Energy-density spectrum of the vacuum around a cosmic string

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The explicit form of the spectrum of the energy density of the vacuum surrounding a cosmic string as would be seen by an observer at rest is calculated. Spin-0, -$\frac{1}{2}$, or -1 massless fields are considered and it is found that the result is independent of the spin value. An interpretation which differs from the one usually found in the literature is also given.

Among the interesting, fundamental issues raised by the possible existence of cosmic strings and the new phenomena predicted to occur as a consequence of the gravitational field of a string lies the effect on the quantum vacuum in the vicinity of a string. Although almost a decade has elapsed since the possibility of string formation was suggested[1] and its possible cosmological relevance assessed,[2–5] most of the research on the influence of a string is very recent, in particular the conclusion that an energy density is "induced" by the string in the surrounding vacuum, even if the spacetime is flat.[6–12]

The most recent analysis known to the authors is mainly concerned with vacuum polarization, particle detectors response, and the stress-energy-momentum tensor.[13,14] However, with the exception of a particular case,[14] the precise form of the energy-density spectrum has not been calculated in general.

It is the purpose of this paper to calculate explicitly the form of the spectrum of the energy density of the vacuum near a cosmic string as would be seen by an observer at rest. The calculations that follow use a simple and straightforward formalism that has been presented elsewhere[15] and which enables a direct and simple interpretation of the results. The analysis presented here is restricted to spin-0, -$\frac{1}{2}$, or -1 massless fields; the consideration of other fields is left for future publications.

The metric describing the geometry of spacetime around a straight cosmic string at distances larger than the radius of the string ρ_0 ($=l_P m_P/m_{\rm GUT}\sim 10^{-29}$ cm for a typical energy scale of grand unified theories[4] $m_{\rm GUT}\sim 10^{15}$ GeV) is given by[6,10,16–21]

$$ds^2=-dt^2+dz^2+d\rho^2+\rho^2 d\phi^2 \ , \tag{1}$$

where ϕ is a cyclical coordinate with a period equal to $2\pi/\nu$, $\nu^{-1}=1-4\mu$, and $\mu c^2/G$ is the constant mass per unit length of the string $[=\lambda^{-1}(m_{\rm GUT}/m_P)^2\sim 10^{-6}$ for $\lambda=10^{-2}$ and $m_{\rm GUT}\sim 10^{15}$ GeV]. This corresponds to a locally flat spacetime that has a conelike singularity at $\rho=0$ with an angle deficit $\delta\phi=8\pi\mu$.

The Wightman functions for a spin-0 field, solutions of the scalar wave equation in the spacetime represented by (1), are given by[7,8,12,22,23]

$$W^{\pm}(x,x')=\frac{1}{8\pi^2\rho\rho'}\frac{\nu}{\sinh\theta}\frac{\sinh(\nu\theta)}{\cosh(\nu\theta)-\cos(\nu\Delta\varphi)} \ , \tag{2}$$

where

$$\cosh\theta=\frac{\Delta z^2+\rho^2+\rho'^2-(t-t'\mp i\epsilon)^2}{2\rho\rho'}$$

and $\Delta\phi=\phi-\phi'$. For an observer at rest, we choose the points $x=(\tau+\sigma/2,\rho,\phi,z)$ and $x'=(\tau-\sigma/2,\rho,\phi,z)$, obtaining

$$W^{\pm}(\sigma)=\frac{1}{8\pi^2\rho^2}\frac{\nu}{\sinh\theta_\pm}\coth\left[\frac{\nu\theta_\pm}{2}\right] , \tag{3}$$

where

$$\sinh(\theta_\pm/2)=i\left[\frac{\sigma\mp i\epsilon}{2\rho}\right] .$$

According to the formalism, we now need the Fourier transform of the Wightman functions:

$$\tilde{W}^{\pm}(\omega)=\int_{-\infty}^{\infty}W^{\pm}(\sigma)e^{i\omega\sigma}d\sigma \ , \tag{4}$$

which, for the present case reduces to (see the Appendix)

$$\tilde{W}^{+}(\omega)=-\frac{\nu}{8\pi^2\rho}I_\nu^+(2\omega\rho) \ , \tag{5a}$$

$$\tilde{W}^{-}(\omega)=0 \ , \tag{5b}$$

where

$$I_\nu^+(t)=-\frac{2\pi t}{\nu}+2\sin(\nu\pi)I_\nu(t) \ ,$$

and I_ν is a function which is defined in the Appendix.

