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Parametrized Post-Post-Newtonian (PP^2N) Formalism for the Solar System

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Abstract

An extension of the parametrized post-Newtonian (PPN) formalism to third order in the expansion parameter m/r is used to derive analytical expressions accurate to the same order for the motion of test particles and photons in the presence of the gravitational field of the sun represented by a static, isotropic metric. The consequences of including higher-order terms are discussed in relation to the so-called classical gravitational tests for the case of general relativity theory. Present observational or experimental data are not accurate enough to detect variations due to the inclusion of higher-order terms but a planned solar probe experiment may provide information that would make such detection possible.

§(1): *Introduction*

The general framework within which general relativity and other theories of gravity, compatible with the experimental and observational tests carried out so far, are tested assumes (i) space-time to be a Riemannian manifold; (ii) the existence of a metric field $g_{ij}(x)$ defining an internal ds, where

$$ds^2 = g_{ij}\, dx^i\, dx^j$$

(iii) the equations of motion for test particles and photons to be derivable from the variational principle

$$\delta \int ds^2 = \delta \int \left(\frac{ds}{d\mu}\right)^2 d\mu^2 = 0$$

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where μ is a parameter along the geodesic lines; and (iv) atomic clocks to keep proper time s. The various theories then differ on the basis of their field equations that determine g_{ij}.

Research carried out on the problem of motion gave rise to an approximation scheme which later evolved into the parametrized post-Newtonian (PPN) framework [1-8]. Over the last decade this formalism has been developed as an extremely useful theoretical tool which allows a clear classification of the theories according to the values they predict for certain parameters, values which can then be used for their comparison with tests [9-12]. Essentially, the formalism consists of assuming the components of g_{ij} to be analytic and expanding them in power series of the parameter m/r around the flat vacuum solution ($m = GM/c^2$ denotes the mass of the field's source and r the distance to its center). The expansion is stopped at an order higher than that of Newtonian mechanics and each term in it is labeled with a coefficient which becomes the corresponding parameter. The equations of motion are then integrated to predict the outcome of experimental and observational tests, the numerical results depending on the values of the different parameters.

In this paper an extension of the PPN analysis to order $(m/r)^3$ is presented. The results given in the PPN formalism are generalized to this order and the consequences of including the higher-order terms are discussed in relation to the observational and experimental tests.

§(2): *Equations of Motion*

For the case of the solar system it can further be assumed that the sun's gravitational field possesses spherical symmetry and that the planets are test particles moving in its presence. According to Birkhoff's theorem, one then has a static and isotropic field [13, 14]. The usual assumption that wave pulses travel along null geodesics $ds = 0$ is actually derivable for electromagnetic wave fronts from the combined Einstein-Maxwell equations [15, 16].

Under assumptions (i) and (ii), the most general proper time interval for a reference system based on the center of the sun can be written in its standard form [17] as

$$ds^2 = B(r)\,dt^2 - A(r)\,dr^2 - H(r)\,d\Omega^2 \tag{1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ and the system of units being used is such that the speed of light c is equal to unity.

Since the field is isotropic, we may consider the orbit of a particle to be confined to the "equatorial plane" $\theta = \pi/2$. Equation (1) is then reduced to

$$ds^2 = B(r)\,dt^2 - A(r)\,dr^2 - H(r)\,d\phi^2$$

Assumption (iii) allows us to derive the equations of motion by finding the extremum of the action, i.e.,

$$0 = \delta \int \left(\frac{ds}{d\mu}\right)^2 d\mu^2 = \delta \int \left[B(r)\left(\frac{dt}{d\mu}\right)^2 - A(r)\left(\frac{dr}{d\mu}\right)^2 - H(r)\left(\frac{d\phi}{d\mu}\right)^2\right] d\mu^2$$

where the geodesic parameter μ is defined by

$$\left(\frac{ds}{d\mu}\right)^2 = k^2, \qquad k^2 = \begin{cases} 0 & \text{for null geodesics} \\ \text{positive constant for test particle geodesics} \end{cases} \tag{2}$$

The usual procedure gives

$$B(r) \frac{dt}{d\mu} = \text{const} = \epsilon \tag{3a}$$

$$H(r) \frac{d\phi}{d\mu} = \text{const} = \tau \tag{3b}$$

$$\frac{d}{d\mu}\left[A(r)\left(\frac{dr}{d\mu}\right)^2\right] = \frac{dr}{d\mu}\left[\frac{dH(r)}{dr}\left(\frac{d\phi}{d\mu}\right)^2 - \frac{dB(r)}{dr}\left(\frac{dt}{d\mu}\right)^2\right] \tag{3c}$$

as the equations of motion.

