ are not necessarily bounded by unity] as follows.

(i) $\Omega_{(m)} = 0 = \Omega_{(r)}$, six points:
(ia) $\Sigma = 1/2$ and $Q_\pm = 4[3\gamma_0 \pm \sqrt{9\gamma_0^2 + 2/15}]/(9\gamma_0^2)$;
(ib) $\Sigma = 1$ and $Q_\pm = 2[3\gamma_0 \pm \sqrt{9\gamma_0^2 + 8/15}]/(9\gamma_0^2)$;
(ic) $\Sigma = -1$ and $Q_\pm = -2[3\gamma_0 \pm \sqrt{9\gamma_0^2 + 8/15}]/(9\gamma_0^2)$.
(ii) $\Omega_{(m)} = 0, \Omega_{(r)} \neq 0$, four points:
(iia) $\Omega_{(r)} = -(12\gamma_0 + 17/15)/(9\gamma_0^2)$, and
$\Sigma_\pm = Q_\pm = \pm 2\sqrt{3\gamma_0 + 8/15}/(3\gamma_0)$;
(iib) $\Sigma_\pm = (4 + 3\gamma_0)/[2 + 3\gamma_0(1 \mp u)]$,
$\Omega_{(r)\pm} = -(1/\gamma_0)(4 + 3\gamma_0)[2 + \gamma_0(1 \pm u)]$
$/[2 + 3\gamma_0(1 \mp u)]$,
and $Q_\pm = (2 \pm 3\gamma_0 u)/(4 + 3\gamma_0)$,
where $u = 2\sqrt{1 + (8\gamma_0/5)(4 + 3\gamma_0)}/(3\gamma_0)$;
$\Sigma = 0 = Q, \Omega_{(r)} = 1$ is a particular solution.
(iii) $\Omega_{(r)} = 0, \Omega_{(m)} \neq 0$, one particular solution:
$\Sigma = 0 = Q$, and $\Omega_{(m)} = 1$.
(iv) $\Omega_{(m)} \neq 0$, $\Omega_{(r)} \neq 0$, two points:
$\Omega_{(m)} = (40 + 79/\gamma_0)/(64\gamma_0)$,
$\Omega_{(r)} = -15(8 + 3\gamma_0)/(128\gamma_0)$,
$\Sigma = \pm\sqrt{5(8 + 3\gamma_0)/(2\gamma_0)}/8$,
and $Q_\pm = \mp 4\sqrt{2(8 + 3\gamma_0)/(45\gamma_0)}$.

The two equilibrium points given by (ia) can be shown to lead to a value for Q which is unphysical in the sense that its magnitude is much too large (in fact, when $\Omega_{(m)} = 0 = \Omega_{(r)}, Q$ would be expected to vanish). The equilibrium points (iv) are also unphysical: since clearly $\Sigma^2 \leq 1/4$ and we get from (iv) that $\Sigma^2 Q^2 = 1/4$, this would imply that $Q^2 \geq 1$. At the equilibrium points (ii) when $\Omega_{(r)} \neq 1$ we have that $\Sigma = 20Q/[\gamma_0(15Q^2 - 32)]$, which leads to a quartic equation for Q; however, for physical values of the parameter γ_0, this equation has no real roots and hence no solutions of physical interest. The two equilibrium points given by the particular solution of (ii) and by (iii), corresponding to FLRW models, namely $(\Omega_{(r)}, \Omega_{(m)}, \Sigma, Q)$ given by (1,0,0,0) and (0,1,0,0), can easily be shown to be saddles. The equilibrium points (ib) and (ic), namely $(\Omega_{(r)}, \Omega_{(m)}, \Sigma, Q) = (0,0,\pm 1, Q_\pm)$, which belong to the invariant set $\chi = 0$ (i.e., correspond to a Bianchi I model with no curvature) have eigenvalues 3, $-1/\gamma_0 + 3\lambda^2 \Sigma Q/2, 2 - 2\Sigma Q, 4 - \Sigma$. In this case, the eigenvalue $-1/\gamma_0 + 3\lambda^2 \Sigma Q/2$ can only be positive if we take the "positive square root" of Eq. (37) (i.e., $Q = Q_+$). However, for $\{\lambda^2, \gamma_0\} \leq 1$, i.e., $\lambda^2 \gamma_0 \leq 1$, it follows that the eigenvalue $2 - 2\Sigma Q_+$ can never be positive. Consequently, this equilibrium point cannot be a source.

We note that there are no sinks at finite values. However, this is to be expected since the models evolve toward maximum expansion at which the variables become unbounded. The models subsequently re-collapse.

Ideal gas case ($\lambda = 2$). Equations (56), (58) and (59) remain unchanged and the governing equation (69) becomes

$$Q' = -\frac{8}{5}\Sigma - \frac{1}{\gamma_0}Q. \tag{71}$$

At an equilibrium point we immediately see that

$$Q = -\frac{8\gamma_0}{5}\Sigma. \tag{72}$$

However, there are no major qualitative changes in the analysis. In particular, there are fewer equilibrium points.

(i) $\Omega_{(m)} = 0 = \Omega_{(r)}$, three points:
(ia) $\Sigma = 1/2$ and $Q = -4\gamma_0/5$;
(ib) $\Sigma = 1$ and $Q = -8\gamma_0/5$;
(ic) $\Sigma = -1$ and $Q = 8\gamma_0/5$.
(ii) $\Omega_{(m)} = 0, \Omega_{(r)} \neq 0$:
$\Sigma = 0 = Q, \Omega_{(r)} = 1$ is a particular solution; otherwise, it is a quadratic equation for Σ which gives another two equilibrium points.
(iii) $\Omega_{(r)} = 0, \Omega_{(m)} \neq 0$, one particular solution:
$\Sigma = 0 = Q$ and $\Omega_{(m)} = 1$.
(iv) $\Omega_{(m)} \neq 0, \Omega_{(r)} \neq 0$, two points that imply:
$\Sigma^2 = 5/(16\gamma_0) \geq 5/16$ (for $\gamma_0 \leq 1$),
which leads to $\Omega_{(m)} + 2\Omega_{(r)} = 1 - 5/(4\gamma_0) < 0$.

VI. ASSUMPTIONS ON THE RELAXATION TIME

In the general dynamical system, Eqs. (56)–(59), the relaxation time $\tau_{\rm rel}$ needs to be specified in order for the governing system of equations to be closed. For the qualitative dynamical analysis we have assumed (68), so that $1/(\tau_{\rm rel}\Theta)$ is a constant, since otherwise we would either not be dealing with an autonomous system, or would be looking at an autonomous but much more difficult dynamical system. However, Eq. (68) is a simplifying assumption that cannot be supported by thermodynamical arguments, perhaps a more realistic assumption would be to consider instead:

$$\tau_{\rm rel} = \frac{\gamma(\tau)}{\Theta/3} = \gamma(\tau) t_H \tag{73}$$

so that Eqs. (57) and (65) become

$$Q' = -\frac{8}{5}\Sigma - \frac{1}{\gamma(\tau)}Q - (\lambda - 2)\Sigma Q^2, \tag{74}$$

$$S' = \frac{2}{\gamma(\tau)}[S_{(e)} - S], \tag{75}$$

where $\gamma = \gamma(\tau)$ is a function that could be suitably adjusted so that $\tau_{\rm rel}$ has a form that is qualitatively analogous to that of microscopic time scales like Eq. (36) or (37), time scales that are physically relevant for the matter source under consideration. The ratio $3/(\tau_{\rm rel}\Theta) = t_H/\tau_{\rm rel} = 1/\gamma$ should provide a comparison of the time scale for the relaxation (transient) effects in the radiative fluid with the time scale of cosmic expansion. Hence this ratio should approach unity as baryonic matter and radiation decouple, so that $\gamma \approx 1$ should be a consistent choice for near decoupling conditions, while $\gamma \gg 1$ or $\gamma \ll 1$ correspond to after decoupling (late times) and much before decoupling (earlier times). Ideally, we should obtain $\tau_{\rm rel}$ from collision integrals associated with each of the various radiative processes occurring in the radiative era,