In general, the energy density is given by[15]

$$\frac{de}{d\omega}=\frac{\omega^2}{\pi}[\tilde{W}^{+}(\omega)+\tilde{W}^{-}(\omega)] \ . \tag{6}$$

Therefore, around a cosmic string,

$$\frac{de}{d\omega}=\frac{\omega^3}{2\pi^2}\left[1-\frac{\nu}{2\pi\omega\rho}\sin(\nu\pi)I_\nu(2\omega\rho)\right]. \qquad (7)$$

Thus, unless ν is an integer, a cosmic string produces a distortion on the usual vacuum energy spectrum, $de/d\omega=\hbar\omega^3/2\pi^2c^3$, in a way which has been thoroughly discussed elsewhere.[15]

If $\omega\rho \ll 1$, Eq. (7) reduces to (see the Appendix)

$$\frac{de}{d\omega}=\frac{\omega^3}{2\pi^2}\nu, \qquad (8a)$$

while for $\omega\rho \gg 1$, the asymptotic form of Eq. (7) is[24]

$$\frac{de}{d\omega}=\frac{\omega^3}{2\pi^2}\left[1-\frac{\nu}{2(\pi\omega^3\rho^3)^{1/2}}\cot(\nu\pi/2)\times\sin(2\omega\rho+\pi/4)\right]. \qquad (8b)$$

Figure 1 shows a plot of $(2\pi^2c^3/\hbar\omega^3)(de/d\omega)$ versus $\omega\rho/c$ for $\nu=1.000004$, a value derived using the estimates mentioned earlier; the integrations in Eq. (7) were performed numerically.

Similarly, the energy density for a spin-$\frac{1}{2}$ field is given by[25]

$$\frac{de}{d\omega}=\frac{iu^\mu}{\pi}\int_{-\infty}^{\infty}\omega e^{i\omega\sigma}\frac{\partial}{\partial x^\mu}[W^+(\tau+\sigma/2,\tau-\sigma/2)-W^-(\tau+\sigma/2,\tau-\sigma/2)]d\sigma. \qquad (9)$$

Since $u^\mu=(1,0)$ for an observer at rest, we only need to calculate the Fourier transform of $\partial W^\pm/\partial t$. Now, since

$$\frac{\partial W^\pm}{\partial t}=\frac{\partial\theta}{\partial t}\frac{dW^\pm}{d\theta},$$

where $\sinh(\theta_\pm/2)=i(\sigma\mp i\epsilon)/2\rho$, it follows that

$$\int_{-\infty}^{\infty}\frac{\partial W^\pm}{\partial t}e^{i\omega\sigma}d\sigma=\int_{\Gamma^\pm}\frac{dW^\pm}{dz_\pm}e^{2i\omega\rho\sin(z_\pm)}dz_\pm =2i\omega\rho\int_{\Gamma^\pm}\cos(z_\pm)W^\pm dz_\pm, \qquad (10)$$

where $\theta_\pm=2iz_\pm$ and $\Gamma^\pm$ are the integration contours shown in Fig. 2. Thus, exactly the same result as in the scalar case is obtained: substituting (10) in (9), the energy spectrum turns out to be as in Eq. (7).

For the electromagnetic (spin-1) field, the energy density is given by[25]

$$\frac{de}{d\omega}=-\frac{1}{\pi}u^\mu u^\nu\int_{-\infty}^{\infty}e^{i\omega\sigma}\frac{\partial^2}{\partial x^\mu\partial x^\nu}\times[W^+(\tau+\sigma/2,\tau-\sigma/2)-W^-(\tau+\sigma/2,\tau-\sigma/2)]d\sigma. \qquad (11)$$

This time, the only term that needs to be calculated is

$$\int_{-\infty}^{\infty}\frac{\partial^2W^\pm}{\partial t^2}e^{i\omega\sigma}d\sigma=\int_{\Gamma^\pm}\frac{d}{dz_\pm}\left[\sec(z_\pm)\frac{dW^\pm}{dz_\pm}\right]\times e^{2i\omega\rho\sin(z_\pm)}dz_\pm =-(2\omega\rho)^2\int_{\Gamma^\pm}\cos(z_\pm)W^\pm dz_\pm. \qquad (12)$$

Substituting in (11), we again obtain the energy spectrum of the scalar case, Eq. (7) and Fig. 1.

The energy density found to exist in the vacuum surrounding a cosmic string can be interpreted in a way

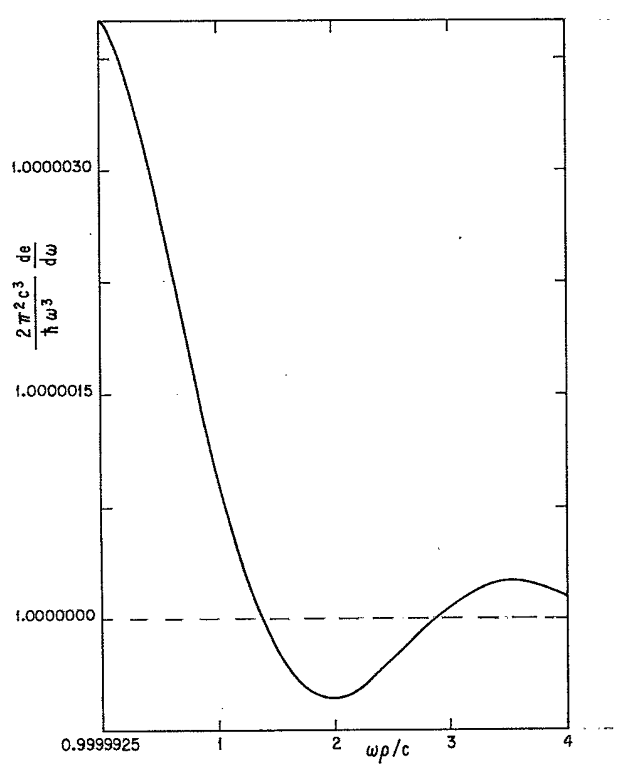

FIG. 1. Energy-density spectrum of the vacuum surrounding a cosmic straight string. The curve has been obtained with $\nu=1.000004$ in agreement with the values mentioned in the text. As a function of ν the curve follows the same general pattern, an increase in ν merely increases the amplitude of the oscillations.