For the motions of test particles (bound orbits) and wave signals (unbound orbits) one rewrites equation (2) in two slightly different forms:

$$\frac{\epsilon^2}{B(r)} - \tau^2 \left[\frac{A(r)}{H^2(r)}\left(\frac{dr}{d\phi}\right)^2 + \frac{1}{H(r)}\right] = k^2$$

and

$$B(r) - A(r)\left(\frac{dr}{dt}\right)^2 - \left(\frac{\tau}{\epsilon}\right)^2 \frac{B^2(r)}{H(r)} = \left(\frac{k}{\epsilon}\right)^2 B^2(r)$$

where use of equations (3a, b) has been made.

Rearranging terms and integrating, we obtain

$$\phi = \phi_i \pm \int_{r(\phi_i)}^{r(\phi)} \left[\frac{A(r)}{H(r)}\right]^{1/2} \frac{dr}{[(\epsilon/\tau)^2 H(r)/B(r) - (k/\tau)^2 H(r) - 1]^{1/2}} \tag{4}$$

and

$$t = t_i \pm \int_{r(t_i)}^{r(t)} \left[\frac{A(r)}{B(r)}\right]^{1/2} \frac{dr}{[1 - (\tau/\epsilon)^2 B(r)/H(r) - (k/\epsilon)^2 B(r)]^{1/2}} \tag{5}$$

In order to eliminate the constants ϵ, τ, and k from expressions (4) and (5) one rewrites equation (2) as

$$\frac{A(r)}{H^2(r)}\left(\frac{dr}{d\phi}\right)^2 + \frac{1}{H(r)} - \left(\frac{\epsilon}{\tau}\right)^2\frac{1}{B(r)} = -\left(\frac{k}{\tau}\right)^2$$

where use of equations (3a, b) has been made.

One must now make a distinction between bound and unbound motion. Recalling that for wave signals ($k = 0$) $dr/d\phi$ vanishes at the distance of nearest approach of the signal to the sun ($r = r_0$ in Figure 1), we see that the previous expression gives

$$\left(\frac{\tau}{\epsilon}\right)^2 = \frac{H_0}{B_0} \tag{6}$$

where $H_0 = H(r_0)$ and $B_0 = B(r_0)$.

For test particles $dr/d\phi$ vanishes at perihelion and aphelion (r_- and r_+, respectively), so that

$$\left(\frac{\tau}{\epsilon}\right)^2 = \frac{B_+^{-1} - B_-^{-1}}{H_+^{-1} - H_-^{-1}} \tag{7a}$$

$$\left(\frac{k}{\tau}\right)^2 = \frac{B_+H_+^{-1} - B_-H_-^{-1}}{B_- - B_+} \tag{7b}$$

$$\left(\frac{k}{\epsilon}\right)^2 = \frac{H_+B_+^{-1} - H_-B_-^{-1}}{H_+ - H_-} \tag{7c}$$

where $H_\pm = H(r_\pm)$ and $B_\pm = B(r_\pm)$.

