Zero-point field in curved spaces

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Boyer's conjecture that the thermal effects of acceleration are manifestations of the zero-point field is further investigated within the context of quantum field theory in curved spaces. The energy-momentum current for a spinless field is defined rigorously and used as the basis for investigating the energy density observed in a noninertial frame. The following examples are considered: (i) uniformly accelerated observers, (ii) two-dimensional Schwarzschild black holes, (iii) the Einstein universe. The energy spectra which have been previously calculated appear in the present formalism as an additional contribution to the energy of the zero-point field, but particle creation does not occur. It is suggested that the radiation produced by gravitational fields or by acceleration is a manifestation of the zero-point field and of the same nature (whether real or virtual).

I. INTRODUCTION

During the past decade there has been considerable investigation of quantum phenomena in curved space-times or in accelerated frames.[1–4] All the new and puzzling effects which have been found are related in some way to the very concept of a vacuum in quantum physics. In quantum field theory, the vacuum involves only virtual quanta and the total energy of the vacuum is infinite, but is is assumed that this has no physical reality. Thus, several complicated (and nonrigorous) techniques have been developed in order to extract physically relevant information from infinite quantities.

In recent years, stochastic electrodynamics has emerged as an alternative interpretation of quantum physics.[5–7] According to this theory, the "vacuum" state is actually formed by a universal random classical electromagnetic field. This random field should exist even at absolute-zero temperature; hence it is termed the zero-point field. Unlike the commonly accepted interpretation of quantum field theory, the zero-point field is assumed to be a physical real field interacting with the matter in the universe.

Boyer[8] noticed the interesting fact that the thermal effects produced by acceleration can be neatly explained within the framework of stochastic electrodynamics. The idea is that any energy spectrum observed in a moving frame appears distorted by the Doppler effect. Now the spectrum of the zero-point field is such that it is invariant under Lorentz transformations from one inertial frame to another; however, it need not be invariant when the transformation is to a noninertial frame. Boyer proved that the field correlation function in a uniformly accelerated frame is exactly the same that would be found in an inertial frame in a thermal bath. Thus the Planckian spectrum observed by a uniformly accelerated detector is a distortion of the zero-point field and is not due to the "creation of particles."

Similar arguments have also been given by Sciama, Candelas, and Deutsch,[9] who pointed out that the thermal effects of acceleration have their origin in the zero-point fluctuations of the quantum field, without implying that particles are created. This point will be followed up in the present article.

Since a gravitational field also produces a shift of energy, it must be expected that the zero-point field will also be distorted by gravity and manifest itself in some peculiar way. This is indeed the case as we shall prove in the present article.

The aim of this work is to explore further Boyer's conjecture by considering curved spaces. We will remain, however, within the framework of quantum field theory as this theory is more familiar, but it must be kept in mind that a formal analogy exists between stochastic electrodynamics and the quantum theory of bosonic fields: the field correlation functions in one theory are related to the Wightman functions in the other theory.[8] For the sake of simplicity, we restrict our attention to spinless fields only. Spin-$\frac{1}{2}$ and spin-1 fields will be considered in future publications.

Section II of this article introduces the energy-momentum current for a scalar field in curved space, and a formalism is developed which permits one to calculate in a relatively simple way the energy and particle number densities of the field. Some specific applications of the formalism are given in Sec. III: uniformly accelerated observers, two-dimensional Schwarzschild black holes, and the Einstein universe are considered. In all these cases, no particles are seen to be created; it is only the zero-point field which appears with an additional energy spectrum (corresponding to previously obtained results). This point is further discussed in Sec. III.

