THE ENERGY-DENSITY SPECTRUM OF VACUUM IN PRISMATIC CAVITIES

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We calculate the explicit form of the energy-density spectrum of scalar vacuum as seen by an observer either at rest or uniformly accelerating, inside a prismatic cavity.

The energy spectrum of the Casimir effect between two parallel planes has been calculated recently by several authors [1], both for inertial and accelerated frames [2]. It is the aim of this work to further explore the energy spectrum of vacuum produced by boundary conditions. In particular, we investigate the case of a prismatic cavity using a previously designed formalism [3] which, besides simplifying both the calculations and the physical interpretation of the results, does not arbitrarily cut off the zero-point energy. This article is restricted to scalar fields only, the analysis of the electromagnetic field will be carried out separately [4].

We consider an observer either at rest or uniformly accelerating inside a prismatic cavity. The system of units used in this work is such that $c=\hbar=1$.

A. Observer at rest. The Wightman functions for a scalar field inside a prismatic cavity with boundaries at $x_2=0$, b and $x_3=0$, a are found using the method of images in a straightforward generalization of the case with two parallel plates [5]. Thus

$$D^{\pm}(x,x')=\frac{1}{4\pi^2}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left(\frac{1}{(x_1-x'_1)^2+(x_2-x'_2-2bm)^2+(x_3-x'_3-2an)^2-(t-t'\mp i\varepsilon)^2}\right.$$
$$-\frac{1}{(x_1-x'_1)^2+(x_2+x'_2-2bm)^2+(x_3-x'_3-2an)^2-(t-t'\mp i\varepsilon)^2}$$
$$-\frac{1}{(x_1-x'_1)^2+(x_2-x'_2-2bm)^2+(x_3+x'_3-2an)^2-(t-t'\mp i\varepsilon)^2}\tag{1}$$
$$\left.+\frac{1}{(x_1-x'_1)^2+(x_2+x'_2-2bm)^2+(x_3+x'_3-2an)^2-(t-t'\mp i\varepsilon)^2}\right).$$

For an observer at rest one can choose x and x' as $(x_0=\tau-\sigma/2,\ x_1,\ x_2=y,\ x_3=z)$ and $(x'_0=\tau+\sigma/2,\ x_1,\ x'_2=y,\ x'_3=z)$, respectively. The Wightman functions then take the form

$$D^{\pm}(\sigma)=-\frac{1}{4\pi^2}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left(\frac{1}{(\sigma\mp i\varepsilon)^2-4a^2n^2-4b^2m^2}-\frac{1}{(\sigma\mp i\varepsilon)^2-4a^2n^2-4(y-bm)^2}\right.$$
$$\left.-\frac{1}{(\sigma\mp i\varepsilon)^2-4b^2m^2-4(z-an)^2}+\frac{1}{(\sigma\mp i\varepsilon)^2-4(y-bm)^2-4(z-an)^2}\right).\tag{2}$$

The corresponding Fourier transform is

$$\tilde{D}^{\pm}(\omega) = -\frac{1}{4\pi^2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (I_1^{\pm} - I_2^{\pm} - I_3^{\pm} + I_4^{\pm}) , \tag{3}$$

where

$$I_i^{\pm} = \int_{-\infty}^{\infty} \frac{e^{i\omega\sigma} d\sigma}{(\sigma \mp i\varepsilon)^2 - B_i^2}$$

and $B_1^2 = 4a^2n^2 + 4b^2m^2$, $B_2^2 = 4a^2n^2 + 4(y-bm)^2$, $B_3^2 = 4b^2m^2 + 4(z-an)^2$, $B_4^2 = 4(y-bm)^2 + 4(z-an)^2$. As shown in the appendix, the functions I_2, I_3 and I_4 integrated over y and z do not contribute to the energy-density per unit length inside the cavity. As for I_1 it is given by, for $\omega > 0$,

$$I_1^{+} = \int_{-\infty}^{\infty} \frac{e^{i\omega\sigma} d\sigma}{(\sigma - i\varepsilon)^2 - B_1^2} = -\frac{2\pi}{B_1} \sin(\omega B_1) ,$$

$$I_1^{-} = \int_{-\infty}^{\infty} \frac{e^{i\omega\sigma} d\sigma}{(\sigma - i\varepsilon)^2 - B_1^2} = 0 .$$