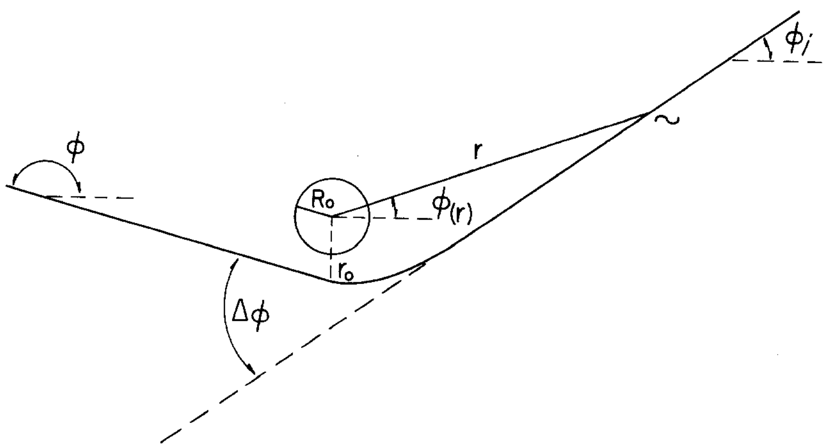

Fig. 1. Quantities referred to in the calculation of the motion of a wave signal ~.

Inserting equations (6) and (7) in equations (4) and (5) one obtains

$$\phi = \phi_i \pm \int_{r(\phi_i)}^{r(\phi)} \left[\frac{A(r)}{H(r)}\right]^{1/2} \frac{dr}{[(B_0/H_0)H(r)/B(r) - 1]^{1/2}} \tag{8a}$$

and

$$t = t_i \pm \int_{r(t_i)}^{r(t)} \left[\frac{A(r)}{B(r)}\right]^{1/2} \frac{dr}{[1 - (H_0/B_0)B(r)/H(r)]^{1/2}} \tag{8b}$$

for wave signals, and

$$\phi = \phi_i \pm \int_{r(\phi_i)}^{r(\phi)} \frac{A^{1/2}(r)}{H(r)} \cdot \left\{\frac{H_-[B^{-1}(r) - B_-^{-1}] - H_+[B^{-1}(r) - B_+^{-1}]}{H_+H_-(B_+^{-1} - B_-^{-1})} - \frac{1}{H(r)}\right\}^{-1/2} dr \tag{9a}$$

and

$$t = t_i \pm \int_{r(t_i)}^{r(t)} \frac{A^{1/2}(r)}{B(r)} \cdot \left\{\frac{1}{B(r)} - \frac{B_-[H^{-1}(r) - H_-^{-1}] - B_+[H^{-1}(r) - H_+^{-1}]}{B_+B_-(H_+^{-1} - H_-^{-1})}\right\}^{-1/2} dr \tag{9b}$$

for test particles.

§(3): *Order of Approximation*

So far, the derived equations of motion are exact and valid for any metric theory of gravity. Here one assumes that the field equations of whatever theory one is considering admit of asymptotically flat vacuum solutions and introduce a variant of the post-post-Newtonian formalism consisting of a series expansion for the three components $g_{00} = B(r)$, $g_{11} = -A(r)$, and $g_{33} = -H(r)$ accurate to third order in the parameter m/r. These expansions are given by

$$A(r) = 1 + 2\gamma \frac{m}{r} + 4\delta \frac{m^2}{r^2} + 8\lambda \frac{m^3}{r^3} \tag{10a}$$

$$B(r) = 1 - 2\alpha \frac{m}{r} + 2\beta \frac{m^2}{r^2} - 2\eta \frac{m^3}{r^3} \tag{10b}$$

$$H(r) = r^2 \tag{10c}$$

In fact, $H(r)$ is not being expanded, rather it is defined in such a way as to identify r, θ, and ϕ with spherical polar coordinates and to interpret r as the distance from the source. The results will then be coordinate independent [18].