II. FORMALISM

Consider a system of chargeless spin-zero particles with mass m in an arbitrary background space-time defined by a metric tensor $g_{\alpha\beta}$. The particles are described by a scalar field $\phi(x)$. In particular, the noninteracting field satisfies the Klein-Gordon equation ($\hbar=1=c$ throughout):

$$(\Box+m^2+\xi R)\phi=0\ , \tag{1}$$

where R is the Ricci scalar and ξ is a certain constant ($\xi=0$ corresponds to minimal conformal coupling, while $\xi=\frac{1}{6}$ makes the Klein-Gordon equation conformally invariant in the case $m=0$). The D'Alembert operator $\Box$ is defined as

$$\Box\phi=\frac{1}{(-g)^{1/2}}\partial_\mu[(-g)^{1/2}g^{\mu\nu}\partial_\nu\phi]\ . \tag{2}$$

Let us define the Lagrangian density

$$\mathscr{L}=-\tfrac{1}{2}(-g)^{1/2}\phi(\Box+m^2+\xi R)\phi\ . \tag{3}$$

It is easy to prove that the Klein-Gordon equation (1) can be obtained from the Euler-Lagrange equations

$$\frac{\partial\mathscr{L}}{\partial\phi}-\partial_\mu\frac{\partial\mathscr{L}}{\partial\phi_{,\mu}}+\partial_\mu\partial_\nu\frac{\partial\mathscr{L}}{\partial\phi_{,\mu,\nu}}=0 \tag{4}$$

(where $\phi_{,\alpha}=\partial_\alpha\phi$, etc.).

Notice that the Lagrangian density $\mathscr{L}$, Eq. (3), differs from the commonly used Lagrangian density

$$\mathscr{L}'=\tfrac{1}{2}(-g)^{1/2}[g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}-(m^2+\xi R)\phi^2] \tag{5}$$

only by the term

$$\tfrac{1}{2}(-g)^{1/2}\partial_\mu(g^{\mu\nu}\phi\phi_{,\nu})\ ,$$

which makes no contribution to the variation of the action. In particular, the energy-momentum tensor $T_{\mu\nu}$ can be obtained by varying either $\mathscr{L}$ or $\mathscr{L}'$ with respect to the metric tensor $g^{\mu\nu}$ and its derivatives, the result being the same. However, the Lagrangian density $\mathscr{L}$ will prove to be more convenient for the purposes of the following analysis.

Consider in general a Lagrangian density $\mathscr{L}(x)$ which depends on the field $\phi(x)$ and its first and second derivatives. If the background metric is kept fixed, the variation of $\mathscr{L}$ when ϕ varies is given by

$$\delta\mathscr{L}=\left[\frac{\partial\mathscr{L}}{\partial\phi}-\partial_\mu\frac{\partial\mathscr{L}}{\partial\phi_{,\mu}}+\partial_\mu\partial_\nu\frac{\partial\mathscr{L}}{\partial\phi_{,\mu,\nu}}\right]\delta\phi+\partial_\mu\left[\left[\frac{\partial\mathscr{L}}{\partial\phi_{,\mu}}+\frac{\partial\mathscr{L}}{\partial\phi_{,\mu,\nu}}\overleftrightarrow{\partial}_\nu\right]\delta\phi\right]\ , \tag{6}$$

where the double-headed arrow indicates differentiation to the right minus differentiation to the left.

Suppose now that the background metric admits a Killing vector ξ^α. Then, under an infinitesimal transformation

$$x^\mu\to x'^\mu=x^\mu+\xi^\mu\ , \tag{7}$$

$g_{\alpha\beta}$ remains invariant, while ϕ and $\mathscr{L}$ vary according to

$$\delta\phi=\xi^\mu\phi_{,\mu}\ , \tag{8}$$

$$\delta\mathscr{L}=\xi^\mu\mathscr{L}_{,\mu}\ . \tag{9}$$

The Killing vector satisfies the equation

$$\xi_{\alpha;\beta}+\xi_{\beta;\alpha}=0\ . \tag{10}$$

Inserting Eqs. (8) and (9) into (6) (and using the fact that $\xi^\alpha{}_{;\alpha}=0$), we arrive to a conservation law for any field ϕ satisfying the Euler-Lagrange equations (4):

$$J^\mu{}_{;\mu}\equiv\frac{1}{(-g)^{1/2}}\partial_\mu[(-g)^{1/2}J^\mu]=0\ , \tag{11}$$

where the conserved current is

$$J^\alpha=-2(-g)^{1/2}\left[\mathscr{L}\xi^\alpha-\left[\frac{\partial\mathscr{L}}{\partial\phi_{,\alpha}}+\frac{\partial\mathscr{L}}{\partial\phi_{,\alpha,\beta}}\overleftrightarrow{\partial}_\beta\right](\xi^\gamma\phi_{,\gamma})\right] \tag{12}$$

and the semicolon denotes covariant derivative.