Thus one gets for the energy density per unit length

$$\frac{dE}{d\omega} = \frac{\omega^3}{2\pi^2} ab \left(1 + \frac{1}{\omega a} \sum_{n=1}^{\infty} \frac{\sin(2\omega an)}{n} + \frac{1}{\omega b} \sum_{m=1}^{\infty} \frac{\sin(2\omega bm)}{m} + \frac{2}{\omega} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin[2\omega(a^2n^2+b^2m^2)^{1/2}]}{(a^2n^2+b^2m^2)^{1/2}}\right) . \tag{4}$$

Since (see ref. [6], formula 1.441.1)

$$\sum_{n=1}^{\infty} \frac{\sin(\Delta n)}{n} = \frac{\pi - \Delta}{2} \equiv f(\Delta), \qquad \text{for } 0 < \Delta < 2\pi ,$$

we can write (4) as

$$\frac{dE}{d\omega} = \frac{\omega^3}{2\pi^2} ab \left(1 + \frac{1}{\omega a} f(2\omega a) + \frac{1}{\omega b} f(2\omega b) + \frac{2}{\omega} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin[2\omega(a^2n^2+b^2m^2)^{1/2}]}{(a^2n^2+b^2m^2)^{1/2}}\right) . \tag{5}$$

The first term in expression (5) represents the zero-point field energy. The second term corresponds to the case of an observer at rest between two parallel plates separated a distance a apart [1]; indeed, dividing (5) by b (in order to have the energy per unit area) and taking the limit $b \to \infty$ but keeping a finite, we obtain the known spectrum of the usual Casimir energy [1]. Similarly, the third term corresponds to the case of an observer at rest between two parallel plates a distance b apart. Finally the last term is an interference term due to the presence of plates in both directions.

In order to see the form of the spectrum, both as a function of the cavity size and of ω, we rewrite (5) using the dimensionless quantities $\tilde{\omega} = a\omega$, $\tilde{E} = a^2 E$ and $\beta = b/a$. We also subtract the zero point term to get

$$\frac{d\tilde{E}}{d\tilde{\omega}} = \beta \frac{\tilde{\omega}^2}{2\pi^2} \left[f(2\tilde{\omega}) + \frac{1}{\beta} f(2\beta\tilde{\omega}) + 2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sin[2\tilde{\omega}(n^2+\beta^2m^2)^{1/2}]}{(n^2+\beta^2m^2)^{1/2}} \right] . \tag{6}$$

The square bracket of expression (6) is plotted in fig. 1 as a function of $\tilde{\omega}$ for $\beta = 10^4$, 10, and 1. As expected, the first graph shows the sawtooth spectrum of two parallel plates. The case $\beta = 10$ (second graph in fig. 1) shows a superposition of the spectrum of each of the plates: for $\tilde{\omega}$ small we get simply the addition of spectra with a ratio of periods approximately equal to β, while for large $\tilde{\omega}$ we observe a mixing between the spectra of the plates generating a complicated behaviour. For $\beta = 1$ (third graph in fig. 1) the behaviour is even more