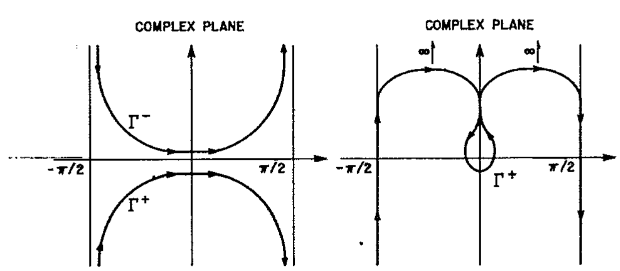

FIG. 2. The integration contours used for evaluating the integrals in the Appendix. For practical calculations, it is convenient to deform Γ^+ as shown.

which agrees with our previous work:[15] the spectrum given by Eq. (7) is a distortion of the (Lorentz-invariant) zero-point field assumed by stochastic electrodynamics to be the vacuum state. The distortion is created by the gravitational field of the string only and no other processes such as energy induction or particle creation are necessarily invoked in order to understand the result. The result being independent of the spin of the field does not seem to be interpretable in a clear and simple way.

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APPENDIX

Integral (4) can be rewritten as

$$\tilde{W}^{\pm}(\omega)=-\frac{\nu}{8\pi^2\rho}\int_{-\infty}^{\infty}\frac{\cot(\nu z_{\pm})e^{its}}{\sin(z_{\pm})\cos(z_{\pm})}ds ,$$

where we have set $s=\sigma/2\rho$, $t=2\omega\rho$, $z_{\pm}=-i\theta_{\pm}/2$, and thus $\sin(z_{\pm})=s\mp i\epsilon$. Since $s=\sin(x+iy)\pm i\epsilon=\sin(x)\cosh(y)+i[\cos(x)\sinh(y)\pm\epsilon]$ we define the curves $\Gamma^{\pm}$ as those on which $\mathrm{Im}(s)=0$, i.e., $\Gamma^{\pm}:\sinh(y)=\mp\epsilon/\cos(x)$, see Fig. 2. The integral then becomes Eqs. (5):

$$\tilde{W}^{\pm}(\omega)=-\frac{\nu}{8\pi^2\rho}\times\begin{cases}\int_{\Gamma^+}\frac{\cot(\nu z)}{\sin(z)}e^{it\sin(z)}dz=I_{\nu}^{+}(t) & \text{where } \sin(z)=\sin(x)\cosh(y),\\ 0 & \text{since } \Gamma \text{ can be closed in the upper semiplane without enclosing any poles}.\end{cases}$$

The integral $I_{\nu}^{+}(t)$ can be evaluated in a simple way. Deforming Γ^+ as in Fig. 2, we get

$$I_{\nu}^{+}(t)=-i\int_{-\infty}^{\infty}\frac{\cot[\nu(-\pi/2+iy)]}{\cosh(y)}e^{-it\cosh(y)}dy$$

$$+i\int_{\infty}^{-\infty}\frac{\cot[\nu(\pi/2+iy)]}{\cosh(y)}e^{it\cosh(y)}dy+2\pi i\,\mathrm{Res}\left[\frac{e^{it\sin(z)}\cot(\nu z)}{\sin(z)}\right]_{z=0}$$

or

$$I_{\nu}^{+}(t)=2\sin(\nu\pi)\int_{0}^{\infty}\frac{\sin[t\cosh(y)]dy}{\cosh(y)[\cosh^2(\nu y)-\cos^2(\nu\pi/2)]}-\frac{2\pi t}{\nu}=-\frac{2\pi t}{\nu}+2\sin(\nu\pi)I_{\nu}(t) , \tag{A1}$$

where it can be readily seen that $I_1^{+}(t)=-2\pi t$, and expanding $\sin[t\cosh(y)]$ for $t\ll1$, one gets[26]

$$I_{\nu}^{+}(t)\approx2t\left[-\frac{\nu}{\pi}+\frac{\sin2\alpha}{2\nu\cos\alpha}\left[\int_0^{\infty}\frac{dx}{\cosh x-\cos\alpha}-\int_0^{\infty}\frac{dx}{\cosh x+\cos\alpha}\right]\right]=-2\pi t ,$$

where $\alpha=\nu\pi/2$, this expression leads to Eq. (8a) and the last expression in (A1) is the equation following Eqs. (5), the rest is just algebra.

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