Inserting equations (10) in equations (8) and integrating one obtains

$$\begin{aligned}\phi = \phi_i \pm \Bigg\{ & \left[1 + \frac{Em^2}{2r_0^2}\left(\frac{1}{2} - \frac{\alpha m}{r_0}\right)\right]\left(\frac{\pi}{2} - \arctan\frac{r_0}{r}\right) \\ & + \frac{m}{r_0 r}\left(\gamma + \frac{(4\delta - \gamma^2) m}{4r} + \left[\frac{J - (2\alpha + \gamma)\beta}{r_0^2} + \frac{J}{2r^2}\right] m^2\right)(r^2 - r_0^2)^{1/2} \\ & + \frac{m}{r_0}\left[\alpha - \frac{(5\alpha + 4\gamma)\alpha m}{4r_0} + \left(\frac{2D - 3\alpha\beta}{r_0} + \frac{D}{r}\right)\frac{m^2}{r_0}\right]\left(\frac{r - r_0}{r + r_0}\right)^{1/2} \\ & + \frac{\alpha^2 m^2 (r - r_0)}{4r_0^2}\,\frac{(r - r_0)^{1/2}}{(r + r_0)^{3/2}} + \frac{\alpha^2 m^3}{2r_0^3} \\ & \cdot\left[\frac{(19\alpha + 3\gamma) r^2}{3} + (14\alpha + 3\gamma) r r_0 + (9\alpha + 2\gamma) r_0^2\right]\frac{(r - r_0)^{1/2}}{(r + r_0)^{5/2}}\Bigg\}_{r_i}^{r} \end{aligned} \tag{11a}$$

and

$$\begin{aligned} t = t_i \pm \Bigg\{ & \left(1 + \frac{Cm^3}{r r_0^2}\right)(r^2 - r_0^2)^{1/2} + (\alpha + \gamma) m \ln\left[\frac{r + (r^2 - r_0^2)^{1/2}}{r_0}\right] \\ & + m\left[\alpha - \frac{(3\alpha + 2\gamma)\alpha m}{2r_0} + \left(\frac{F}{r_0} + \frac{G}{r}\right)\frac{m^2}{2r_0}\right]\cdot\left(\frac{r - r_0}{r + r_0}\right)^{1/2} \\ & + \frac{Em^2}{2r_0}\left(1 - \frac{\alpha m}{r_0}\right)\left(\frac{\pi}{2} - \arcsin\frac{r_0}{r}\right) \\ & + \frac{\alpha^2 m^2 r}{2r_0}\left[1 - \frac{(3\alpha + \gamma) m}{r_0}\right]\frac{(r - r_0)^{1/2}}{(r + r_0)^{3/2}} + \frac{\alpha^3 m^3 r^2}{2r_0^2}\,\frac{(r - r_0)^{1/2}}{(r + r_0)^{5/2}}\Bigg\}_{r_i}^{r} \end{aligned} \tag{11b}$$

where

$$C = 4\lambda + 2\alpha\delta - 2\beta\gamma + \tfrac{3}{2}\gamma\alpha^2 + \eta - 4\alpha\beta + \tfrac{5}{2}\alpha^3 - 2\delta\gamma - \tfrac{1}{2}\alpha\gamma^2 + \tfrac{1}{2}\gamma^3 \tag{12a}$$

$$D = 4\alpha^3 + 2\alpha^2\gamma - 2\alpha\beta + 2\alpha\delta - \tfrac{1}{2}\alpha\gamma^2 + \eta \tag{12b}$$

$$E = 8\alpha^2 + 4\delta + 4\alpha\gamma - 4\beta - \gamma^2 \tag{12c}$$

$$F = 4\eta - 20\alpha\beta + 30\alpha^3 + 11\alpha^2\gamma + 8\alpha\delta - 2\alpha\gamma^2 \tag{12d}$$

$$G = 4\alpha\delta + 5\alpha^2\gamma - 8\alpha\beta + 11\alpha^3 - \alpha\gamma^2 + 2\eta \tag{12e}$$

$$J = \tfrac{1}{3}(\gamma^3 - 4\gamma\delta + 8\lambda) \tag{12f}$$

For the motion of test particles, forced to move in bound orbits by the gravitational interaction, one has the familiar result of Newtonian mechanics that the typical kinetic energy and the typical potential energy are of the same order of magnitude. That is to say that $v^2 \sim m/r$, where v is the typical spatial velocity of any test particle. Therefore, if the expansion of expression (1) is to be homogeneous, terms involving the square of the spatial velocity of the particles $(dx^l/dt)^2$ need only be expanded up to second order in m/r.