In particular, for the Lagrangian density (3), the current turns out to be

$$J_\alpha=-\phi\overleftrightarrow{\partial}_\alpha(\xi^\beta\phi_{,\beta})\ . \tag{13}$$

It is a straightforward exercise to check that the divergence of this particular four-vector J_α is indeed zero, provided that ϕ and ξ_α satisfy Eqs. (1) and (10), respectively. [The following formulas must be used in the demonstration:

$$\phi_{,\alpha;\beta}{}^{;\beta}-(\Box\phi)_{,\alpha}=R^\beta{}_\alpha\phi_{,\beta}\ , \tag{14}$$

$$\xi_{\alpha;\beta}{}^{;\beta}=-R^\beta{}_\alpha\xi_\beta\ ; \tag{15}$$

here $R^\beta{}_\alpha$ is the Ricci tensor.]

The Minkowski space-time admits four linearly independent constant Killing vectors associated to the Poincaré group. As a consequence, a tensor $T^{\alpha\beta}$ can be defined through the equation

$$J^\alpha=T^{\alpha\beta}\xi_\beta\ , \tag{16}$$

and the conservation of the current J^α implies that $T^{\alpha\beta}{}_{;\alpha}=0$, since the ξ_α's are linearly independent. From Eqs. (13) and (16) it explicitly follows that

$$T_{\alpha\beta}=-\tfrac{1}{2}\phi\overleftrightarrow{\partial}_\alpha\overleftrightarrow{\partial}_\beta\phi \tag{17}$$

in Cartesian coordinates. This form of the energy-momentum tensor has been occasionally used; it occurs, for instance, in the covariant definition of the Wigner function[10] (in fact, the formalism of the present article was inspired from this last approach). However, $T_{\alpha\beta}$ as defined by Eq. (17) has no obvious counterpart in curved space. One can try to define

$$T'_{\alpha\beta}=-\tfrac{1}{2}\phi\overleftrightarrow{\nabla}_\alpha\overleftrightarrow{\nabla}_\beta\phi \tag{18}$$

(in obvious notation) but the divergence of this tensor is

$$T'_{\alpha\beta}{}^{;\beta}=R_\alpha{}^\beta\phi_{;\beta} \tag{19}$$

which does not vanish in general; moreover, the trace of $T'_{\alpha\beta}$ is not zero for a massless field.

However, as long as a timelike Killing vector exists, J_α

given by (13) provides a definition of the energy-momentum four-vector which is sufficient for many practical purposes. The fact that J_α has vanishing divergence guarantees that the total energy

$$E=\int J_\mu d\sigma^\mu \tag{20}$$

is independent of the particular three-dimensional space (whose normal vector is $d\sigma^\mu$) over which the integration is performed.

The orbit of a timelike Killing vector can be identified with the world line $x^\alpha = x^\alpha(\tau)$ of an observer whose four-velocity is

$$\frac{dx^\alpha}{d\tau}=U^\alpha=(\xi^\mu\xi_\mu)^{-1/2}\xi^\alpha , \tag{21}$$

where τ is his proper time. The energy density can be defined unambiguously as

$$e=U^\alpha J_\alpha , \tag{22}$$

and from Eq. (13)

$$e=(\xi^\mu\xi_\mu)^{1/2}\left[-\phi\frac{d^2\phi}{d\tau^2}+\frac{d\phi}{d\tau}\frac{d\phi}{d\tau}\right] , \tag{23}$$

where $U^\beta\nabla_\beta = d/d\tau$.

The procedure for quantizing the field ϕ is straightforward. Starting from the Lagrangian density $\mathscr{L}$ of Eq. (3), with ϕ interpreted as an operator, we arrive at the definition

$$J_\alpha=-\langle\phi\overleftrightarrow{\partial}_\alpha(\xi^\beta\phi_{,\beta})\rangle \tag{24}$$

for the vacuum expectation value of the energy-momentum four-vector, and

$$e=\tfrac{1}{2}(\xi^\mu\xi_\mu)^{1/2}\left\langle-\phi\frac{d^2\phi}{d\tau^2}-\frac{d^2\phi}{d\tau}\phi+2\frac{d\phi}{d\tau}\frac{d\phi}{d\tau}\right\rangle \tag{25}$$

for the vacuum energy density.