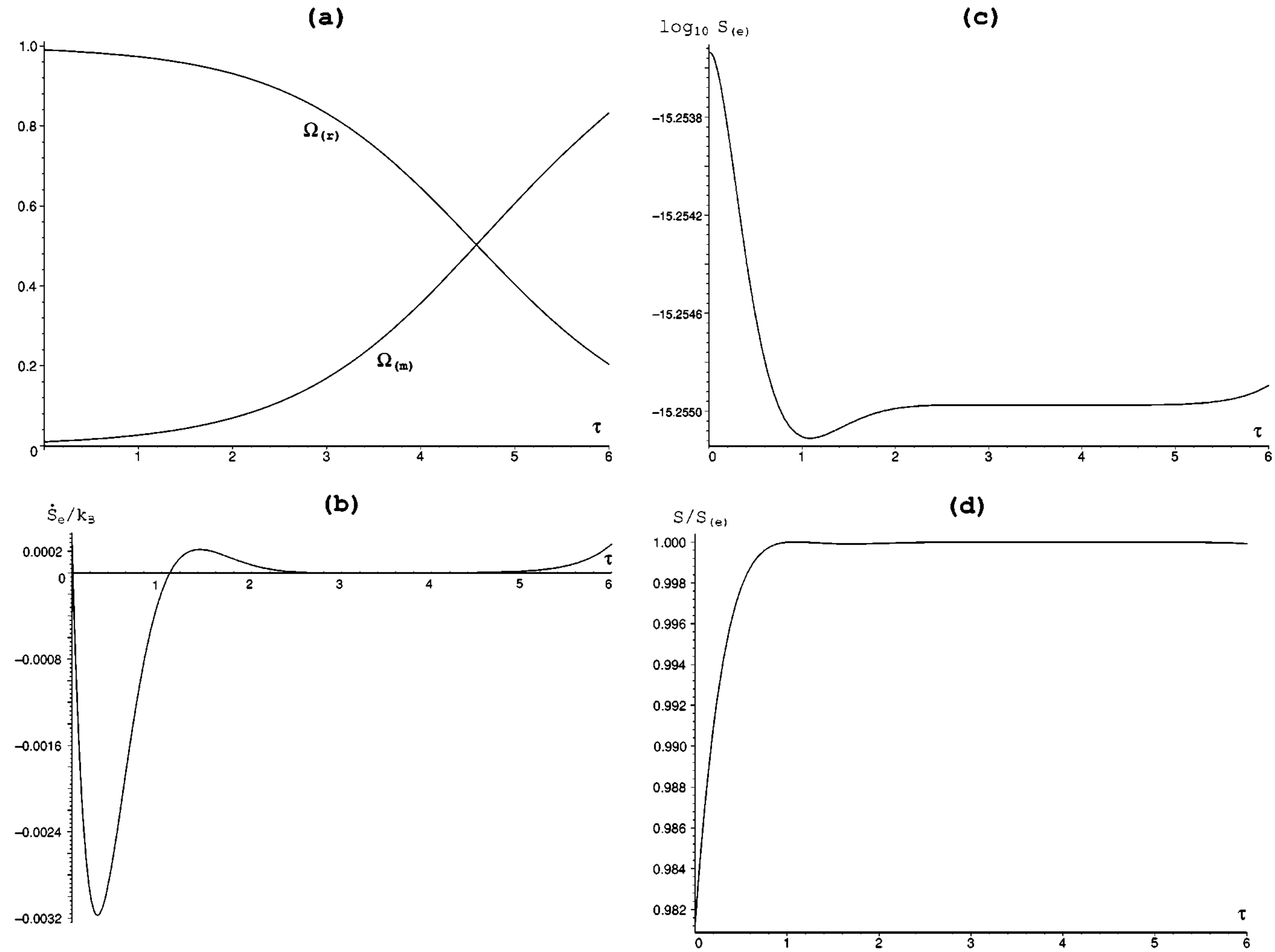

FIG. 1. A physically plausible evolution. These figures illustrate the fulfillment of conditions 1, 2 and 3 for a physically plausible evolution given in Sec. VIII B. The function $S_{(e)}$ in (c) is given in cgs units and is almost equal to the equilibrium photon entropy. These figures (as well as those of Fig. 2) were obtained using initial conditions (104), with $\tau_{\rm rel}=0.7$. Notice in (b) that $\dot{S}_{(e)}$ becomes negative (though small) near the initial time. This behavior does not denote an unphysical situation since $\dot{S}>0$ holds throughout the evolution.

but such an endeavor would merit a separate paper by itself and will not be attempted here. Instead, we will consider $\tau_{\rm rel}$ as an ''effective'' relaxation time, encompassing the different radiative processes. We will examine the non-transient limit (or near-Eckart regime) and use the dynamical equations themselves in order to suggest a suitable form for $\tau_{\rm rel}$. The discussion of this section will be complemented and tested numerically in Sec. VIII.

A. The "near-Eckart" and the transient regimes

In order to examine the relaxation process as $S\to S_{(e)}$, we will assume that $\gamma(\tau)$ in Eq. (73) is a smooth function so that we can always expand it in the form $\gamma\approx\gamma(0)+\gamma'(0)\tau+\gamma''(0)\tau^2/2$. Therefore, at early times $\tau\approx 0$ we can always associate the constant γ_0 in Eq. (68) as $\gamma_0=\gamma(0)$, so that the corresponding form of $\tau_{\rm rel}$ is approximately correct at least near $\tau=0$. Also, at early times we must have $\gamma(0)<1$, and so $1/\gamma_0\gg 1$, while $\Sigma(0)$ and $Q(0)$ are necessarily small quantities. Hence the terms Σ and ΣQ^2 in Eq. (74) will be much smaller than $Q(0)/\gamma_0$ and so we have that near $\tau=0, Q'\approx -Q/\gamma_0$, so that

$$Q\approx Q(0)\exp\left(\frac{-\tau}{\gamma_0}\right),$$

$$S\approx S_{(e)}-\frac{15}{8}\lambda_0 Q^2(0)\exp\left(\frac{-2\tau}{\gamma_0}\right), \tag{76}$$

where λ_0 is given by Eq. (53). Since the process of relaxation to equilibrium can be characterized as the decay of the dissipative flux $Q\to 0$ as S grows and asymptotically approaches the equilibrium state given by Eq. (35), the numerical value of γ_0 in Eq. (76) may be interpreted as a measure of the ''rate of transiency'' in terms of how fast or slow the system accomplishes this relaxation in comparison with the timescale provided by t_H. We can then identify two possible situations.

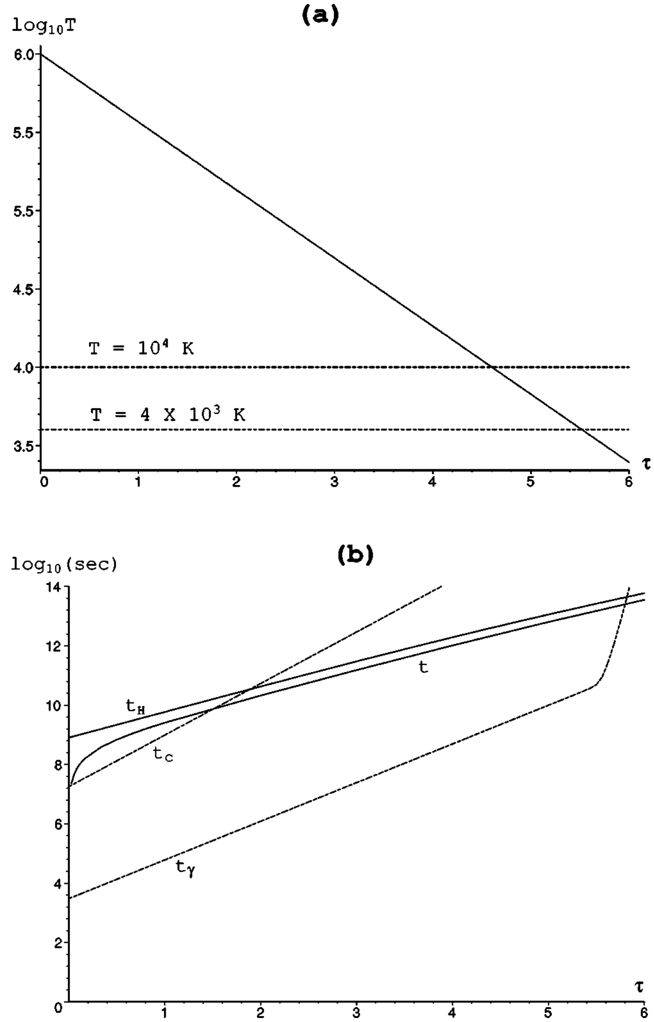

FIG. 2. Range of validity of the models. The range $0 \leqslant \tau \leqslant 6$ corresponds to the temperature range of the radiative era, between $T_i = 10^6$ K and the baryon-radiation decoupling temperature 4×10^3 K (roughly at $\tau = 5.5$). A logarithmic plot of the various time scales used in the paper (in seconds) are depicted in Fig. 6(b): the Thomson and Compton scatterings (t_γ, t_c), the Hubble time t_H and physical time t. Notice how the decoupling temperature occurs at the same τ as the condition $t_\gamma = t_H$, while radiation-matter equality, $\Omega_{(r)} = \Omega_{(m)}$ (see Fig. 1a) corresponds to $T = 10^4$ K (roughly $\tau = 4.6$).

The near-Eckart regime. If $\gamma_0 \ll 1$, then Eq. (76) indicates a very fast relaxation with a very abrupt decay of Q and S to $S_{(e)}$. In the very limit $\gamma_0 \to 0$ (so that $\tau_{\rm rel} \to 0$ as well) we have $Q \to Q(0)\delta(\tau)$, so that the relaxation is infinitely fast, in agreement with the non-causal nature of Eckart's classical theory. The fast relaxation associated with a very small γ_0 implies a very short duration of the relaxation process (small $\tau_{\rm rel}$), indicating that the approximation $\gamma = \gamma_0$, as well as the expressions in Eq. (76) for $Q(\tau)$ and $S(\tau)$, are approximately valid for the whole evolution time. Hence, since Q practically vanishes very quickly, we have for most of the evolution time that $S \approx S_{(e)}$, agreeing with the fact that in Eckart's theory the entropy is rigorously and unambiguously given by its equilibrium form (the hypothesis of "local equilibrium," see section 1.3.1 of [23]). The near-Eckart regime is appropriate to describe a given radiative process for which microscopic time scales are much smaller than the cosmological expansion time scale t_H. See Sec. VIII D.

The transient regime. If $\gamma_0 < 1$ but $\gamma_0 \approx O(10^{-1}) - O(1)$, then Q and S also decay but the relaxation process is much slower, hence the term "transient." In this case, the expressions in Eq. (76) and the assumption $\gamma = \gamma_0$ are only good approximations for $\gamma, Q(\tau)$ and $S(\tau)$ near $\tau = 0$. In general, we must use

$$Q \approx Q(0)\exp\left[-\int \frac{d\tau}{\gamma(\tau)}\right],$$

$$S \approx S_{(e)} - \frac{15}{8}\lambda_0 Q^2(0)\exp\left[-\int \frac{2\,d\tau}{\gamma(\tau)}\right], \qquad (77)$$

leading to Eq. (75) and to

$$S'' \approx S''_{(e)} - \frac{2(1+\gamma')(S_{(e)} - S)}{\gamma^2}, \qquad (78)$$

so that the fulfillment of $S' > 0$ and $S'' < 0$ can be examined in terms of $\gamma(\tau)$ and $S_{(e)}$. Sufficient (but not necessary) conditions follow by demanding that γ is a monotonically increasing function $(\gamma' > 0)$ and $S'_{(e)} > 0, S''_{(e)} < 0$. Another condition on γ is furnished by Eq. (28), which together with Eqs. (29), (52), (54), (55), (60), and (73) yield

$$2 + \gamma' Q^2 + \gamma\left[\frac{16}{5}\Sigma Q - \left(1 + 2\Sigma^2 + \frac{\Omega_{(m)}}{2} + \Omega_{(r)}\right)Q^2\right] > 0, \qquad (79)$$

a condition that should be tested numerically for any given choice of $\tau_{\rm rel}$. The relaxation time $\tau_{\rm rel}$ can be approximated by a given $\gamma = \gamma_0$, but as the decoupling era is reached, γ must increase to $O(1)$ allowing for $\tau_{\rm rel}$ to overtake t_H.