Using this simplification one can now rewrite equations (9) as

$$\phi = \phi_i \pm K_1^{-1/2} \int_{r_i}^{r} \frac{1 + \gamma(m/r) + (4\delta - \gamma^2)(m^2/2r^2)}{r^2\,[(1/r_- - 1/r)(1/r - 1/r_+)(1 + K_3/r)]^{1/2}}\, dr \tag{13a}$$

and

$$t = t_i \pm K_2^{-1/2} \int_{r_i}^{r} \frac{r^2 + (\gamma + 2\alpha)\, mr + Mm^2 + Nm^3/r}{r^2\,[(1/r_- - 1/r)(1/r - 1/r_+)(1 + K_3/r)]^{1/2}}\, dr \tag{13b}$$

where

$$K_1 = \frac{r_+^2(1 - B_+^{-1}) - r_-^2(1 - B_-^{-1})}{r_+ r_-(B_+^{-1} - B_-^{-1})} \tag{14a}$$

$$K_2 = \frac{r_+^2(1 - B_+^{-1}) - r_-^2(1 - B_-^{-1})}{r_+ r_-(H_+^{-1} - H_-^{-1})} \tag{14b}$$

$$K_3 = \frac{2(\eta - 4\alpha\beta + 4\alpha^3)\, m^3 (r_+^2 - r_-^2)}{r_+ r_-[r_+^2(1 - B_+^{-1}) - r_-^2(1 - B_-^{-1})]} \tag{14c}$$

$$M = \tfrac{1}{2}\,(4\delta - \gamma^2 + 4\alpha\gamma + 8\alpha^2 - 4\beta) \tag{14d}$$

$$N = \alpha(2M - 4\beta) + 2(\eta - \beta\gamma) \tag{14e}$$

With the introduction of a new variable ψ given by

$$\frac{1}{r} \equiv \frac{1}{L} - \frac{1}{\Gamma} \sin\psi$$

where

$$\frac{1}{L} \equiv \frac{1}{2}\left(\frac{1}{r_+} + \frac{1}{r_-}\right) \quad \text{and} \quad \frac{1}{\Gamma} \equiv \frac{1}{2}\left(\frac{1}{r_-} - \frac{1}{r_+}\right)$$

one can integrate expressions (13) and obtain the variables ϕ and t, which in terms of the original variable r are given by

$$\phi = \phi_i \pm K_1^{-1/2} \left\{ \left(TP + \frac{X}{4\Gamma^2} \right) \arcsin\left[\frac{r(r_+ + r_-) - 2r_+ r_-)}{r(r_+ - r_-)} \right] \right.$$
$$+ \left(UP - \frac{TQK_3}{2} - \frac{Y}{6\Gamma^2} \right) \frac{[(r - r_-)(r_+ - r)]^{1/2}}{(r_+ r_-)^{1/2} r}$$
$$- \frac{X}{8} \frac{[r(r_+ + r_-) - 2r_+ r_-] [(r - r_-)(r_+ - r)]^{1/2}}{(r_+ r_-)^{3/2} r^2}$$
$$\left. - \frac{Y}{48} \frac{[r(r_+ + r_-) - 2r_+ r_-]^2 [(r - r_-)(r_+ - r)]^{1/2}}{(r_+ r_-)^{5/2} r^3} \right\}_{r_i}^{r} \tag{15a}$$