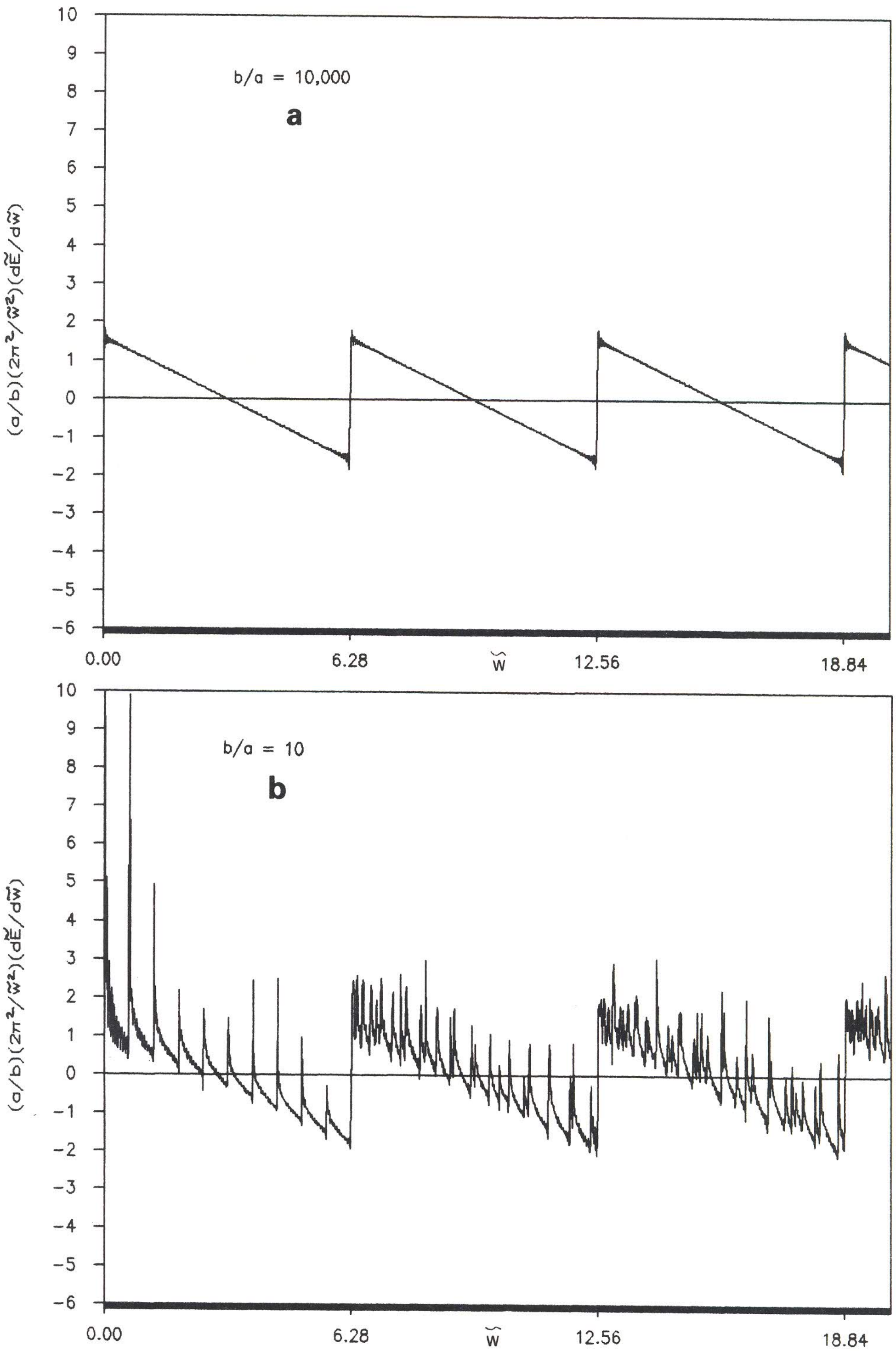

Fig. 1. Spectrum of the energy density of vacuum per unit length inside a prismatic cavity of rectangular section $a\times b$, as seen by an observer at rest. (a) $\beta=10000$; (b) $\beta=10$ and (c) $\beta=1$.