B. Dynamical relaxation times

Let us consider the following ansatz:

$$\frac{1}{\gamma} = -\frac{8}{5}\frac{\Sigma}{Q}[1 + \zeta(\tau)], \qquad (80)$$

which provides an exact relation for the relaxation time in Eq. (73). In regimes in which the Eckart theory is a good approximation, we can assume $\gamma \approx \gamma_0 \ll 1$, so that Eq. (80) implies $Q \propto \Sigma$ but $|\Sigma/Q| \gg 1$ and ζ is "small." Such regimes would correspond to a radiative process that takes place in time scales smaller than a mean collision time, thus decaying very fast to equilibrium. In case we wish to consider processes taking place on time scales comparable and larger than main collision times, then transient effects are important and the near-Eckart regime is no longer appropriate. In particular, we can construct an expression for $\tau_{\rm rel}$ that acts as an "effective" relaxation time that encompasses the relaxation times for the main radiative processes acting in the radiative era under consideration. Having this idea in mind, a reasonable and more general expression for ζ can be obtained from the dynamical equations themselves. Consider the conditions discussed above for a near-Eckart regime, and assume that

$$Q=-\mu_0\Sigma, \tag{81}$$

where μ_0 is a non-negative constant. We obtain an expression analogous to Eq. (80) by substituting Eq. (81) into the (full EIT) evolution equations (57) and (59), leading to the consistency requirement

$$\frac{1}{\gamma}=\frac{1}{\gamma_0}+a_0\Omega_{(m)}-b_0\Sigma^2+\left(1-\mu_0-\frac{1}{\Sigma}\right)\chi, \tag{82}$$

where

$$\frac{1}{\gamma_0}=1+\mu_0+\frac{8}{5\mu_0}\geqslant 1+4\sqrt{\frac{2}{5}}>1,$$

$$a_0\equiv\frac{1}{2}-\mu_0,\quad b_0=1+(\lambda-3)\mu_0, \tag{83}$$

and χ is the curvature term $\chi\equiv 1-\Omega_{(m)}-\Omega_{(r)}-\Sigma^2$ given by Eq. (61), a term that is very small and can be neglected. This is an extremely simple dynamical relation for the relaxation time, and for a wide range of conditions it might be a very good approximation. In addition, it has some important physical properties. For early times in which $\chi,\chi/\Sigma,\Omega_{(m)},\Sigma^2$ are very small, we have a near-Eckart regime associated with $\gamma\approx\gamma_0\ll 1$, as expected. This is appropriate for the Compton scattering, the dominant radiative process in the early part of the period under consideration, a process that quickly thermalizes and ceases to be effective. At later times, as $\Omega_{(m)}$ increases toward a value of order unity at recombination (and to a lesser extent, the curvature term also grows), the constants γ_0,a_0,b_0 in Eq. (83), as well as the initial conditions, can be selected in such a way that γ, given by Eq. (82), increases sufficiently as to allow $\tau_{\rm rel}$ to overtake t_H and to approach the characteristic timescale of Thomson scattering in Eq. (37). We show in Sec. VIII that adequate parameters and initial conditions can be found so that $\tau_{\rm rel}$ associated with Eq. (82) has the expected behavior (see Sec. VIII E).

VII. TRUNCATED THEORY

The discussion so far has been based on the full transport equation. In order to appreciate the effect of considering the truncated transport equation, it is useful to rewrite Eq. (26) with η and $\tau_{\rm rel}$ given by Eqs. (29) and (73) in terms of Q and Σ, thus allowing the full and truncated equations to appear jointly. This yields

$$Q'=-\frac{8}{5}\Sigma-\left[\frac{1}{\gamma(\tau)}-4(1-\epsilon_0)\right]Q-(\lambda\epsilon_0-2)\Sigma Q^2, \tag{84}$$

where the full and truncated theories are, respectively, given by $\epsilon_0=1$ and $\epsilon_0=0$. Assuming a transient regime, instead of Eq. (76) we have near $\tau=0$:

$$Q\approx Q(0)\exp\left[-\frac{\tau}{\gamma_0}+4(1-\epsilon_0)\tau\right],$$

$$S\approx S_{(e)}-\frac{15}{8}\lambda_0 Q^2(0)\times\exp\left\{2\left[-\frac{\tau}{\gamma_0}+4(1-\epsilon_0)\tau\right]\right\}, \tag{85}$$

so that using the truncated transport equation ($\epsilon_0=0$) introduces a large linear term ($\propto 4\tau$) that is absent in the full theory. This linear term could change dramatically the form of Q and the relaxation of S to $S_{(e)}$. As we show below, the truncated transport might lead to $Q\Sigma$ being positive (implying that $\dot S_{(e)}<0$), so even at this level (early times) there might be problems of consistency in the truncated approach of EIT.

A comprehensive analysis in the case of the truncated theory, similar to that presented in Sec. V, can be undertaken. The evolution Eqs. (56), (58), and (59) remain unchanged, while the evolution Eq. (57) for Q must be replaced by the truncated equation that follows by setting $\epsilon_0=0$ in Eq. (84)

$$Q'=-\frac{8}{5}\Sigma-\left(\frac{1}{\gamma_0}-4\right)Q+2\Sigma Q^2. \tag{86}$$

At an equilibrium point

$$2\Sigma Q^2-\left(\frac{1}{\gamma_0}-4\right)Q-\frac{8}{5}\Sigma=0. \tag{87}$$

However, we immediately note from Eq. (87) that close to an equilibrium point for small Σ and Q,

$$\Sigma\simeq\frac{5}{8}\left(4-\frac{1}{\gamma_0}\right)Q, \tag{88}$$

so that for $\gamma_0\geqslant\frac{1}{4}$ we have that Σ and Q have the same sign (unlike the non-truncated theory case), and hence it is immediately clear that there will be a different qualitative dynamical behavior in the truncated theory.

In particular, close to the FLRW equilibrium point $(\Omega_{(r)},\Omega_{(m)},\Sigma,Q)=(1,0,0,0)$ (which we will use to determine the initial conditions in our numerical analysis in the following section), from a calculation of the corresponding eigenvalues we have that

$$\Omega_{(m)}\propto e^{\tau},\qquad (1-\Omega_{(r)})\propto e^{2\tau}, \tag{89}$$

$$\Sigma,Q\propto e^{\alpha_+\tau}+\beta e^{\alpha_-\tau}, \tag{90}$$

from Eqs. (56)–(59) in the non-truncated case, where β is a constant, and

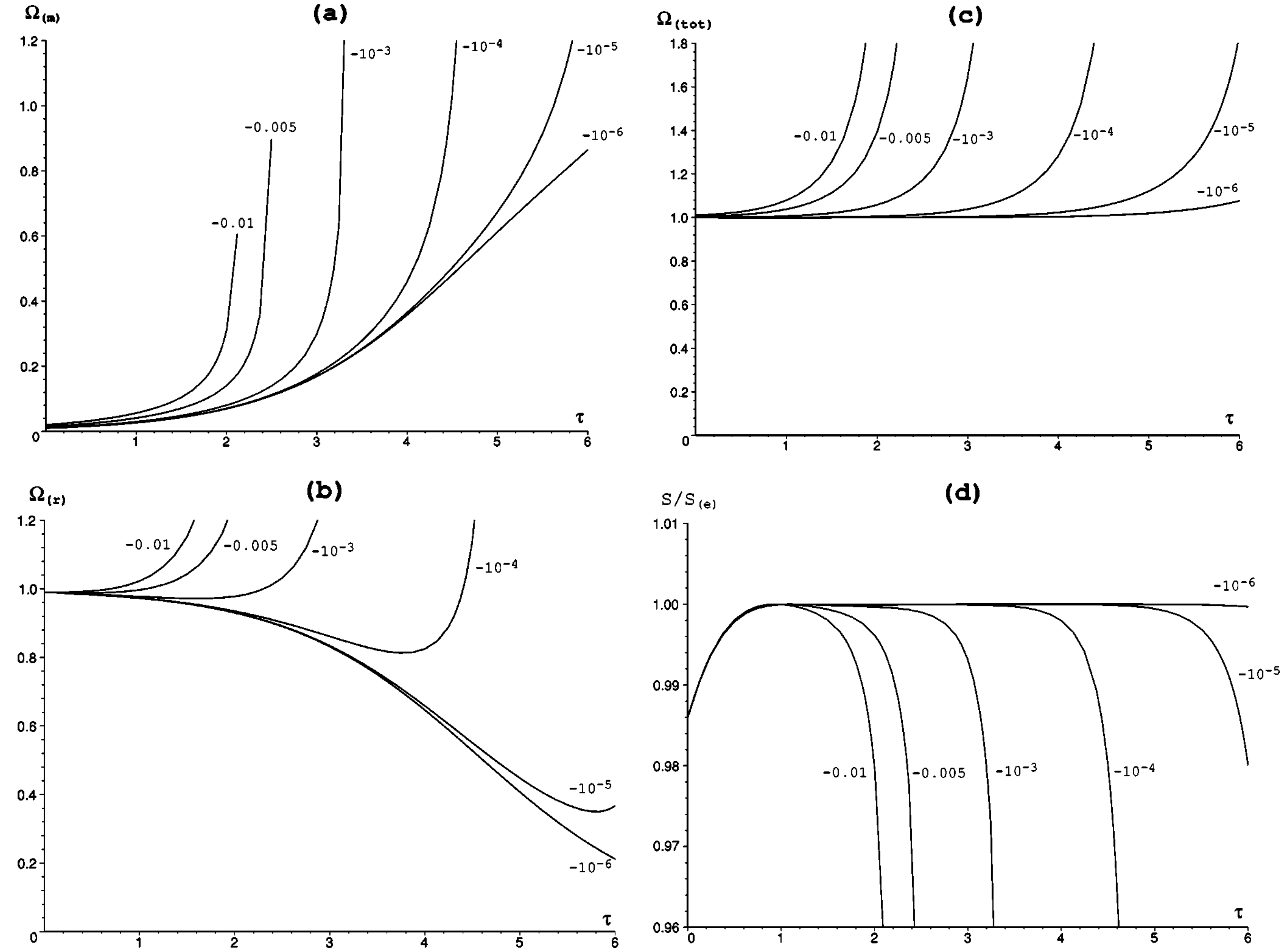

FIG. 3. Sensitivity to $\chi(0)=0$. Positive curvature. These figures display $\Omega_{(m)},\Omega_{(r)},\Omega_{(tot)}=\Omega_{(m)}+\Omega_{(r)}$ and $S/S_{(e)}$ for initial conditions (105) with $-0.01<\delta<0$. The numerical values of δ appear next to each curve. Notice how $\Omega_{(m)},\Omega_{(r)}$, and $\Omega_{(tot)}$ branch upward and $S/S_{(e)}$ downward, with the branching τ smaller for $|\delta|$ larger. This branching up corresponds to $\Theta\to 0$, marking the re-collapsing stage of the models. Comparing (a), (b), and (c) with (d), and with Figs. 2a and 2b, it is evident that a physically plausible evolution for the duration of the radiative era $(0<\tau<6)$ and for the appropriate temperature ranges is only possible for $|\delta|<10^{-5.5}$.

$$\alpha_{\pm}=\frac{1}{2\gamma_0}[\gamma_0-1\pm\sqrt{(\gamma_0-1)^2-4\gamma_0(1+8\gamma_0/5)}]. \tag{91}$$

For physical values $\gamma_0\leqslant 1$, both $\alpha_{\pm}$ have negative real parts and as noted earlier, this FLRW equilibrium point is a saddle (e.g., for $\gamma_0=1/2,\alpha_{\pm}=-1/2\pm i\sqrt{67/20}$).