and

$$t = t_i \pm K_2^{-1/2} \left\{ - \frac{L\Gamma^2 Z}{(\Gamma^2 - L^2)} \frac{[(r - r_-)(r_+ - r)]^{1/2}}{(r_+ r_-)^{1/2}} + \frac{\Gamma}{(\Gamma^2 - L^2)^{1/2}} \right.$$
$$\cdot \left[2(\gamma + 2\alpha) Zm + \frac{L^2 \Pi}{\Gamma^2 - L^2} \right] \arctan\left(\frac{r - r_-}{r_+ - r} \right)^{1/2}$$
$$+ \left[m^2 \Omega P + \frac{1}{4\Gamma^2} \left(\Lambda - \frac{15 W m^3 K_3{}^3}{128\Gamma^4} \right) \right] \arcsin\left[\frac{r(r_+ + r_-) - 2r_+ r_-}{r(r_+ - r_-)} \right]$$
$$- \left[(\gamma + 2\alpha) \Xi m - \left(3 - \frac{5K_3}{2L} \right) \frac{K_3{}^2}{4} \right] \arctan\left[\frac{r_+(r - r_-)}{r_-(r_+ - r)} \right]^{1/2}$$
$$- \left[\frac{(\gamma + 2\alpha)}{8} \left(3 - \frac{10K_3}{L} \right) mK_3{}^2 + \left(\frac{QK_3\Omega}{2} - PWm \right) m^2 + \frac{5K_3{}^3}{16} + \frac{\Sigma}{12\Gamma^2} \right]$$
$$\cdot \frac{[(r - r_-)(r_+ - r)]^{1/2}}{(r_+ r_-)^{1/2} r} - \frac{1}{8} \left[\Lambda - \frac{5}{8} \left(\gamma + 2\alpha + \frac{3Wm^2}{4\Gamma^2} \right) mK_3{}^3 \right]$$
$$\cdot \frac{[r(r_+ + r_-) - 2r_+ r_-] [(r - r_-)(r_+ - r)]^{1/2}}{(r_+ r_-)^{3/2} r^2}$$
$$- \frac{\Sigma}{96} \frac{[r(r_+ + r_-) - 2r_+ r_-]^2 [(r - r_-)(r_+ - r)]^{1/2}}{(r + r_-)^{5/2} r^3}$$
$$\left. + \frac{5Wm^3 K_3{}^3}{512} \frac{[r(r_+ + r_-) - 2r_+ r_-]^3 [(r - r_-)(r_+ - r)]^{1/2}}{(r_+ r_-)^{7/2} r^4} \right\}_{r_i}^{r} \tag{15b}$$

where K_1, K_2, and K_3 are given by equations (14a, b, c) and

$$P = 1 - \frac{K_3}{2L} + \frac{3K_3{}^2}{8L^2} - \frac{5K_3{}^3}{16L^3} \tag{16a}$$

$$Q = 1 - \frac{3K_3}{2L} + \frac{15K_3{}^2}{8L^2} \tag{16b}$$

$$R = 1 - \frac{5K_3}{2L} \tag{16c}$$

$$T = 1 + \frac{\gamma m}{L} + \frac{(4\delta - \gamma^2)\, m^2}{2L^2} \tag{16d}$$

$$U = \gamma m + \frac{(4\delta - \gamma^2)\, m^2}{L} \tag{16e}$$

$$V = (4\delta - \gamma^2)\, m^2 \tag{16f}$$

$$W = 2\alpha\delta + \alpha^2\gamma - 4\alpha\beta + \frac{3}{2}\alpha^3 + \frac{\alpha\gamma^2}{2} + \eta \tag{16g}$$

$$X = VP - UQK_3 + \frac{3TRK_3{}^2}{4} \tag{16h}$$

$$Y = K_3 \left(VQ - \frac{3URK_3}{2} + \frac{5TK_3{}^2}{4} \right) \tag{16i}$$

$$Z = LP + \frac{QK_3}{2} + \frac{3RK_3{}^2}{8L} + \frac{5K_3{}^3}{16L^2} \tag{16j}$$

$$\Pi = 2\Gamma^2 P + LQK_3 + \frac{3R(2L^2 - \Gamma^2) K_3{}^2}{4L^2} + \frac{5(3L^2 - 2\Gamma^2) K_3{}^3}{8L^3} \tag{16k}$$

$$\Lambda = m^2 K_3 \left(\frac{3R\Omega K_3}{4} - WQm \right) \tag{16l}$$

$$\Sigma = m^2 K_3{}^2 \left(\frac{5\Omega K_3}{2} - 3WRm \right) \tag{16m}$$

$$\Xi = K_3 \left(Q + \frac{9RK_3}{4L} + \frac{5(2\Gamma^2 + L^2) K_3{}^2}{16\Gamma^2 L^2} \right) \tag{16n}$$

$$\Omega = C + \frac{Wm}{L} \tag{16o}$$

One has thus derived expressions for the trajectories of photons [equations (11)] and test particles [equations (15)] accurate to third order in m/r.

