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THE IMPRESSIVE COMPLEXITY IN THE *NAUTILUS POMPILIUS* SHELL

A. A. CASTREJÓN PITA and J. R. CASTREJÓN PITA
Centro de Investigación en Energía, UNAM
Ap. Postal 34, 62580 Temixco, Morelos, México

A. SARMIENTO GALÁN
Instituto de Matemáticas, UNAM
Av. Universidad s/n, 62200 Chamilpa, Morelos, México
ansar@matcuer.unam.mx

R. CASTREJÓN GARCÍA
Instituto de Investigaciones Eléctricas
Av. Reforma 113, 62490 Temixco, Morelos, México

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Abstract

The complexity of the *Nautilus pompilius* shell is analyzed in terms of its fractal dimension and its equiangular spiral form. Our findings assert that the shell is fractal from its birth and that its growth is dictated by a self-similar criterion (we obtain the fractal dimension of the shell as a function of time). The variables that have been used for the analysis show an exponential dependence on the number of chambers/age of the cephalopod, a property inherited from its form.

Keywords: Self-Similarity; Fractality; Adaptation; Nature Dynamics.

1. INTRODUCTION

Fractal analysis is being applied with increasing frequency to living organisms, trying to explain some of the complex forms found in nature. An astonishing example reveals that *Ammonites* continuously increased their complexity up to the point in which they became extinct.[1] It is our purpose to study in this paper the amazing complexity of a close relative of the *Ammonites*, the *Nautilus pompilius.*

This pelagic species is a native of the western Indopacific ocean (30° N latitude to 30° S latitude

and 90° to 185° W longitude[2]), and usually lives at a depth that varies from 50 to 480 meters (temperature ranges from 24 to 8°. The *Nautilus* reaches sexual maturity at least 15 years after hatching and then produces ten to 15 eggs per year (it is not known if the female breeds more than once), and it may live for up to 20 years.

The shell is mother-of-pearl lined and pressure resistant (it implodes at approximately 800 m); its hardness has been the basis of various ornamental handicrafts.[3] The vulnerability due to their slow reproduction rate, and the fact that its exploitation has increased so much at present, make it a possible new addition to the large list of endangered species. But the most striking characteristic of this thin, two-layered, and spirally coiled shell is its internal subdivision in a series of successive chambers (phragmocone), starting from the very moment of hatching when there are already seven chambers present in the shell. As the cephalopod grows and requires more space, it creates a new chamber by sealing the space behind it with a calcareous septum and moves to live at the open, bigger end of the shell. The rate at which a new chamber is created varies, at the beginning it seems to take longer for the mollusc to seal the 8th chamber but later on, the process takes from 43 to 77 days per chamber[4] and lasts up to the completion of approximately 39 sealed chambers[5] plus the open space where the mollusc lives;[6] these changes in the growth rate are easily understood in terms of the food availability and other environmental variables. The sealing of the chambers however, is not complete, there is a small duct in the center of each wall, called siphuncle, that allows the living fossil to keep control of the pressure inside every previous chamber and thus to regulate its buoyancy;[7,8] the heyday of the nautiluses is estimated to be around 500 million years ago.

A transversal cut of the shell (Fig. 1) shows a perplexing spiral geometry, not found in any other natural object; this is a black and white image where the borders have been prepared to facilitate the box-counting analysis. The hemishell is 96.1×106.2 mm and 32.2 mm wide; the number of chambers is 30. Most amazing is the fact that its growth appears to be self-similar, and thus for the shell to possess a fractal dimension. We now proceed to confirm that this is indeed so.