It is also possible to define a current

$$n_\alpha=-i\langle\phi\overleftrightarrow{\partial}_\alpha\phi\rangle \tag{26}$$

which is divergenceless, but has no counterpart in the nonquantum theory. Loosely speaking, the scalar

$$\begin{aligned}n&\equiv U^\alpha n_\alpha\\&=-i\left\langle\phi\frac{d\phi}{d\tau}-\frac{d\phi}{d\tau}\phi\right\rangle\end{aligned} \tag{27}$$

is the "particle number density" of the vacuum, but this interpretation is not direct, as we shall see in the following.

We are now in a position to develop a formalism which permits direct evaluation of the energy density or the particle number density measured by a detector moving along a Killing orbit. We first consider Eq. (27) and write the average as

$$\begin{aligned}n&=-i\int_{-\infty}^{\infty}d\sigma\,\delta(\sigma)\left\langle\phi(\tau+\tfrac{1}{2}\sigma)\frac{d\phi}{d\tau}(\tau-\tfrac{1}{2}\sigma)-\frac{d\phi}{d\tau}(\tau+\tfrac{1}{2}\sigma)\phi(\tau-\tfrac{1}{2}\sigma)\right\rangle\\&=-2i\int_{-\infty}^{\infty}d\sigma\frac{d\delta}{d\sigma}(\sigma)\langle\phi(\tau+\tfrac{1}{2}\sigma)\phi(\tau-\tfrac{1}{2}\sigma)\rangle .\end{aligned} \tag{28}$$

A partial integration has been performed to obtain the last member, and of course

$$\phi(\tau\pm\tfrac{1}{2}\sigma)\equiv\phi[x^\mu(\tau\pm\tfrac{1}{2}\sigma)] .$$

Hereafter, only massless fields will be considered for the sake of simplicity, although the inclusion of a massive field presents no formal difficulty.

We now use the definition of the Wightman functions $D^\pm(x^\mu,x'^\mu)$, evaluated at two points, $x^\mu=x^\mu(\tau+\frac{1}{2}\sigma)$ and $x'^\mu=x^\mu(\tau-\frac{1}{2}\sigma)$, along a given world line:

$$D^\pm(\tau+\tfrac{1}{2}\sigma,\tau-\tfrac{1}{2}\sigma)=\langle\phi(\tau\pm\tfrac{1}{2}\sigma)\phi(\tau\mp\tfrac{1}{2}\sigma)\rangle . \tag{29}$$

With the representation

$$\delta(\sigma)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\,e^{i\omega\sigma} \tag{30}$$

for the delta function, and defining the Fourier transforms of the Wightman functions

$$\widetilde{D}^{\pm}(\omega,\tau)\equiv\int_{-\infty}^{\infty}d\sigma\,e^{i\omega\sigma}D^\pm(\tau+\tfrac{1}{2}\sigma,\tau-\tfrac{1}{2}\sigma) , \tag{31}$$

we obtain the final form of Eq. (28):

$$n=\frac{1}{\pi}\int_0^\infty d\omega\,\omega[\widetilde{D}^{+}(\omega,\tau)-\widetilde{D}^{-}(\omega,\tau)] . \tag{32}$$

Consider now the energy density, Eq. (25). Performing the same manipulation as above it follows that

$$e=\frac{1}{\pi}(\xi^\mu\xi_\mu)^{1/2}\int_0^\infty d\omega\,\omega^2[\widetilde{D}^{+}(\omega,\tau)+\widetilde{D}^{-}(\omega,\tau)] . \tag{33}$$

Now, ω is the frequency measured by a detector with proper time τ and four-velocity U^μ. Accordingly, Eq. (32) implies that the particle density $f(\omega,\tau)$ is given by

$$f(\omega,\tau)=\frac{1}{(2\pi)^2\omega}[\widetilde{D}^{+}(\omega,\tau)-\widetilde{D}^{-}(\omega,\tau)] \tag{34}$$

and Eq. (33) implies that the energy density *per* mode is

$$de=(\xi^\mu\xi_\mu)^{1/2}\frac{\omega^2}{\pi}[\widetilde{D}^{+}(\omega,\tau)+\widetilde{D}^{-}(\omega,\tau)]d\omega . \tag{35}$$

Thus, the particle density must be evaluated with the vacuum expectation value of the field commutator (i.e., the Pauli-Jordan-Schwinger function) while the energy density involves the anticommutator (i.e., the Hadamard function). Various examples will be considered in the following sections which will elucidate the above statements.