§(4): *Classical Gravitational Tests*

The expressions derived for the trajectories of test particles and photons may now be used to calculate the values of the measurements performed in the actual experimental and observational tests [17, 19-21]. Every metric theory would predict different results since the set of parameters $\Delta = \{\alpha, \beta, \gamma, \delta, \eta, \lambda\}$ is different for each theory. Since the post-Newtonian calculations have already been made for most of the metric theories [17, 20, 21], the following discussion is restricted to the case of general relativity and illustrates how the realized extension modifies the previous calculations. The values for Δ in general relativity are $\{1, 0, 1, 1, 0, 1\}$.

4.1. Deflection of Electromagnetic Signals by the Sun. Figure 1 shows a light ray ~ approaching the sun from a very great distance. The total change in ϕ as r decreases from infinity to its minimum value r_0 and then increases again to infinity is just twice its change from ∞ to r_0, that is, $2|\phi(r_0) - \phi(\infty)|$. If the trajectory were a straight line, this would just equal π; hence the deflection of the orbit from a straight line is

$$\Delta\phi_l = 2|\phi(r_0) - \phi(\infty)| - \pi$$

If this is positive then the angle ϕ changes by more than 180°, that is, the trajectory is bent toward the sun; if $\Delta\phi_l$ is negative, then the trajectory is bent away from the sun.

Setting $r = r_0$ and $r_i = \infty$ in equation (11a) and evaluating gives

$$\begin{aligned}\phi(r_0) = \phi(\infty) \pm \Bigg\{ &-\frac{\pi}{2} - (\alpha+\gamma)\frac{m}{r_0} - \left[\frac{E\pi}{8} - \alpha(\alpha+\gamma)\right]\frac{m^2}{r_0^{\,2}} \\ &+ \left[\frac{\alpha E\pi}{4} - \alpha\left(\frac{67\alpha^2}{6} - 9\beta + \frac{9\alpha\gamma}{2} + 4\delta - \gamma^2\right)\right. \\ &\left. - \gamma\left(\frac{\gamma^2}{3} - \beta - \frac{4\delta}{3}\right) - \frac{8\lambda}{3} - 2\eta\right]\frac{m^3}{r_0^{\,3}} \Bigg\}\end{aligned}$$

Hence to third order in m/r the deflection is

$$\begin{aligned}\Delta\phi_l = 2(\alpha+\gamma)\frac{m}{r_0} &+ \left[\frac{E\pi}{4} - 2\alpha(\alpha+\gamma)\right]\frac{m^2}{r_0^{\,2}} - \left[\frac{\alpha E\pi}{2} - \alpha\left(\frac{67\alpha^2}{3} - 18\beta\right.\right. \\ &\left.\left. + 9\gamma\alpha + 8\delta - 2\gamma^2\right) - 2\gamma\left(\frac{\gamma^2}{3} - \beta - \frac{4\delta}{3}\right) - \frac{16\lambda}{3} - 4\eta\right]\frac{m^3}{r_0^{\,3}}\end{aligned}$$

Taking $m = 1.47664 \times 10^5$ cm and $r_0 = R_\odot = 6.9598 \times 10^{10}$ cm one obtains

$$\Delta\phi_l = 1.7505(10764962082) \text{ arc sec}$$

the digits in brackets are uncertain due to our limited knowledge of m and $R_\odot$.

4.2. Advance of the Planets' Perihelia. Consider now a test particle bound in an orbit around the sun. For this type of motion, the change in ϕ as r decreases from r_+ to r_- is the same as the change in ϕ as r increases from r_- to r_+, so the total change in ϕ per revolution is $2|\phi(r_+) - \phi(r_-)|$. This change would equal 2π if the orbit were a closed ellipse, so in general the orbit precesses in each revolution by an angle

$$\Delta\phi_p = 2|\phi(r_+) - \phi(r_-)| - 2\pi$$

If $\Delta\phi_p$ is positive (negative) the whole orbit should precess in the same (opposite) direction as the motion of the test particle. Equation (15a) with $r = r_+$ and $r_i = r_-$ gives

$$\phi(r_+) = \phi(r_-) \pm \pi K_1^{-1/2} \left(TP + \frac{X}{4\Gamma^2}\right)$$

Hence to third order in m/r the precession per revolution is

$$\Delta\phi_p = 2\pi \left[K_1^{-1/2} \left(TP + \frac{X}{4\Gamma^2}\right) - 1\right]$$

which for the cases of Mercury, Venus and Icarus gives the values 0.429799, 0.862494×10^{-1}, and 0.100827 arc sec/year, respectively.