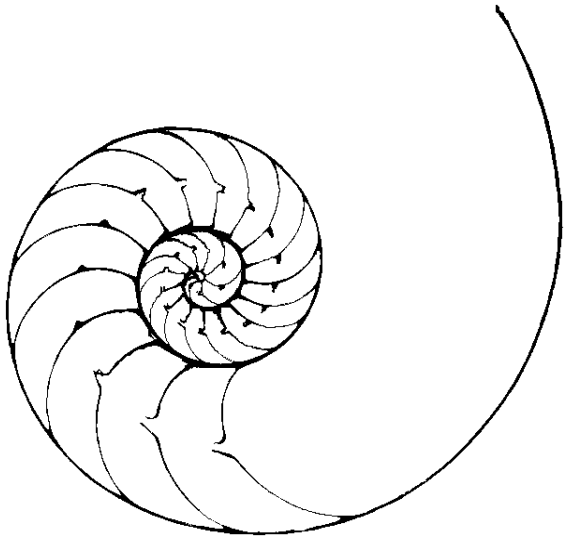


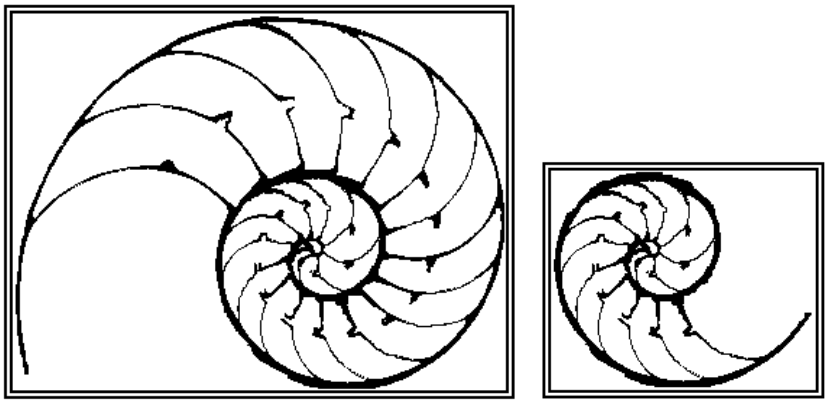
Fig. 1 Black and white image of a transversal cut of a *Nautilus pompilius* shell (see text).

2. METHOD

The digital image in Fig. 1 was obtained by placing half of the shell directly on a scanner bed; the cutting was performed going through half of the shell as accurately as possible. All measurements are performed on the digital images, in pixel units, and the conversion factor is given by the scanner resolution (72 pixels per inch). The borders of the edges in the hemishell were previously tinted to gain contrast and improve definition, and thus, making the contour threshold treatment unnecessary. It must be said that we are assuming a perfect symmetry of the shell with respect to this cut.

As usually done when applying the box-counting method, it is necessary to define a criterion for the size of the boxes, in particular for the maximum possible size of the grid. This last value is easily determined: the use of boxes bigger than the image size would produce constant values from there on and thus a breakdown of the method. The box-counting method is applied to the original image and the fractal dimension of the whole shell is obtained via a linear fit to the data.[9,10]

The previous selection however, means that the maximum possible size of the grid will change when analyzing portions instead of the whole image; this is precisely what happens if, in order to test the observed self-similarity, we analyze the fractal



(a) **(b)**

Fig. 2 Figure showing the images obtained by digitally altering the number of chambers in the shell: **(a)** 25 chambers and **(b)** 15 chambers.

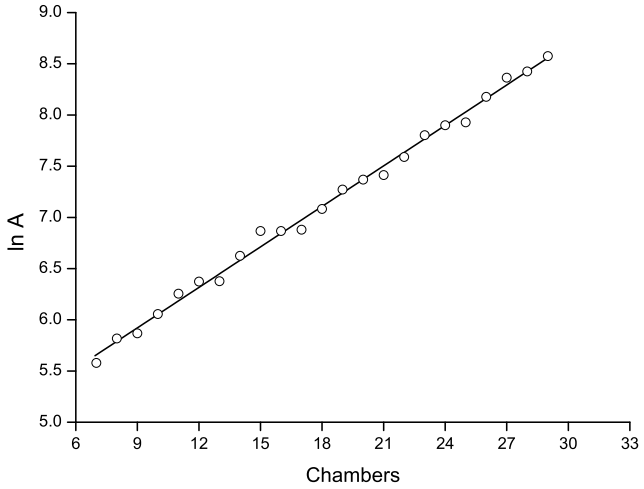


Fig. 3 Area (A) of the rectangle in which the *Nautilus* is inscribed as a function of the number of chambers in the shell.

dimension of smaller fragments of the image, i.e. if we check that its complex structure is the same regardless of the scale used to measure it. To accomplish this test, we proceeded as follows; once the box-counting method had been applied to the whole, bigger image, the last chamber was digitally eliminated from the initial image and the method re-applied to the new image after adjusting the maximum possible size to the new, smaller image size (Fig. 2). This procedure was repeated up to the point in which there were only the original seven chambers in the shell. The area of the circumscribed rectangle was calculated for each step and the results are shown in Fig. 3 in mm^2.