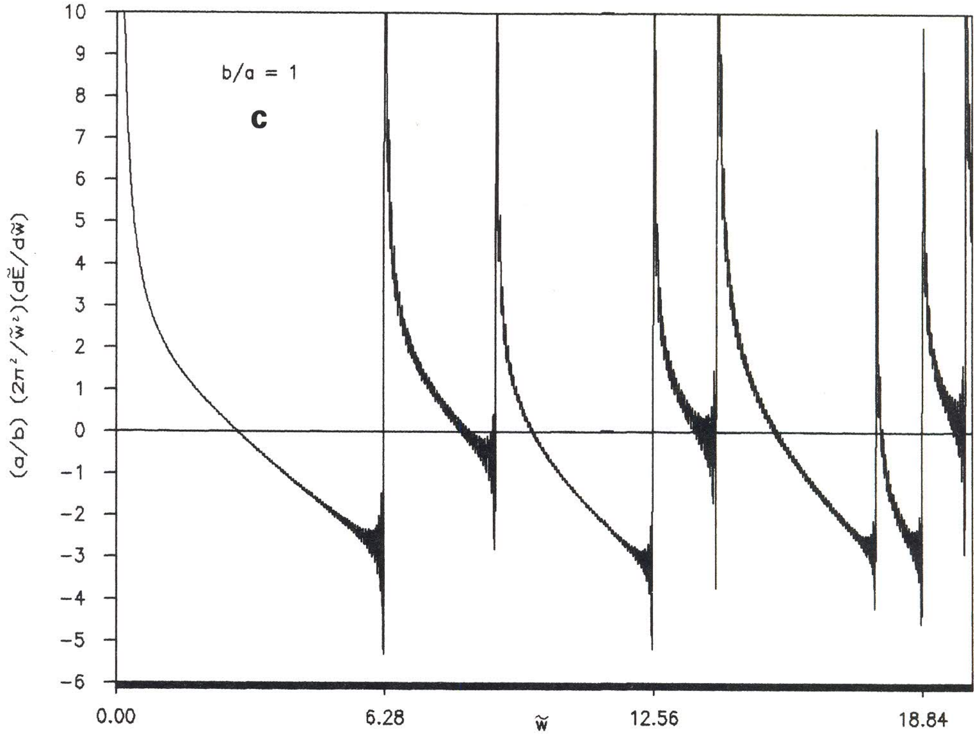

Fig. 1. (continued).

complicated, showing some kind of *vacuum turbulence*; a characterization of this semi-random behaviour is currently in progress.

The total energy per unit length inside the cavity has been calculated previously [7]. In our case, we subtract from (4) the zero point energy density and integrate over ω to obtain (see ref. [6], formulae 3.411.1 and 3.951.12)

$$E=-\frac{\pi^2(a^4+b^4)}{720a^3b^3}+\frac{ab}{4\pi^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(a^2n^2+b^2m^2)^2}. \tag{7}$$

Again we recover the correct expression for the Casimir energy in the limit of only two parallel plates ($b\gg a$ or $b\ll a$). The dimensionless total energy per unit length $\tilde{E}=a^2E$ goes as $-\pi^2/720\beta^3$ when $\beta\sim 0$ and as $-\pi^2\beta/720$ when $\beta\sim\infty$.

B. Accelerated observer. The world-line of an accelerated observer is given by $t=\alpha^{-1}\,\mathrm{sh}[\alpha(\tau\pm\sigma/2)]$ and $x=\alpha^{-1}\,\mathrm{ch}[\alpha(\tau\pm\sigma/2)]$ where α is the acceleration. The Wightman functions, eqs. (1), then reduce to

$$D^{\pm}(x,x')=\frac{1}{16\pi^2}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left(\frac{1}{a^2n^2+b^2m^2-\alpha^{-2}\,\mathrm{sh}^2(\alpha\sigma/2\mp i\varepsilon)}-\frac{1}{a^2n^2+(y-bm)^2-\alpha^{-2}\,\mathrm{sh}^2(\alpha\sigma/2\mp i\varepsilon)}\right.$$

$$\left.-\frac{1}{(z-an)^2+b^2m^2-\alpha^{-2}\,\mathrm{sh}^2(\alpha\sigma/2\mp i\varepsilon)}+\frac{1}{(z-an)^2+(y-bm)^2-\alpha^{-2}\,\mathrm{sh}^2(\alpha\sigma/2\mp i\varepsilon)}\right). \tag{8}$$