4.3. Time Delay of Electromagnetic Signals. From Figure 1 we see that the time required for an electromagnetic signal to go from one point $(r_1, \theta_1 = \pi/2, \phi_1)$ to a second point $(r_2, \theta_2 = \pi/2, \phi_2)$ is given by

$$t(r_1, r_2) = \begin{cases} t(r_1, r_0) + t(r_0, r_2) & \text{if } |\phi_1 - \phi_2| > \pi/2 \\ t(r_1, r_0) - t(r_0, r_2) & \text{if } |\phi_1 - \phi_2| < \pi/2 \\ t(r_1, r_0 = r_2) & \text{if } |\phi_1 - \phi_2| = \pi/2 \end{cases}$$

The time required for light to go from r to r_0 is given by equation (11b). The leading term $[r^2 - r_0{}^2]^{1/2}$ is what one should expect if light traveled in straight lines at unit velocity, the other terms produce a general-relativistic time delay.

For a light signal going from the earth to Mercury being near superior conjunction (so that the signal just grazes the sun) and back, the maximum round-trip time delay is given by

$$\Delta t = 2\,\{t(r_\oplus, R_\odot) + t(R_\odot, r_♀) - [r_\oplus^2 - R_\odot^2]^{1/2} - [r_♀^2 - R_\odot^2]^{1/2}\}$$

Using equation (11b), inserting the values for m, $R_\odot$, and Δ mentioned above and taking $r_♀ = 5.9197 \times 10^{12}$ cm and $r_\oplus = 1.4959 \times 10^{14}$ cm one obtains

$$\Delta t = 7.7281(253) \times 10^6 \text{ cm} = 2.5778(247) \times 10^{-4} \text{ sec}$$

where the brackets enclose digits which are uncertain due to limited knowledge of m and $R_\odot$. The value in seconds has also an uncertainty due to the value for the speed of light ($c = 2.997925 \times 10^{10}$ cm/sec).

4.4. Experimental and Observational Measurements. The experimental and observational values for the three discussed effects are

$\Delta\phi_l = 1.761 \pm 0.016$ arc sec (References 22–24)

$\Delta\phi_p = 43.16 \pm 0.21$ arc sec/century for the planet Mercury (Reference 25)

$\Delta t \simeq 250 \pm 1.5\ \mu$sec (References 26 and 27)

These values are, respectively, 1.006, 1.004, and 0.9698 times the values predicted by general relativity using the expressions derived here. From these calculations it is apparent that the first relativistic correction essentially is m/r orders of magnitude less than the Newtonian value, that the second correction is in turn m/r orders of magnitude less than the first one and that the same relation exhibits the third correction with respect to the second one. The corresponding predicted values being reported would then be m/r orders of magnitude more accurate than the calculations carried out using the post-Newtonian formalism if our determinations of the values of the planetary orbital and solar elements were such as to make such accurate calculations possible.

§(5): *Conclusion*

The improvement reported above is far greater than the obtainable accuracy with present experiments and observations. For a discussion on the phenomena that limit the observational or experimental resolution and the accuracy achieved so far the reader is referred to the original reports [22-27].

Advances in technology as well as the gathering of more data are expected to provide us with more accurate experiments and more precise observational results in the near future. There is, however, a planned solar mission which may provide values accurate at least to second order in m/r [28-33]. An extension of the analysis to order $(m/r)^{3/2}$ for modeling the experimental results of such a mission has been performed by Hechler [34]. The derivation of more general expressions than those already published and the calculations that will predict the outcome of the mentioned solar mission are currently under progress and will be published later on [35].

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