III. SOME EXAMPLES

A. Flat space

The simplest (almost trivial) example is that of an observer at rest in flat space. The Wightman functions for the Minkowski space are

$$D^{\pm}(x,x')=-\frac{1}{4\pi^2}[(t-t'\mp i\epsilon)^2-|\mathbf{x}-\mathbf{x}'|^2]^{-1} . \quad (36)$$

The space-time admits a timelike Killing vector $\xi^\alpha=(1,0)$ which is also the four-velocity of the observer at rest with world line $t=\tau$ and $\mathbf{x}=0$. Thus for this particular observer,

$$D^{\pm}(\tau+\tfrac{1}{2}\sigma,\tau-\tfrac{1}{2}\sigma)=-\frac{1}{4\pi^2}(\sigma\mp i\epsilon)^{-2} , \quad (37)$$

and the Fourier transforms are

$$\tilde{D}^{+}(\omega,\tau)=\frac{\omega}{2\pi} ,$$
$$\tilde{D}^{-}(\omega,\tau)=0 , \quad (38)$$

for $\omega>0$. Then, according to Eqs. (34) and (35)

$$f(\omega,\tau)=(2\pi)^{-3} \quad [=(2\pi\hbar)^{-3}] , \quad (39)$$

$$de=\frac{\omega^3}{2\pi^2}d\omega \quad \left[=\frac{\hbar\omega^2}{2\pi^2c^3}d\omega\right] . \quad (40)$$

The first equation merely expresses the fact that there is one particle in each cell of phase space; this is a consequence of the normalization used for the wave function and has no physical meaning. The second equation is the well-known formula for the zero-point energy.

Consider now a uniformly accelerated observer with world line defined by

$$t=\alpha^{-1}\sinh(\alpha\tau) ,$$
$$x=\alpha^{-1}\cosh(\alpha\tau) , \quad (41)$$

where τ is its proper time and α its acceleration. The four-velocity of this observer is

$$U^\alpha=\alpha(x,t,0,0) , \quad (42)$$

which is also a Killing vector: the one associated to a Lorentz boost. Inserting (41) in (36) it follows that

$$D^{\pm}(\tau+\tfrac{1}{2}\sigma,\tau-\tfrac{1}{2}\sigma)=-\frac{\alpha^2}{16\pi^2}\operatorname{csch}^2[\tfrac{1}{2}\alpha(\sigma\mp i\epsilon)] . \quad (43)$$

The Fourier transforms of these functions can be evaluated using the formula[11]

$$\csc^2(\pi x)=\frac{1}{\pi^2}\sum_{k=-\infty}^{\infty}(x-k)^{-2} . \quad (44)$$

The final result is

$$\tilde{D}^{+}(\omega,\tau)=\frac{\omega}{2\pi}\frac{e^{2\pi\omega/\alpha}}{e^{2\pi\omega/\alpha}-1} ,$$
$$\tilde{D}^{-}(\omega,\tau)=\frac{\omega}{2\pi}\frac{1}{e^{2\pi\omega/\alpha}-1} . \quad (45)$$

Then, according to Eqs. (34) and (35),

$$f(\omega,\tau)=(2\pi)^{-3} , \quad (46)$$

$$de=(\xi^\mu\xi_\mu)^{1/2}\frac{\omega^3}{\pi^2}\left[\frac{1}{2}+\frac{1}{e^{2\pi\omega/\alpha}-1}\right]d\omega . \quad (47)$$