The last three terms do not contribute to the energy density per unit length inside the cavity (see appendix); thus, the Fourier transforms of (8) contain only integrals of the form

$$\int_{-\infty}^{\infty} \frac{e^{i\omega\sigma}\, d\sigma}{\operatorname{sh}^2(\alpha\sigma/2)-(\alpha\rho)^2},$$

where $\rho=(a^2n^2+b^2m^2)^{1/2}$. The poles of the function are located at $P_{1,2}=2\alpha^{-1}[\pm(-1)^k \operatorname{arcsh}(\alpha\rho)+i\pi k]$, where k is an integer. The non-vanishing contributions are

$$\tilde{D}^{\pm}(\omega)=\pm\frac{2}{\pi}\frac{1}{1-e^{\mp 2\pi\omega/\alpha}}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{\sin[2\omega\alpha^{-1}\operatorname{arcsh}(\alpha\rho)]}{\rho[1+(\alpha\rho)^2]^{1/2}}, \qquad (9)$$

which lead to an energy density per unit length inside the cavity given by

$$\frac{dE}{d\omega}=\frac{\omega^3}{\pi^2}\left(\frac{1}{2}+\frac{1}{e^{2\pi\omega/\alpha}-1}\right)\left(ab+\frac{b}{\omega}\sum_{n=1}^{\infty}\frac{\sin[2\omega\alpha^{-1}\operatorname{arcsh}(\alpha an)]}{n[1+(\alpha an)^2]^{1/2}}\right.$$
$$\left.+\frac{a}{\omega}\sum_{m=1}^{\infty}\frac{\sin[2\omega\alpha^{-1}\operatorname{arcsh}(\alpha bm)]}{m[1+(\alpha bm)^2]^{1/2}}+\frac{2ab}{\omega}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin[2\omega\alpha^{-1}\operatorname{arcsh}(\alpha\rho)]}{\rho[1+(\alpha\rho)^2]^{1/2}}\right). \qquad (10)$$

The Planckian spectrum in the first bracket can be identified with the thermal spectrum due to the acceleration α. The interpretation of the terms in the second bracket is parallel to that for the observer at rest: the first term corresponds to the vacuum, each one of the following two terms corresponds to a single pair of plates alone [2] and the last term is an *interference* term. Fig. 2 shows the dimensionless form of (10) (the dimensionless acceleration is $\tilde{\alpha}=a\alpha$) as function of $\tilde{\omega}$. The behavior is even more complicated than in the case of an observer at rest, suggesting again some sort of *vacuum turbulence*.

Expression (10) gives the correct limiting cases: first, if $\alpha\to 0$, we obtain the result (4); second, if $\beta\to\infty$