In the truncated case we have that Eqs. (89) and (90) are satisfied close to the FLRW equilibrium point, but now with

$$\alpha_{\pm}^{\rm tr}=\frac{1}{2\gamma_0}[5\gamma_0-1\pm\sqrt{(5\gamma_0-1)^2+4\gamma_0(12\gamma_0/5-1)}]. \tag{92}$$

We first note that for $\gamma_0\geqslant 1/5$, at least one of the $\alpha^{\rm tr}$ has a positive real part which leads to a change of stability (indeed, for $1/5\leqslant\gamma_0\leqslant 5/12$, this equilibrium point is a source). Physically, this means that in the truncated case Σ and Q in Eq. (90) have a growing mode, and hence their magnitudes increase leading to a breakdown in the physical model (and the time period for which the assumptions are valid). This breakdown is seen in the numerics (Sec. VIII F). As a comparison, for $\gamma_0=1/2$, $\alpha_{\pm}\simeq\frac{3}{2}(1\pm 1.1)$; the growing mode which evolves approximately as $e^{3\tau}$, leads to a rapid increase in the magnitudes of Σ and Q and the models fail after a relatively short cosmological time (see Sec. VIII F).

VIII. NUMERICAL INTEGRATION

From the dynamical analysis carried out in Sec. V, there are no sources (at finite values) in the physical regime; that is, during the time period for which the various physical assumptions used here are valid there are no past attractors. It is reasonable to assume, as conditions prevailing at the beginning of the regime we are interested in, that the universe is approximately isotropic and spatially homogeneous (e.g., almost FLRW with $|Q|\ll 1$ and $\Sigma\ll 1$) and that the radiation component is dominant. These assumptions are consistent with current observations, and we are also assuming that some mechanism (such as, for example, inflation) at early times has driven the universe toward this configuration. Thus the model is close to the particular solution mentioned in (ii) with $(\Omega_{(r)},\Omega_{(m)},\Sigma,Q)$ given by (1,0,0,0). As noted above, this equilibrium point is a saddle; however, it is a ‘‘stronger’’ attractor than other saddles in that it has more positive eigenvalues. Also, this equilibrium point lies in the invariant subspace $\chi=0$, associated with the zero curvature Bianchi I sub-case. Since $(\Omega_{(r)},\Omega_{(m)},\Sigma,Q)=(1,0,0,0)$ is an equilibrium point lying in an invariant set, if a model is driven toward this point in phase space it can stay an arbitrarily long time close to this point (i.e., the universe can spend an extended period close to this FLRW model). The universe will then eventually begin to evolve away from this configuration and from the invariant set as time evolves. Therefore, we shall assume initial conditions for our numerical integration based on the fact that the universe starts evolving from situations close to this equilibrium point.

A. Initial conditions

Since we must have (for physical reasons) $Q<0$ and $Q\Sigma<0$, initial conditions close to $(\Omega_{(r)},\Omega_{(m)},\Sigma,Q)=(1,0,0,0)$ can be given by

$$\Omega_{(r)}(0)=1-\epsilon,\quad \Omega_{(m)}(0)=\epsilon-\delta>0,$$
$$\Sigma(0)=\Sigma_i>0,\quad Q(0)=Q_i<0, \tag{93}$$

where $\epsilon,\delta,\Sigma_i,Q_i$ are real small constants. The initial values of ‘‘total omega’’: $\Omega_{(tot)}=\Omega_{(m)}+\Omega_{(r)}$ and χ given by Eq. (61) are

$$\Omega_{(tot)}(0)=1-\delta,\qquad \chi(0)=\delta-(\Sigma_i)^2, \tag{94}$$

so that the value of δ reflects the deviation of $\Omega_{(tot)}$ from unity, while the deviation from the invariant set $\chi=0$ depends on δ and $(\Sigma_i)^2$.

The factor γ. As discussed earlier, during the interactive range we are interested in, the various time scales must satisfy

$$t_H>\tau_{\rm rel}>t_c>t_\gamma, \tag{95}$$

where $t_H=3/\Theta,\tau_{\rm rel}$ follows from Eq. (73) and t_c,t_γ are given by Eqs. (36) and (37). Considering that $(\Theta_i/3)^2\approx(1/3)\kappa a T_i^4$ and inserting the constants appearing in Eqs. (36) and (37), we arrive at the following initial values:

$$\log_{10}(t_H)|_i\approx 8.8,\quad \log_{10}(t_c)|_i\approx 7.6,$$
$$\log_{10}(t_\gamma)|_i\approx 3.4, \tag{96}$$

thus suggesting the range

$$0.06<\gamma_i<1 \tag{97}$$

for the initial value $\gamma(0)=\gamma_i$.[2] If assuming that $\gamma=\gamma_0$ for all times, then the near-Eckart regime can be associated with $\gamma_0\leqslant O(10^{-2})$, while a transient regime follows by setting $\gamma_0\approx O(1)<1$.

The constants Σ_i *and* Q_i. From the evaluation of Eq. (53) at $\tau=0$, the constant Q_i^2 is proportional to the initial deviation of the photon entropy from its equilibrium value:

$$\frac{15\lambda_0}{8S_{(e)}}Q_i^2\approx Q_i^2\approx 1-\frac{S(0)}{S_{(e)}(0)}, \tag{98}$$

where we have used Eqs. (34) and (35) so that $\lambda_0/S_{(e)}\approx O(1)$, with λ_0 given by Eq. (53). Both numbers Σ_i,Q_i are initial ratios of off-equilibrium and anisotropic variables (Π_{ab},σ_{ab}) with respect to equilibrium and isotropic variables $(p^{(r)},\Theta/3)$. Since we are choosing initial conditions close to a saddle point associated with near FLRW conditions and we must assume near thermal equilibrium, then $\Sigma_i,|Q_i|$ must necessarily be small numbers $(\ll 1)$. A maximal bound on $|Q_i|$ and $|\Sigma_i|$ can be fixed from CMB observations [33], making it reasonable to take

$$|Q_i|<0.01-0.1,\quad |\Sigma_i|<0.001-0.01. \tag{99}$$

However, we will comment further ahead on the sensitivity of the functions to these initial values.

The constants ϵ *and* δ. The values for ϵ and δ are restricted by the ratio of photons to WIMP’s. Considering the neutralino as the WIMP particle with $m\sim 100$ GeV, using the ideal gas law and Eq. (18) yields

$$\frac{\epsilon-\delta}{1-\epsilon}=\frac{\Omega_{(m)}(0)}{\Omega_{(r)}(0)}=\frac{mc^2}{3k_BT_i}\nu_{\rm w}\approx 0.013, \tag{100}$$

leading to the following constraint:

$$\delta\approx 1.013\epsilon-0.013, \tag{101}$$

which must be satisfied by all initial conditions compatible with $T_i\approx 10^6$ K and with the observational constraints $\Omega_{\rm w}\sim 0.3\pm 0.1$ and $h\simeq 0.7$. Further restrictions on ϵ and δ follow by demanding that $\Omega_{(r)}$ decreases and $\Omega_{(m)}$ increases at the initial time $\tau=0$. From the expressions for $\Omega'_{(m)}$ and $\Omega'_{(r)}$ in the differential equations, these conditions imply

$$-1-4\Sigma_i^2<-\epsilon-\delta<-2\Sigma_i|Q_i|-4\Sigma_i^2,$$

so that $\epsilon+\delta>0$ must hold, leading to the following minimal values of ϵ and δ:

$$\epsilon>0.0064,\quad \delta>-0.0065. \tag{102}$$

The condition that $\Omega_{(r)}$ decreases at $\tau=0$, together with $Q_i<0$ and $\Sigma_i>0$, imply

$$0<\Sigma_i<\frac{1}{4}\sqrt{Q_i^2+4(\epsilon+\delta)}-\frac{|Q_i|}{4}. \tag{103}$$

[2] We distinguish between the constant initial value γ_i and the case in which $\gamma=\gamma_0$ for all the evolution period.

B. A physically plausible evolution and the range of validity of the models

It is important to specify a criterion in order to distinguish a physically plausible evolution for the models. We define such an evolution by the following conditions that must hold all along the range of validity discussed previously:

(1) $\Omega_{(m)}$ increases while $\Omega_{(r)}$ decreases. The transition from a radiation- to a matter-dominated epoch occurs within the radiative era. However, the ratio $\Omega_{(m)}/\Omega_{(r)}$ must remain finite in all the validity range.

(2) S must be an increasing and convex function, tending asymptotically to the equilibrium photon entropy given by Eq. (35).

(3) $\dot{S}_{(e)} \propto -\Sigma Q$ must be very small. Ideally, we should have $\dot{S}_{(e)}>0$, though this condition might fail to hold as long as $\dot{S}>0$ holds (see [23] for examples).

(4) Initially, we have Eqs. (95), (96), and (97), but then at later stages t_c (the Compton scattering time scale) is no longer relevant, while t_γ and $\tau_{\rm rel}$ should overtake t_H, so that the baryon-photon decoupling is defined as $t_\gamma = t_H$ and should occur at $\tau=\tau_D$ such that $T(\tau_D)=T_D \simeq 4\times 10^3$ K.

We will not be concerned with the evolution of the models after the radiative era, since the assumptions regarding a hydrodynamic description of the radiative fluid break down. After the baryon-photon decoupling, an appropriate treatment of cosmic matter requires a different theoretical framework based on kinetic theory [34].