From the above procedure, we can also obtain the value of the different intersections of the various straight lines with the vertical axis, i.e. the ordinates of each one of the lines obtained by the self-similar test. Since these lines correspond to a different chamber number each, the result is a function that can be used to predict the position and time of appearance of the new chamber (and thus corroborate the average time mentioned earlier).

Now, according to the well known box-counting method, the fractal dimension of an object, D, is defined as:[9]

$$D = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln 1/\varepsilon} \tag{1}$$

where $N(\varepsilon)$ is the number of boxes of a square grid of side-size ε required to cover the object in question. This definition comes from the scaling law $N(\varepsilon) = C(1/\varepsilon)^D$, in which our knowledge on integer-dimension objects is clearly expressed (one needs c/ε boxes of side-length ε for each one of the D dimensions of the object to be covered, where c accounts for the "length" in that dimension, and the ε-independent constant C is merely the product of the c's). From this scaling law, a linear relation is obtained:

$$\ln N(\varepsilon) = D \ln(1/\varepsilon) + \ln C \tag{2}$$

and from it, definition (1) follows (the ε-independent, constant term becomes negligible as $\varepsilon \to 0$). We then see that the ordinate of the linear relation (2) can be interpreted as the natural logarithm of the number of pixels of the original image

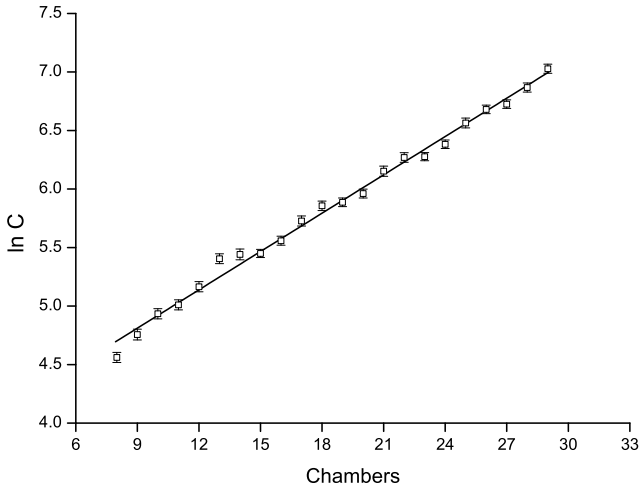


Fig. 4 Ordinates (C) of the lines obtained via the box-counting method as a function of the number of chambers.

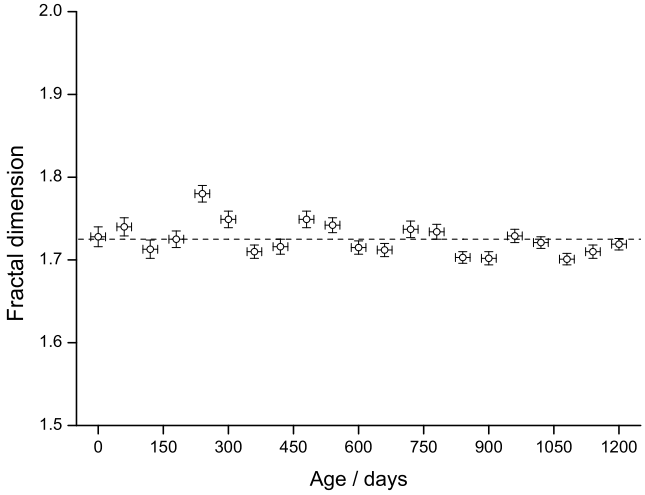


Fig. 5 Fractal dimension of the shell as a function of time, the age is measured in days after hatching, starting with the eighth chamber.

(in practice, the grid cannot be made smaller than a pixel when $\varepsilon \to 0$).