Again, there is one particle in each phase-space cell, but now the energy of the zero-point field has an additional Planckian term. This suggests that an accelerated detector does not count newly created particles; what it actually detects is the zero-point field which, by effect of the acceleration, manifests itself as a Planck spectrum. The above results and their interpretation are in agreement with the conclusions reached by Boyer within the framework of stochastic electrodynamics.[8]

B. Curved space

As the first example of the application of our formalism to a curved space-time, we consider a two-dimensional Schwarzschild black hole. The metric is

$$ds^2=\frac{2M}{r}e^{-r/2M}du\,dv , \quad (48)$$

where M is the mass of the hole, r is the radial coordinate, and u and v are Kruskal coordinates defined through the equation

$$uv=-(4M)^2\left[\frac{r}{2M}-1\right]e^{r/2M} . \quad (49)$$

The metric admits a timelike Killing vector ξ^α with magnitude $(\xi^\mu\xi_\mu)^{1/2}=(1-2M/r)^{1/2}$. The world line of a detector at rest at $r=r_0$, say, is given by the parametric equations

$$u=e^{a\tau} ,$$
$$v=-be^{a\tau} , \quad (50)$$

where τ is the proper time,

$$a\equiv(1-2M/r_0)^{-1/2}/4M ,$$

and

$$b\equiv16M^2(r_0/2M-1)\exp(r_0/2M) .$$

Clearly, the four-velocity of such a detector is parallel to the Killing vector ξ^α. The Wightman function is given by[12]

$$D^{\pm}(u,v,u',v')=-\frac{1}{8\pi}\ln[(u'-u\mp i\epsilon)(v'-v\mp i\epsilon)] . \quad (51)$$

Using Eqs. (50) we find that

$$D^{\pm}(\tau+\tfrac{1}{2}\sigma,\tau-\tfrac{1}{2}\sigma)=-\frac{1}{4\pi}\ln[2b\sinh(\tfrac{1}{2}a\sigma\mp i\epsilon)] , \quad (52)$$

from where it follows that[13]

$$\tilde{D}^{+}(\omega,\tau)=\frac{1}{2\omega}\frac{e^{2\pi\omega/a}}{e^{2\pi\omega/a}-1} , \quad (53a)$$

$$\tilde{D}^{-}(\omega,\tau)=\frac{1}{2\omega}\frac{1}{e^{2\pi\omega/a}-1} , \quad (53b)$$

for $\omega>0$. Thus, according to Eqs. (32) and (33),

$$n=\frac{1}{2\pi}\int_0^\infty d\omega\ , \tag{54}$$

$$e=\left[1-\frac{2M}{r_0}\right]^{1/2}\frac{2}{\pi}\int_0^\infty d\omega\,\omega\left[\frac{1}{2}+\frac{1}{e^{2\pi\omega/a}-1}\right]. \tag{55}$$

Just as in the case of uniform acceleration, we find one particle *per* phase-space cell—which is a consequence of the normalization used—and we obtain the energy of the zero-point field with an additional Planckian term.

Thus an observer at rest detects a thermal bath with a temperature

$$k_BT=(8\pi M)^{-1}(1-2M/r_0)^{-1/2}\ ,$$

which is in complete agreement with well-known results. [The extra factor $(1-2M/r_0)^{1/2}$ in Eq. (55) comes from the term $(\xi^\mu\xi_\mu)^{1/2}$ and guarantees that $de/d\omega$ is finite at the horizon.] The analogy with the case of uniform acceleration is quite clear, and we can now advance the following interpretation: a gravitational field deforms the energy spectrum of the zero-point field and makes it appear as a thermal spectrum.

The second example we consider is that of an Einstein universe. The metric is

$$ds^2=R^2[d\eta^2-d\chi^2-\sin^2\chi(d\theta+\sin^2\theta\, d\phi^2)]\ , \tag{56}$$

where R is the constant radius of the universe. There is a timelike Killing vector $\xi^\alpha=\delta_0^\alpha$ $(x^0=\eta)$ such that $\xi^\mu\xi_\mu=R^2$. The Wightman function is[14]

$$D^{\pm}(x^\mu,x'^\mu)=\frac{1}{8\pi^2R^2}[\cos(\eta-\eta'\mp i\epsilon)-\cos(\chi-\chi')]^{-1} \tag{57}$$

[it reduces to the form given by Eq. (36) in the limit $R\rightarrow\infty$].