Consider a transient regime [$\gamma \approx \gamma_0 \approx O(1) < 1$], together with "test" initial conditions given by Eqs. (93) with $\delta = 0$, satisfying Eqs. (99), (101), (102), and (103), hence $\epsilon = 0.0128$. For the time being, Σ_i will be taken to be two orders of magnitude smaller than Q_i. This yields the following initial conditions lying very near $\chi(0)=0$:

$$\Omega_{(m)}(0)=0.0128,\ \ \Omega_{(r)}(0)=0.9872,\ \ \Sigma(0)=0.001,$$
$$Q(0)=-0.1, \tag{104a}$$

so that

$$\Omega_{(tot)}(0)=1, \quad \chi(0)=-10^{-6}. \tag{104b}$$

In order to illustrate how the different variables should behave in a physical evolution taking place in the appropriate time scale, we integrate the system of governing equations (56), (58), (59), and (69) for Eq. (104) and $\gamma=\gamma_0=0.7$. The results are displayed in Figs. 1 and 2. As shown by Figs. 1(a), 1(b), 1(c) and 1(d), the functions $\Omega_{(m)}, \Omega_{(r)}, \Sigma, Q, S_{(e)}$ and S comply with conditions 1, 2 and 3 of the physical evolution mentioned above (we shall discuss condition 4 in Sec. VIII E). Figure 2a depicts joint logarithmic plots of the various time scales t_γ, t_c, t_H, and physical time t, respectively, given by Eqs. (36), (37), (66) and (67), while the radiation temperature T is displayed in Fig. 2b. By comparing Figs. 1 and 2, it is evident that the range of validity of the models is roughly $0 \leqslant \tau < 6$, corresponding to 10^6 K$>T>10^3$ K, with $t \approx 10^5$ years (the physical time for the radiative era), while the transition from radiation to matter dominance taking place at about $\tau=4$ ($T\approx 10^4$ K). Numerical integration of the governing equations for initial conditions different from Eq. (104) might yield important qualitative changes in the state variables plotted in Fig. 1, like $\Omega_{(m)}, \Omega_{(r)}$ or S, but not of those plotted in Fig. 2, such as T or the time scales (36), (37), (66), $\tau_{\rm rel}$ or physical time t. Therefore, Fig. 2 provides a general estimation of the range of validity of the models for a wide range of initial conditions. We will consider more general initial conditions in the following section.

C. Sensitivity to deviations from $\chi=0$

We will test the effect of initial deviations from the invariant set $\chi=0$ on state functions obtained by numerical integration of the governing equations. We consider initial conditions as in Eq. (93), keeping Σ_i, and Q_i fixed, but now we take $\delta \neq 0$:

$$\Omega_{(m)}(0)=0.0128-\delta,\ \ \Omega_{(r)}(0)=0.9872,$$
$$\Sigma(0)=0.001,\ \ Q(0)=-0.1,$$
$$\Omega_{(tot)}(0)=1-\delta, \quad \chi(0)=\delta-10^{-6}. \tag{105}$$

Since δ can be given in terms of ϵ by Eq. (101) and $\epsilon = 0.0128$ corresponds to $\delta=0$, testing values of ϵ near $\epsilon = 0.0128$ determines the initial deviation from $\Omega_{(tot)}=1$ and $\chi=0$. Notice that δ can be positive or negative, respectively, for $\epsilon>0.0128$ or $\epsilon<0.0128$, so that $\chi(0)=0$ if $\delta=\Sigma_i^2 = 10^{-6}$, while $\chi(0)$ is positive/negative if $\delta>\Sigma_i^2=10^{-6}$ or $\delta<\Sigma_i^2=10^{-6}$ [though, from Eq. (61), curvature has the opposite sign to χ]. We integrate numerically the governing equations for initial conditions (105), $\gamma=\gamma_0=0.7$ (transient regime) and for various values of δ. The resulting forms of $\Omega_{(m)}, \Omega_{(r)}, \Omega_{(tot)}$, and S are, respectively, plotted in Figs. 3a–3d (for positive curvature $\chi<0$) and in Figs. 4a–4d (for negative curvature $\chi>0$). These figures clearly show that a physical evolution is only possible for initial conditions that deviate very slightly from $\chi(0)=0$ (less than $\approx 10^{-5.5}$), leading to orbits that remain very close to the invariant set $\chi=0$. If we fix ϵ and δ (for any combination of values compatible with Eqs. (101), (102) and (103)) and vary Σ_i, so that deviation from $\chi(0)=0$ is governed by Σ_i, we obtain exactly the same behavior displayed by Figs. 3 and 4, leading to the same conclusion: a physical evolution is only possible for initial conditions for which $|\chi(0)| \lesssim 10^{-5.5}$.

D. The near-Eckart regime

We examine the near-Eckart regime by assuming $\gamma=\gamma_0 = 0.001 \ll 1$, together with initial conditions (104). As shown in Fig. 5a, the functions $Q(\tau)$ and $S(\tau)$ clearly have the form (76), indicating a quick relaxation in terms of an abrupt exponential decay (in about $\tau\approx 0.01$). Figure 5b shows how the equilibrium entropy $S_{(e)}$ tends to a constant value in about $\tau \approx 0.02$, a longer time than the relaxation of $S/S_{(e)}$, thus indicating that the effective relaxation time is provided by $S_{(e)}$, and $S'_{(e)}>0$ holds for all the evolution, in agreement

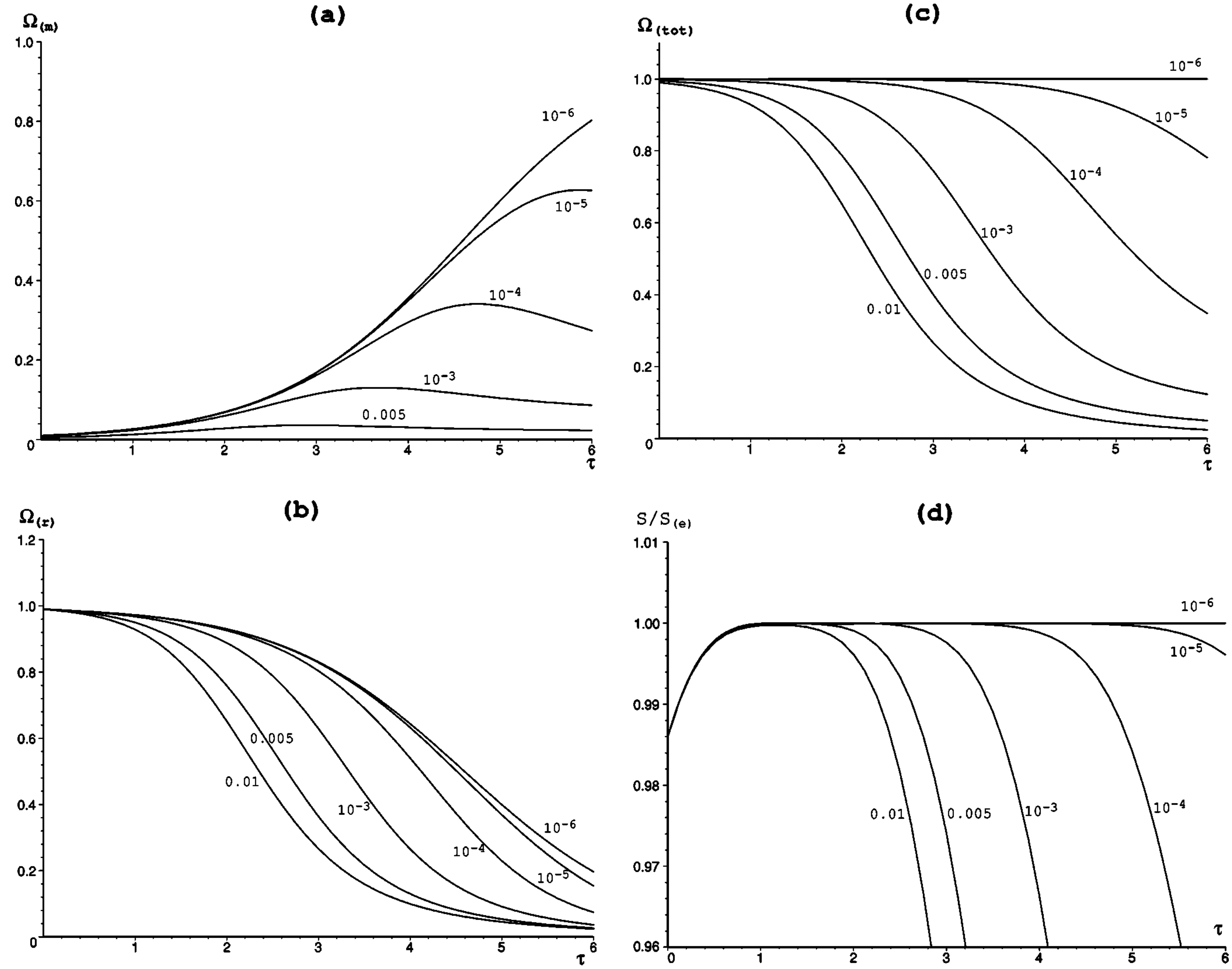

FIG. 4. Sensitivity to $\chi(0)=0$, negative curvature. Analogues of Fig. 3 for $\delta>0$. A similar branching of the functions $\Omega_{(m)},\Omega_{(r)},\Omega_{(tot)},S/S_{(e)}$ is observed even though the models do not re-collapse. Again, the function $S/S_{(e)}$ satisfies the conditions for a physically plausible evolution only if initial conditions are given by $\delta<10^{-5.5}$. Notice that the $\Omega_{(m)}(\tau)$ curves in (a) with $\delta\approx10^{-5.5}$ bend downward, so that for $\tau\gg6$ they tend to the currently accepted values of present CDM abundance: $\Omega_{(m)}\approx0.3$.

with the "local equilibrium hypothesis" (see Sec. VI A). Figure 5c reveals how $(5/8)Q/\Sigma\to\gamma_0=0.001$, so that Eckart's transport equation $[P+2\eta\sigma=0$ with η given by Eq. (29)$]$ is approximately valid once the quick relaxation is over. A good approximation to the relaxation time in a near-Eckart regime is given by the assumption (73) with $\gamma=-(5/8)\gamma_0(Q/\Sigma)$ and $\gamma_0=0.001$. Figure 5d illustrates that this is a correct assumption, since the obtained $\tau_{\rm rel}$ overtakes t_H in the very short period that coincides with the duration of the relaxation process ($\tau\approx0.02$).

The plots of the functions $\Omega_{(m)}(\tau)$ and $\Omega_{(r)}(\tau)$ are identical to those that would have resulted in the transient regime had we chosen initial conditions (104) and a much larger value of γ_0. The function $\Sigma(\tau)$ is affected, becoming almost constant for very small γ_0. This is reasonable, since Eq. (56) does not contain Q, while Eqs. (58) and (59) do contain this function, but $|Q(\tau)|$ and Q' decay very fast becoming almost zero for most of the time range and so the differential equations for the functions $\Omega_{(m)}$ and $\Omega_{(r)}$ (but not Σ) are practically unaffected. The effect of varying γ_0 (as long as we have small values, <0.01) is simply to make the decay of Q and $S'_{(e)}$ slightly more or less abrupt (depending on whether γ_0 is smaller or larger than 0.001) and has no noticeable effect on numerical curves of other functions.