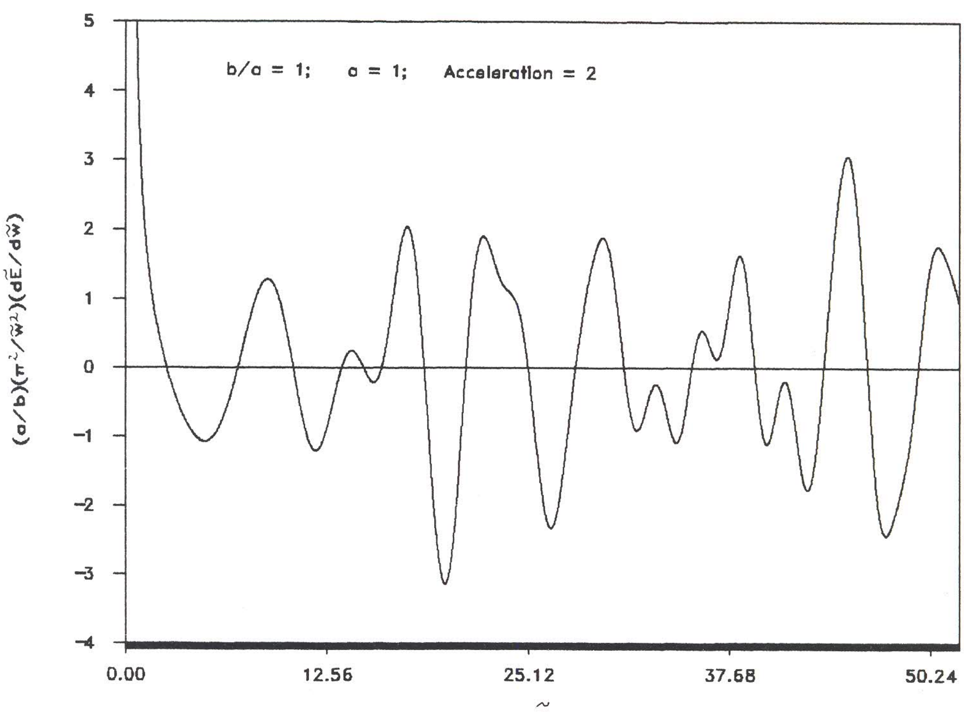

Fig. 2. Spectrum of the energy density of vacuum per unit length inside a prismatic cavity of rectangular section $a\times b$, as seen by a uniformly accelerating observer.

keeping a finite, we get the expression of ref. [2] for an accelerated observer between two infinite parallel plates separated a distance a.

In order to obtain the total energy, one has to integrate over frequency ω. This integration can be done directly by using the formula for the inverse Fourier transform. The final result is

$$E=-\frac{\pi^2(a^4+b^4)}{720a^3b^3}-\frac{ab}{4\pi^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(a^2n^2+b^2m^2)^2}+\frac{ab\alpha^4}{240\pi^2}. \qquad (11)$$

This total energy contains a *thermal* term proportional to α^4, which is also present outside both plates, giving a total null pressure. As a function of $\beta=b/a$, the dimensionless total energy per unit length $\tilde{E}=a^2E$ goes as $-\pi^2/720\beta^3$ when $\beta\sim 0$ and as $-\pi^2\beta/720+\tilde{\alpha}/240\pi$ when $\beta\sim\infty$.

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Appendix

The integrals $\int_0^a \sum_{n=-\infty}^{\infty} f(x-an)\,\mathrm{d}x$ do not contribute to the energy per unit area between the plates since

$$\int_0^a \sum_{n=-\infty}^{\infty} f(x-an)\,\mathrm{d}x=\sum_{n=-\infty}^{\infty}\int_{-an}^{a(n-1)} f(x)\,\mathrm{d}x=\tfrac{1}{2}\sum_{n=-\infty}^{\infty}\left(\int_0^{-a(n-1)} f(x)\,\mathrm{d}x-\int_0^{-an} f(x)\,\mathrm{d}x\right)=0\,,$$

due to the properties of linearity, translational invariance and rescaling invariance of the integral.

The other integrals are particular cases of

$$\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\mu x}\,\mathrm{d}x}{[\mathrm{sh}(x)-\lambda_1][\mathrm{sh}(x)-\lambda_2]}\,,$$

where the roots λ_i are given by

$$2\lambda_{1,2}=\pm\alpha[a^2n^2-\beta^2m^2\,\mathrm{sh}^2(\alpha\tau)]^{1/2}+\mathrm{i}m\beta\alpha\,\mathrm{ch}(\alpha\tau)\,,$$

for $n=0, 1, 2,...,$ and $m=0, \pm 1, \pm 2,...$; the poles are given by $P_{1,2}=(-1)^k\,\mathrm{arcsh}(\lambda_{1,2})+\mathrm{i}\pi k$, for $k=0, \pm 1, \pm 2,...$. The different particular cases correspond to $n=0$, $m=0$, or both. The integration procedure followed for each case depends on the location of the poles which depends on the values of the roots; these roots depend in turn on the values for n and m. The general integral is thus dependent on the value of an with respect to $\beta|\mathrm{sh}(\alpha\tau)|m$, and it does not seem possible to express it in a simple and compact form.

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