To obtain adequate images for the use of the whole relation (2) is not an easy task, most of the known fractal objects do not have well-defined features like borders or surfaces.[11] In this case however, we have all the images that were digitally generated to test the self-similarity of the previously tinted shell (enabling us to avoid any contour threshold analysis[11]), and thus all the data to build the graph in Fig. 4, where we have plotted the image sizes (in mm^2) of the shell circumscribed by a rectangle up to the next sealed chamber (Fig. 2) which correspond to the ordinates in the linear relation (2) (it is perhaps worth recalling that the box-counting method only takes into account the pixels associated to the contour of the shell when the size of the box is unity). We have also used an average value for the time required for the construction of a new chamber in order to obtain the fractal dimension of the shell as a function of time (Fig. 5), this average value is 60 ± 17 days per chamber.

Having corroborated the fractal dimension of the shell at various scales and the advantage of using the whole linear relation (2), we can now count the number of pixels (instead of boxes) inside each chamber; the results are plotted in Fig. 6, where a linear relation is also clearly seen.

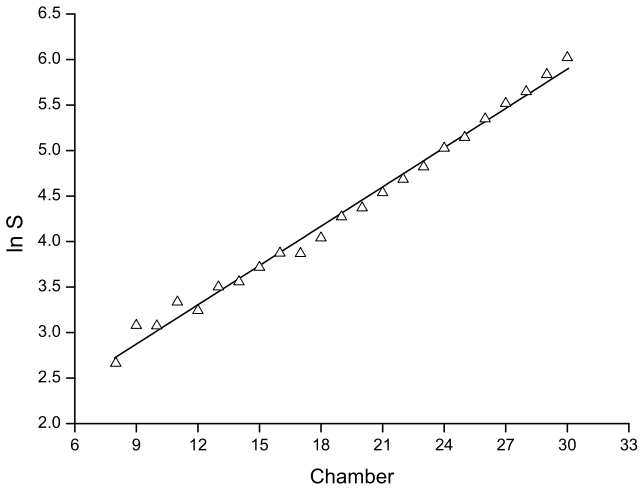


Fig. 6 Area of each chamber in pixels^2 (S) as a function of the number of chambers.

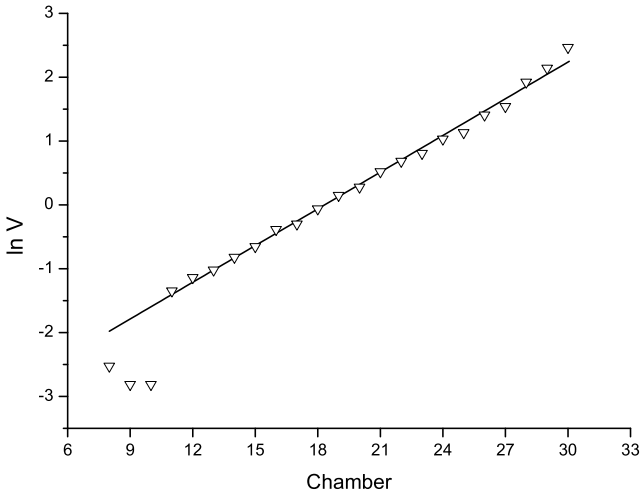


Fig. 7 Volume of each chamber in ml (V) as a function of the number of chambers.

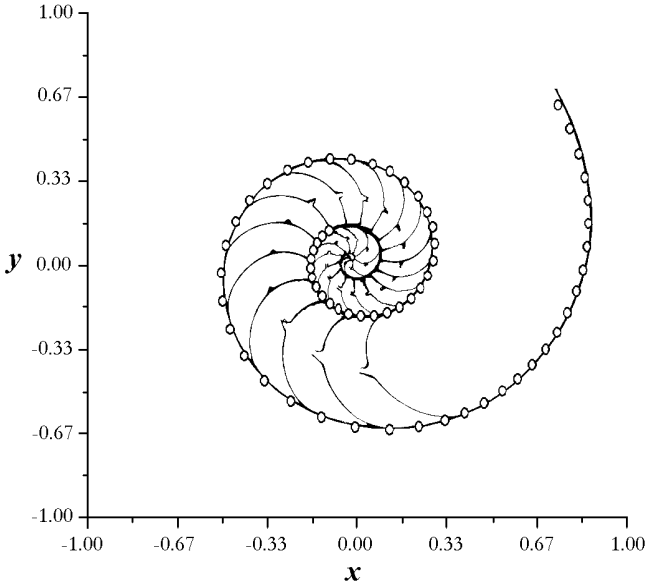


Fig. 8 Equiangular spiral (open circles) superimposed on the image of the *Nautilus* shell.