E. Testing the relaxation times numerically

The assumption $\gamma=\gamma_0<1$ leads to a reasonable relaxation time only in the earlier stages of the evolution, while a choice $\gamma=\gamma_0>1$ might work for later conditions (near matter-radiation decoupling) but not for earlier times. This can be appreciated in Fig. 6b, since the relaxation times obtained for different values of $\gamma=\gamma_0$ would be curves parallel to t_H. As we mentioned before, γ_0 controls the rate of decay of Q and S, but it is still interesting to check if other functions are sensitive to changes in the numerical constant value of γ_0. Assuming initial conditions (104) and integrating the dynamical system for various values of $\gamma_0 > 0.1$ (thus excluding the non-transient zone) shows that the other functions (such as $\Omega_{(m)}, \Omega_{(r)}$ or Σ) are essentially identical to those shown in Figs 1 to 4, being thus unaffected by the numerical value of γ_0.

Since γ constant is not very realistic, we test now the expression for $\tau_{\rm rel}$ that follows from Eqs. (82) and (83) in Sec. VI B. Ignoring the curvature term, we have

$$\gamma = \frac{\mu_0}{8/5 + \mu_0(1+\mu_0) + \mu_0(1-2\mu_0)\Omega_{(m)}/2 - \mu_0[1+(\lambda-3)\mu_0]\Sigma^2}. \tag{106}$$

It is evident that choosing a sufficiently small μ_0 yields $\gamma_i \approx 5\mu_0/8 \ll 1$, an initial value characteristic of a near-Eckart regime. As expected, numerical tests with $\mu_0 \ll 1$ lead to practically identical curves as those corresponding to the near-Eckart regime (Figs. 5a, 5b and 5c). However, for $\mu_0 > 1/2$, the factor multiplying $\Omega_{(m)}$ in the denominator of Eq. (106) can become negative for some τ, thus opening the possibility that the denominator might become small and so γ might increase to $\approx O(1)$ as τ reaches later times related to the baryon-photon decoupling era. In order to explore this possibility, we integrated the dynamical system for initial conditions (104) and under the assumption of γ given by Eq. (106), with various values of $\mu_0 > 1/2$. As shown by Fig. 6a, the choice $\mu_0 > 15$ leads to γ diverging as (approximately) $\tau \to 7$, this leads to $\tau_{\rm rel}$ overtaking t_H (Fig. 6b) around $\tau \approx 6.8$. It would have been nicer to have $\tau_{\rm rel}$ overtaking t_H at an earlier time (say $\tau \approx 5-5.5$) as required by condition 4 for a physically plausible evolution, but we feel that the form of $\tau_{\rm rel}$ associated with Eq. (106) is a reasonable approximation to a physical relaxation time that acts as an "effective" relaxation time for the radiative era. Finally, another consequence of dealing with a more reasonable form for $\tau_{\rm rel}$ is the fact that $\dot{S}_{(e)} \propto -\Sigma Q$ being positive and very small (condition 3 of a physically plausible evolution) is better satisfied than in the case of constant γ_0 (compare Figs. 1b and 6c).

F. The truncated equation

Considering initial conditions (104) and $\gamma = \gamma_0 = 0.7$, the integration of the governing equations (56), (58), (59), and (86) yield the curves for $Q, \Sigma, \Omega_{(m)}$, and $\Omega_{(r)}$ depicted in Figs. 7a, 7b and 7c. These figures reveal that a physical evolution fails to occur, as the growing modes predicted by the qualitative analysis clearly emerge, making all these functions to undertake an unphysical growth. We tested other values of γ_0 and γ given by Eq. (106), as well as other initial conditions and obtained very similar curves to those of Figs. 7a–7c, all failing to comply with the criteria for a physically plausible evolution.

G. A baryonic scenario without CDM

If we had considered only the radiative fluid (baryons, photons and electrons) as the matter source of the models, then the bulk of the rest mass energy density would have been due to the baryons, so that $n^{(m)}$ would have been identified with $n_{\rm b}$ instead of $n_{\rm w}$ and $\Omega_{(m)}$ would correspond to the baryonic $\Omega_{\rm b}$. It is interesting to examine numerically the consequences of this "baryonic scenario." Considering the baryon-photon number ratio in Eq. (15), the baryonic scenario implies replacing Eqs. (100) and (101) by

$$\frac{\epsilon-\delta}{1-\epsilon} = \frac{m_{\rm b}c^2}{3k_B T_i}\nu_{\rm b} \approx 0.002, \quad \delta = 1.002\epsilon - 0.002, \tag{107}$$

while initial conditions are then given by Eqs. (104) and (105), but $\delta = 0$ now corresponds to $\epsilon = 0.00199$ instead of $\epsilon = 0.0128$. Intuitively, we do not expect a major qualitative change in the resulting graphs, though it is reasonable to expect that $\Omega_{(m)}$ will be smaller and $\Omega_{(r)}$ larger, since baryons have less rest mass density (by one or two orders of magnitude) than WIMP's and so it should take longer for baryons to dominate over radiation. The numerical curves that result are as expected intuitively, with $\Omega_{(m)}, \Omega_{(tot)}$, and $S/S_{(e)}$ having very similar forms as the curves of Figs. 3 and 4, with $\Omega_{(r)}$ decreasing slightly slower than in the case with WIMP's. Since the obtained curves for $\Omega_{(m)}$ in the baryonic scenario are so close to those obtained in Figs. 3 and 4 for the case with WIMP's, these curves yield baryon abundances that are clearly incompatible with the bounds placed by cosmic nucleosynthesis ($\Omega_{\rm b} \sim 10^{-2}$).

IX. DISCUSSION AND CONCLUSION

We have studied a class of dissipative Kantowski-Sachs models describing the cosmological evolution during the radiative era characterized by radiative processes involving baryons, electrons and photons, considered as a single dissipative "radiative" fluid. We also assumed the presence of CDM, in the form of a non-relativistic gas of WIMP's (lightest neutralinos). Although this gas does not interact with the radiative fluid, it provides the bulk of the rest mass energy density and thus it strongly influences the dynamics of the models and the resulting values of cosmological time scales, such as t_H. On the other hand, the radiative fluid provides the bulk of thermal and dissipative effects, related to the rate of change and relaxation of the radiation entropy to its equilibrium value.

After defining new normalized variables $\Omega_{(m)}, \Omega_{(r)}, \Sigma, Q$, a set of evolution equations has been derived based upon

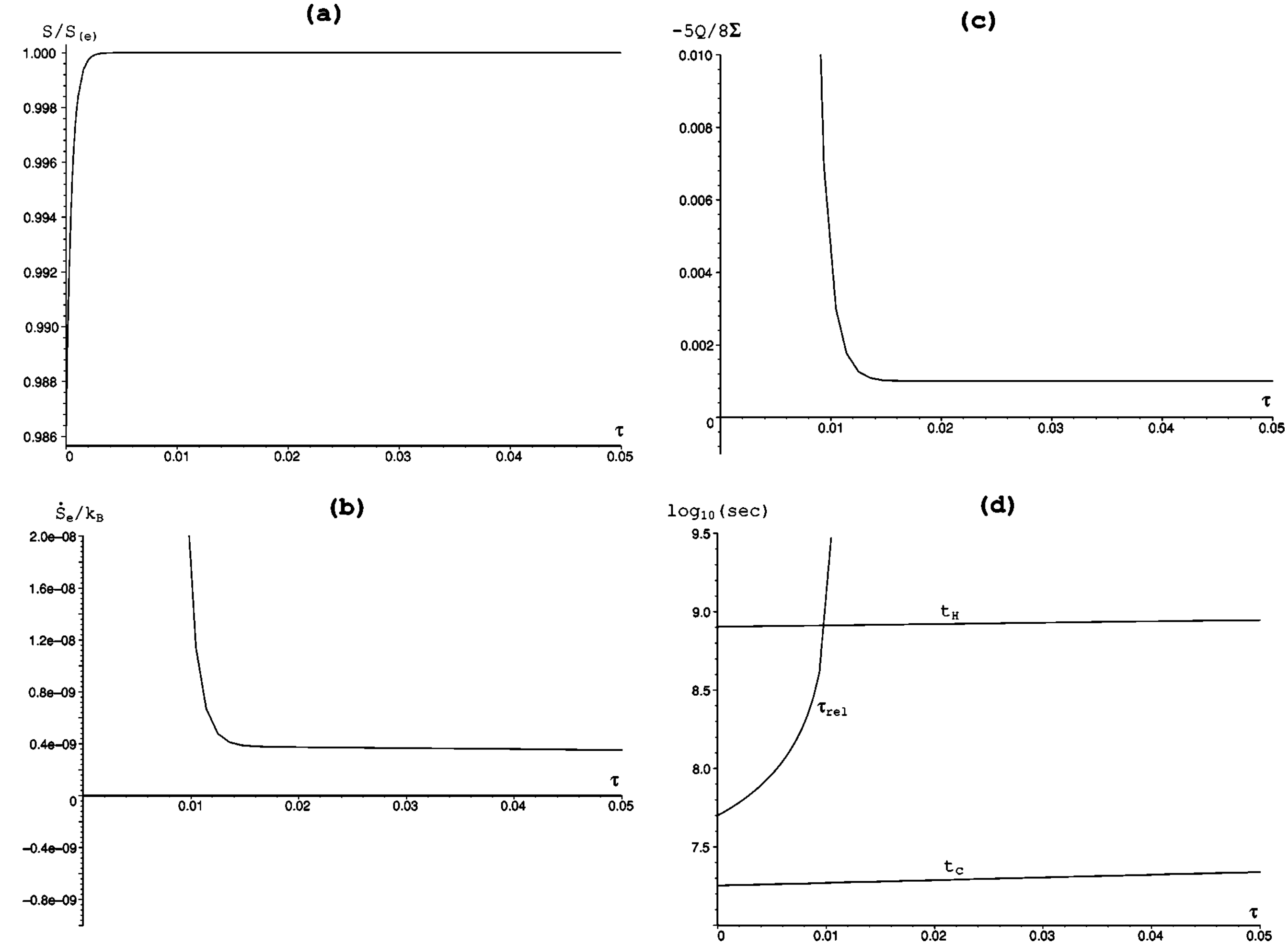

FIG. 5. "Near-Eckart" regime. Functions $S/S_{(e)}$, $\dot{S}_{(e)}/k_B$, and $-(5/8)(Q/\Sigma)$ correspond to initial conditions (104) with $\gamma_0 = 0.001$.$S/S_{(e)}$ in (a) relaxes much faster ($\tau \approx 0.002$) than in Figs. 1d, 3d or 4d, associated with the transient regime. The decay of $S_{(e)}$ in (b) takes longer ($\tau \approx 0.01$) than that of S, hence $S_{(e)}$ is an adequate entropy function in agreement with the "local equilibrium" hypothesis characteristic of the Eckart regime ($\dot{S}_{(e)} > 0$ holds throughout the evolution). (c) shows how $-(5/8)(Q/\Sigma) \to \gamma_0 = 0.001$, indicating that Eckart's transport equation is approximately valid for $\tau > 0.02$. In (d) we used initial conditions (102) and $\gamma = -(5/8)(Q/\Sigma)\gamma_0$ with $\gamma_0 = 0.001$ (instead of $\gamma = \gamma_0 = 0.7$), which yields an excellent approximation to the relaxation parameter of a near-Eckart regime, overtaking t_H at the relaxation time scale $\tau \approx 0.01$; this time scale, however, is too short in comparison with the relaxation time scale of the Compton scattering also shown in (d), hence this and other radiative processes must be studied within a transient regime.

appropriate thermodynamical laws and equations of state. The qualitative and numerical study of these evolution equations has clarified various aspects of the dynamical behavior of the models, their physical viability, as well as a peculiar sensitivity to certain initial conditions related to deviation from the invariant set $\chi = 0$. We discuss below the main features emerging from previous sections.