A detector at rest follows the Killing orbit and its proper time is $\tau=R\eta$. Thus,

$$D^{\pm}(\tau+\tfrac{1}{2}\sigma,\tau-\tfrac{1}{2}\sigma)=-\frac{1}{16\pi^2R^2}\csc^2\left[\frac{\sigma\mp i\epsilon}{2R}\right]. \tag{58}$$

Using formula (43) we obtain the Fourier transform of Eq. (58) by standard methods,

$$\tilde{D}^{+}(\omega,\tau)=\frac{\omega}{2\pi}\left[1+2\sum_{k=1}^{\infty}\cos(2\pi kR\omega)\right], \tag{59a}$$

$$\tilde{D}^{-}(\omega,\tau)=0\ , \tag{59b}$$

for $\omega>0$. To proceed further, the following formulas are useful:

$$\int_0^\infty d\omega\,\omega^2\cos(\omega x)=-\pi\delta''(x)\ , \tag{60a}$$

$$\int_0^\infty d\omega\,\omega^3\cos(\omega x)=6x^{-3} \tag{60b}$$

[strictly speaking, the series in Eq. (59a) must be interpreted as a distribution, and Eqs. (60) as the Fourier transforms of distributions[15]]. Inserting Eqs. (59) in Eqs. (32) and (33), using Eqs. (60) and the value of the Riemann function $\zeta(4)=\pi^4/90$, we finally find that the density of particles in phase space is again $f=(2\pi)^{-3}$ as in flat space, but the total energy density (integrated over all frequencies) is

$$e=e_{\text{flat}}+\frac{1}{240\pi^2R^3}\ , \tag{61}$$

where e_{flat} is the (formally infinite) contribution (40) of the zero-point field in flat space. Equation (61) is in agreement with previously found results,[16] except for a factor 2. This discrepancy comes from the fact that we are using the Wightman function which has poles all along and above the whole real ω axis, while the energy-momentum tensor is usually evaluated in the current literature with the Feynman Green's function which has poles along and above the positive real ω axis. Thus, with our formalism we are picking up twice as much residues at the poles when evaluating the Fourier transform, Eqs. (59). However, we do obtain the correct value for the zero-point field, a fact that gives consistency to the final result.

IV. DISCUSSION

The results obtained in Secs. II and III can be summarized as follows. Starting from the Lagrangian of a scalar field, we obtained the conserved energy-momentum four-vector directly from the Noether theorem and the existence of a timelike Killing vector. This four-vector led in a natural way to a formula for the energy density in terms of the vacuum expectation value of the symmetrized product of the field operator. This point is important, since other authors have used the positive-frequency Wightman function or the Feyman Green's function for the calculation of the energy density; the only justification for the use of these functions is that the negative-frequency contributions to the energy are eliminated; however, in light of our own analysis, this is equivalent to arbitrarily cutting the zero-point energy.

The conclusion of the present article is that the radiation produced by a black hole or an accelerated frame is of the same nature as the zero-point field. Whether this field is real or virtual remains an open question. We expect that the study of realistic fields with spin will help to further clarify this problem.

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[12]Birrell and Davies, Ref. 4, p. 281.
[13]We are using the formula

$$\int_{-\infty}^{\infty} d\sigma\, e^{i\omega\sigma} \ln|\sinh(\tfrac{1}{2}a\sigma \mp i\epsilon)| = \mp\frac{2\pi}{\omega}(e^{\pm 2\pi\omega/a}-1)^{-1}$$

for $\omega > 0$.
[14]Birrell and Davies, Ref. 4, p. 123.
[15]Formulas (59) follow from the Fourier transform of the distribution $\omega^n\theta(\omega)$, where θ is the step function. Strictly speaking,

$$\mathscr{F}[\omega^n\theta] = n!Pf(2\pi i\xi)^{-n-1} + i(-1)^n\pi(2\pi i)^{-n-1}\delta^n(\xi) ,$$

where $\mathscr{F}$ denotes the Fourier transform; see, e.g., Jean Levoine, *Transformation de Fourier des Pseudo-Fonctions* (CNRS, Paris, 1963), p. 85.
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