All these data enables us to predict the size of the new chambers and the time of their appearance, i.e. those chambers that would have formed in a living specimen. The easiest way is to measure the volume of each chamber and in order to achieve this, the hemishell was levelled on a Sartorius balance (5 mg precision) and each chamber was filled up with water by depositing one by one, 0.005 ml drops; the volume is then multiplied by two to account for the other half of the shell. The internal surface of the shell inhibits the formation of menisci and thus, the level of water inside each shell is a flat surface that rises uniformly. The results are shown in Fig. 7.

As a final check, we fitted an equiangular spiral to the shell:[12]

$$r = e^{\delta\theta} \tag{3}$$

where (r, θ) are the usual polar coordinates and δ is a parameter that can be determined by the quotient of the shell distance from the center in any direction and the same distance after a whole turn:

$$\ln\left(\frac{r_1}{r_2}\right) = 2\pi\delta\,. \tag{4}$$

The value obtained from this quotient $\delta = (1 + \sqrt{5})/2$, is the well-known *golden ratio* and the resulting spiral is superimposed on the shell image and shown in Fig. 8; the spiral is represented by empty circles starting at the closing wall of the eighth chamber.

3. RESULTS AND PREDICTIONS

Applying the Intersection Theorem,[11] we find that the fractal (box-counting) dimension of the original *Nautilus* shell shown in Fig. 1 is 2.635 ± 0.006, and the average of the self-similar fractal dimension of the shell (Fig. 5), obtained by the method exemplified in Fig. 2, is 2.730 ± 0.019; this is an average over the life of the particular *Nautilus* and clearly depends on the accuracy of the available data on the shell growth. The lower value for fractal dimension of the original shell with respect to the average, is due to the fact that the shell extension where the mollusc lives is included in the original image (Fig. 1).

The other variables used in the preceding analysis show an exponential dependence on the number of chambers or, equivalently, on the age of the cephalopod:

$$y = \alpha_i e^{\beta_i x} \tag{5}$$

where y is one of the properties described in Figs. 3, 4, 6 or 7, α_i is the exponential of the ordinate in the linear relation shown in figure i, β_i is the slope in the corresponding relation (Fig. i), and x is the number of chambers in the image under analysis (or equivalently, the age of the specimen). The relation obtained in Fig. 3 was subsequently used as the criterion for the box-counting interval. The results for the ordinates are: $\alpha_3 = 113.863 \pm 0.039$ mm^2, $\alpha_4 = 46.016 \pm 0.039$ mm^2, $\alpha_6 = 4.852 \pm 0.050$ mm^2, and $\alpha_7 = 0.030 \pm 0.072$ ml; for the slopes the values are: $\beta_3 = 0.132 \pm 0.002$, $\beta_4 = 0.109 \pm 0.002$, $\beta_6 = 0.143 \pm 0.003$, and $\beta_7 = 0.192 \pm 0.003$. The correlation factors for the linear relations in the above mentioned figures are: 0.996, 0.996, 0.992, and 0.997 (in this case the linear fit was performed from the 11th chamber onwards, see Fig. 7), respectively.

From our findings and within the shell growth accuracy, we can also predict the appearance of a new chamber, the 31st for the analyzed specimen, with a volume of 11.347 ± 0.010 ml or a transversal area of 415.71 ± 0.005 mm^2 at 60 ± 17 days after the last one had been completed.

4. CONCLUSIONS

In the previous analysis, we have shown that the shell of the *Nautilus pompilius* possesses a fractal

dimension, that its value is 2.635 ± 0.006 (2.730 ± 0.019 on average), and that it does not depend on the number of chambers (or, equivalently, the age) used to calculate it. This establishes the self-similar structure of the shell at any scale/time, and how its growth follows the same self-similar criterion. Hence, we propose the measurement of the predicted appearance of the new chambers in living *Nautilus*, even if only in laboratory specimens. Recalling that the *Ammonites*, close relatives of the *Nautilus*, kept changing their structure and disappeared from the face of the earth, one could also conjecture that the preservation of the structure in the *Nautilus* meant an evolutive advantage for this species. This facet might help in the conservation of this complex living fossil.

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