The definition of the phase space variables $\Omega_{(r)}, \Omega_{(m)}, \Sigma, Q$ leads in a natural way to expressing the relaxation time $\tau_{\rm rel}$ as proportional to $t_H = 3/\Theta$ [see Eqs. (68) and (73)]. The understanding of the relaxation process can be accomplished by studying the effect of different choices of the proportionality factor, $\gamma(\tau) > 0$, on the exponential decay of the dissipative stress (related to Q) and of the photon entropy S to its equilibrium value $S_{(e)}$. We have identified a "near-Eckart" regime if this decay is abrupt ($\gamma \ll 1$, and becoming instantaneous in the limit $\gamma \to 0$, so that $\tau_{\rm rel} \to 0$), while a "transient regime" [$\gamma \approx O(1) < 1$] can be associated with a slower decay. Both the near-Eckart and the transient regimes are compatible with a physically plausible evolution. The difference between the two regimes is the time scale of their relaxation process: for the transient regime this time scale can be comparable with the duration of the radiative era, for the near-Eckart regime it is much shorter (about eight orders of magnitude in physical time). This is well illustrated by the differences in evolution time scales between Fig. 5 and Figs. 1–4. Comparing Figs. 2b and 5d, it is evident that the relaxation time scale of the near-Eckart regime ($\approx 10^8$ sec) is much shorter than that of the Compton scattering ($\approx 10^{11}$ sec). Therefore the near-Eckart regime yields a relaxation that is too swift and so it is inadequate to examine the two main radiative processes of the radiative era: the Compton scattering and (more so) the Thomson scattering. It is important to mention this fact in view of recent claims [35]

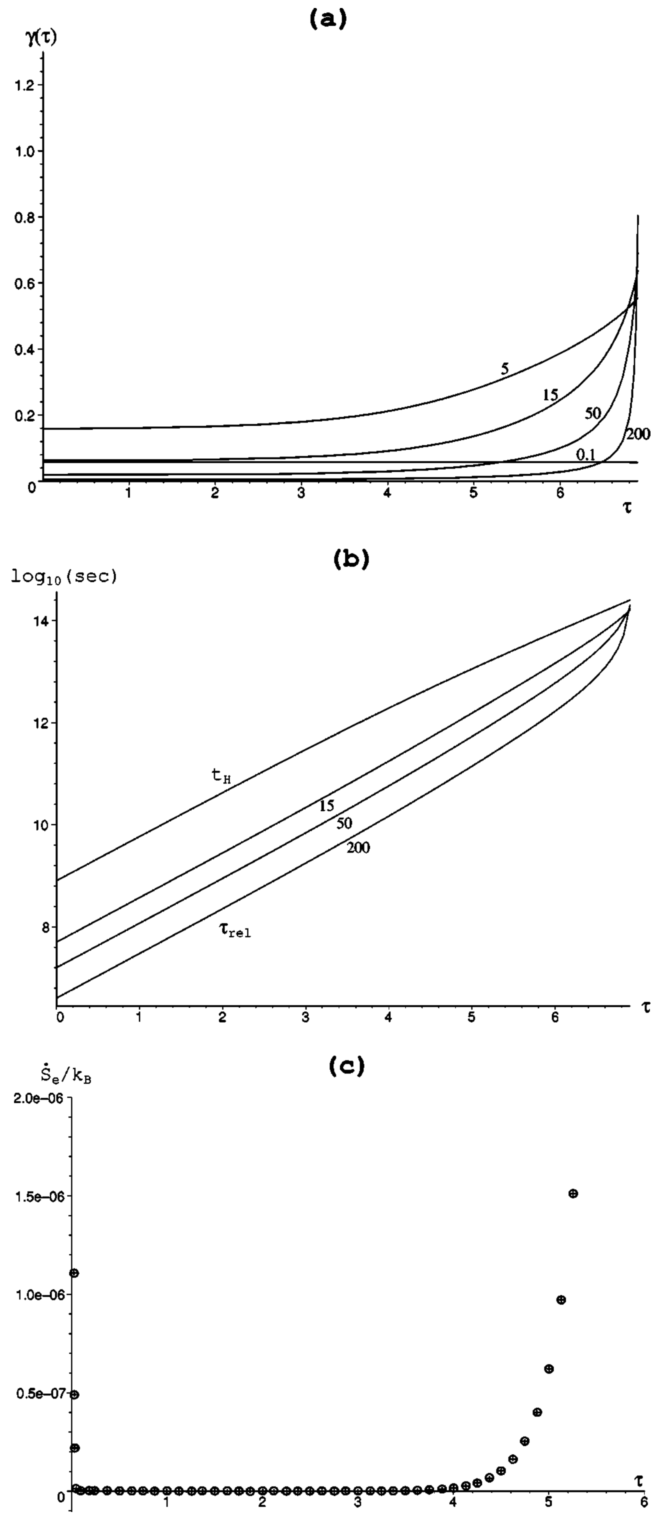

FIG. 6. Dynamical relaxation time. (a) displays γ obtained from integrating the governing equations for the form given by Eq. (106), the numbers next to each curve correspond to the chosen numerical values of μ_0; γ diverges for $\mu_0>15$. (b) displays the corresponding form for $\tau_{\rm rel}$ which overtakes t_H for $\tau\approx6.8$. (c) shows $\dot{S}_{(e)}/k_B$ for $\mu_0=200$. A comparison with Fig. 1b shows that requiring $\dot{S}_{(e)}/k_B$ to be small is better satisfied with γ given by Eq. (104) than with a constant value $\gamma=0.7$ (as in Fig. 1b).

that a transient theory of irreversible thermodynamics is not really necessary (see [36] for a comprehensive discussion). It is evident that the cosmological study of radiative processes in pre-decoupling times needs to be accomplished with a transient regime.

As revealed by Figs. 1–4, a physically plausible evolution is possible for all the duration of the radiative era for initial conditions given by an initial state very close to $\chi(0)=0$, hence lying very close to the invariant set of zero curvature $\chi=0$. All models complying with a physical evolution begin their evolution near the equilibrium point $(\Omega_{(r)},\Omega_{(m)},\Sigma,Q)=(1,0,0,0)$, a saddle with positive eigenvalues (i.e., stable) associated with the FLRW sub-case of the models, and the proceeding evolution remains very close to the invariant set $\chi=0$. This is an extremely interesting feature of these models, as it relates a geometric property of KS solutions with the constraints imposed by the physics and by observational evidence, since recent data from CMB observations indicates $\Omega_{(tot)}\approx1$. By looking at the curves of $\Omega_{(m)}$ and $\Omega_{(tot)}$ with $\delta\lesssim10^{-5}$ in Figs. 4a and 4c (negative curvature), it is evident that for $\tau>6$ these curves decrease from their values $\Omega_{(m)}\sim1$ and $\Omega_{(tot)}\sim1$ around $\tau=6$. Had we plotted these curves for larger values of τ, extending to the present era ($\tau\approx15$), we would have obtained $\Omega_{(m)}\sim\Omega_{(tot)}\sim0.3$, in agreement with the currently accepted value of $\Omega_{(m)}$, but not of $\Omega_{(tot)}$. Of course, the estimated contribution to $\Omega_{(tot)}$, today, for non-relativistic matter (CDM plus baryons) is only ≈0.3, with the remaining two-thirds of the critical density probably related to a Λ-type "dark energy" interaction whose precise nature and properties are still uncertain. However, this discrepancy with regards to $\Omega_{(tot)}$ today is not surprising since we did not consider any Λ-type interaction, and so it does not affect our results. Also, the models we are considering are only valid for a specific range of cosmological times: $10^3\lesssim z\lesssim10^6$, in which this "dark energy" would likely not have been dominant. Still, the close link between a physically plausible evolution and $\Omega_{(tot)}$ near unity is remarkable.

It is interesting to compare our results to those reported previously [37] dealing with the perfect fluid sub-case of the KS models examined in this paper (though these models did not consider a CDM component). As reported in [37], there are numerical solutions in which both matter and radiation normalized densities, $\Omega_{(m)},\Omega_{(r)}$, decay to zero as the models re-collapse and approach a crunch singularity. By looking at the forms of $\Omega_{(m)}$ and $\Omega_{(r)}$ in Fig. 4, it is evident that such evolution is similar to that depicted by curves associated with initial conditions $\delta\gtrsim10^{-3}$. However, the evolution that results from these initial conditions fails to comply with our physical criteria, since the entropy S is no longer a convex function for all of the time range (see Fig. 4d) and starts decreasing at too early times. In the perfect fluid case, these examples satisfy an appropriate equation of state and all of the energy conditions and also the photon entropy is simply $S_{(e)}$ and is constant for all times, hence there is no physical reason to discard these curves (other than remarking that such behavior of $\Omega_{(m)},\Omega_{(r)}$ is not observed in the real universe). However, for the dissipative source under examination here, S and $S_{(e)}$ are not constant and the conditions for a

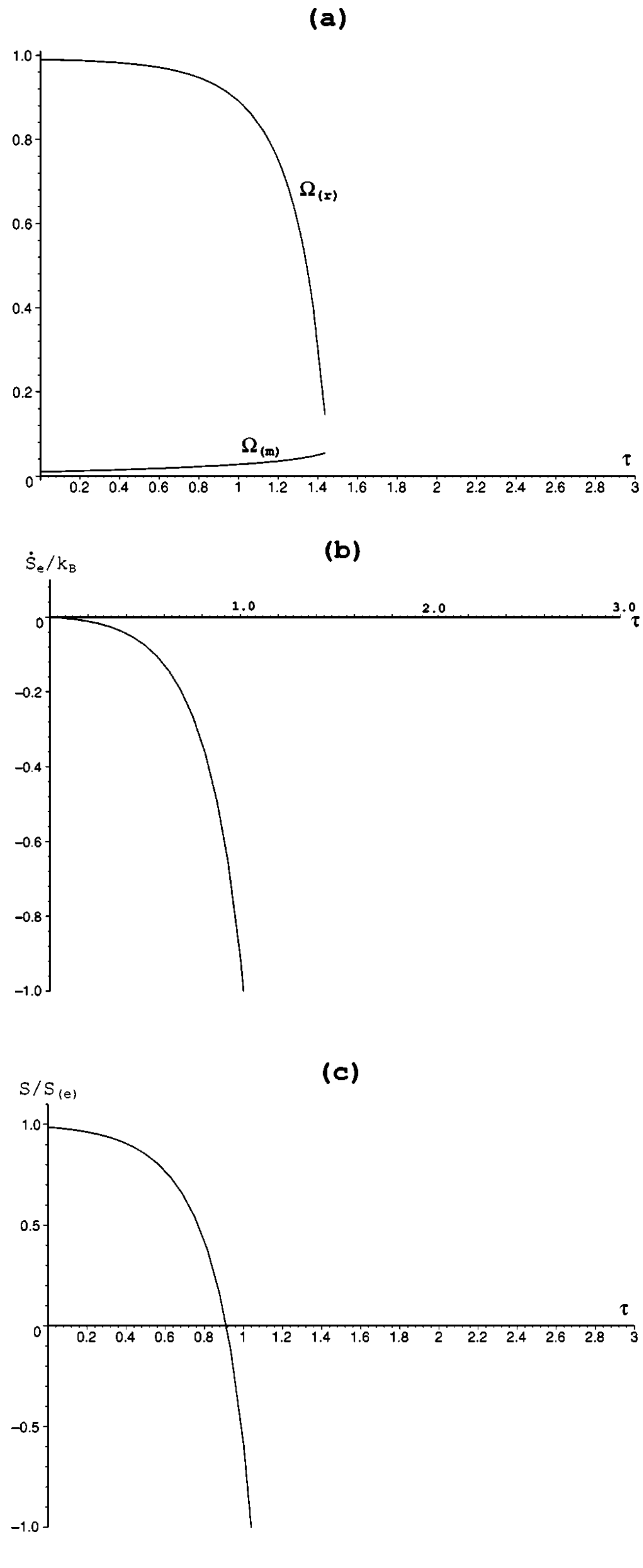

FIG. 7. The truncated transport equation. Curves obtained by integrating the governing equations with initial conditions (104) and $\gamma_0 = 0.7$, but using Eq. (84) (''truncated'' transport equation) instead of Eq. (57) (full transport equation). The exponential growth terms mentioned in Sec. VII lead to an unphysical evolution characterized by negative $\dot{S}_{(e)}$ and $\dot{S}$ during all the evolution range.

physical evolution are more stringent, hence we can apply clear physical criteria to discard these perfect fluid ''strange'' cases.

For initial conditions near the equilibrium point $(\Omega_{(r)},\Omega_{(m)},\Sigma,Q)=(1,0,0,0)$ [as in Eq. (104)], the numerical curves of the ''equilibrium variables'' $\Omega_{(m)}$ and $\Omega_{(r)}$ are not affected by the choice of $\gamma=\gamma_0$, the constant proportionality factor between $\tau_{\rm rel}$ and t_H. This is a consequence of the fact that for these initial conditions the values of Σ, Q and $S-S_{(e)}$ remain small during all of the evolution. Therefore, the evolution equations for $\Omega_{(m)}$ and $\Omega_{(r)}$ [Eqs. (56) and (58)] are practically unaffected by the presence of Σ and Q, and so are insensitive to the rate of transiency given by γ constant. Since Σ and Q govern the deviation from the FLRW equilibrium point, this decoupling of $\Omega_{(m)}$ and $\Omega_{(r)}$ from Σ and Q is then a consequence of the evolution of the system always remaining close to thermal equilibrium. We tested this behavior for a particular form of γ [Eq. (106), Sec. VI B]: the curves for $\Omega_{(m)},\Omega_{(r)},\Sigma$, and Q are practically identical with those that follow from choosing $\gamma=0.7$ in Figs. 1 to 4. However, as shown in Figs. 6a and 6b, for large values of μ_0 defined by Eq. (83), the obtained relaxation time $\tau_{\rm rel}$ behaves similarly to what one would expect of a relaxation parameter for the radiative era. Although it has become common practice to simply equate $\tau_{\rm rel}$ with a microscopic interaction time, like t_γ or t_c, the relaxation time is not a microscopic but a mesoscopic or even macroscopic quantity (though it must be qualitatively analogous to interaction times [36]). Since it can be extremely cumbersome to evaluate $\tau_{\rm rel}$, it is useful to have a concrete example where this relaxation parameter can be adequately approximated by the same dynamical equations associated with the models.

Finally, by means of qualitative arguments supported by the numerical analysis, we have shown in Sec. VII (Fig. 7), that the truncated equation does not comply with a physically plausible evolution. This is an important result, since we have found a concrete example in which a truncated transport equation leads to unphysical evolution of dissipative fluxes. Although this conclusions strictly applies to the KS models under consideration, we must point out that one should be very cautious when applying these equations to other models and other equations of state.

A possible extension of this work would be to consider instead of CDM other forms of dark matter, such as ''warm'' dark matter (WDM) or axions. Another possibility is to include, together with dark matter, a scalar field associated with ''dark energy.'' Another route to generalize the present work is use a class of metrics associated with a geometry that is less restrictive than KS, for example the non-static spherical symmetry (perhaps under the assumption of self-similarity). We regard the present analysis of the Kantowski-Sachs models as a first step toward an understanding of the dynamics of cosmic matter in more general and physically motivated inhomogeneous models.

ACKNOWLEDGMENTS

This work has been partially supported by the Universidad Nacional Autónoma de México (UNAM) under grant DGAPA-IN-122498.

[1] E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley, Reading, MA, 1990).
[2] T. Padmanabhan, *Structure Formation in the Universe* (Cambridge University Press, Cambridge, UK, 1995).
[3] J.A. Peacock, *Cosmological Physics* (Cambridge University Press, Cambridge, UK, 1999).
[4] J. Bernstein, *Kinetic Theory in the Expanding Universe* (Cambridge University Press, Cambridge, UK, 1988).
[5] G. Efstathiou, in *Physics of the Early Universe*, Proceedings of the Thirty-Sixth Scottish Universities Summer School in Physics, edited by J.A. Peacock, A.F. Heavens, and A.T. Davies (IOP, Bristol, UK, 1990).
[6] A.A. Coley and B.O.J. Tupper, J. Math. Phys. **27**, 406 (1986).
[7] M.D. Pollock and N. Caderni, Mon. Not. R. Astron. Soc. **190**, 509 (1980); J.A.S. Lima and J. Tiomno, Gen. Relativ. Gravit. **20**, 1019 (1988); R.A. Sussman, Class. Quantum Grav. **9**, 1891 (1992).
[8] V. Méndez and D. Pavón, Mon. Not. R. Astron. Soc. **782**, 753 (1996).
[9] S. Weinberg, Astrophys. J. **168**, 175 (1971).
[10] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
[11] P.J.E. Peebles, *The Large Scale Structure of the Universe* (Princeton University Press, Princeton, NJ, 1980).
[12] R.A. Sussman and J. Triginer, Class. Quantum Grav. **16**, 167 (1999).
[13] D. Kramer, H. Stephani, M.A.H. MacCallum, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, UK, 1980).
[14] A. Krasiński, *Inhomogeneous Cosmological Models* (Cambridge University Press, Cambridge, UK, 1997).
[15] John Ellis, Summary of DARK 2002: 4th International Heidelberg Conference on Dark Matter in Astro and Particle Physics, Cape Town, South Africa, 2002, astro-ph/0204059.
[16] G. Jungman, M. Kamionkowski, and K. Griest, Phys. Rep. **267**, 195 (1996).
[17] S. Khalil, C. Muñoz, and E. Torrente-Lujan, New J. Phys. **4**, 27 (2002).
[18] W.A. Hiscock and L. Lindblom, Phys. Rev. D **31**, 725 (1985).
[19] W.A. Hiscock and L. Lindblom, Phys. Rev. D **35**, 3723 (1987).
[20] W. Israel, Ann. Phys. (N.Y.) **100**, 310 (1976).
[21] W. Israel and J. Stewart, Ann. Phys. (N.Y.) **118**, 341 (1979).
[22] D. Pavón, D. Jou, and J. Casas-Vázquez, Ann. Inst. Henri Poincare, Sect. A **36**, 79 (1982).
[23] D. Jou, J. Casas-Vázquez, and G. Lebon, *Extended Irreversible Thermodynamics*, 2nd ed. (Springer, Berlin, 1996).
[24] W.A. Hiscock and L. Lindblom, Contemp. Math. **71**, 181 (1991).
[25] N. Udey and W. Israel, Mon. Not. R. Astron. Soc. **199**, 1137 (1982).
[26] D. Pavón, D. Jou, and J. Casas-Vázquez, J. Phys. A **16**, 775 (1983).
[27] D. Jou and D. Pavón, Astrophys. J. **291**, 447 (1985).
[28] P. T. Landsberg and D. Evans, *Mathematical Cosmology* (Oxford University Press, Oxford, UK, 1979).
[29] R. Maartens and J. Triginer, Phys. Rev. D **56**, 4640 (1997).
[30] W. Zimdahl, Mon. Not. R. Astron. Soc. **280**, 1239 (1996).
[31] R.A. Sussman and D. Pavón, Phys. Rev. D **60**, 104023 (1999).
[32] A. Coley and M. Goliath, Phys. Rev. D **62**, 043526 (2000).
[33] R. Maartens, G.F.R. Ellis, and W.R. Stoeger, Phys. Rev. D **51**, 1525 (1995).
[34] G.F.R. Ellis and H. van Helst, "Cosmological Models," Cargèse Lectures 1998, gr-qc/9812046.
[35] R. Geroch, "On hyperbolic 'theories' of relativistic dissipative fluids," gr-qc/0103112.
[36] L. Herrera and D. Pavón, Physica A **307**, 121 (2002).
[37] E. Weber, J. Math. Phys. **27**, 1578 (1986).