On Milnor's fibration Theorem for real and complex singularities*

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Milnor's fibration theorem for complex singularities is a key-stone in singularity theory. This is a result about the topology of the fibres of analytic functions near their critical points. This result, proved in the late 1960's, has given rise to a vast literature and it has gripped the attention of many people for a long time.

In this expository article we give an introduction to Milnor's theorem as well as to some of its generalisations and refinements that have appeared in the literature. We focus on explaining the main ideas in each of the topics we envisage and we refer to the literature for details and complete proofs.

This article does not pretend to be comprehensive of the subject. In fact this is threaded towards the end of presenting certain aspects in which I have been particularly interested in the last few years, and in which I have done some contributions, both by myself and in collaboration with several colleagues. Yet, we do cover in this work a rather wide spectrum that allows us to give an overall picture of some of the developments in this branch of singularity theory.

We first motivate the fibration theorem by proving it in a concrete example, the case of the celebrated Pham-Brieskorn singularities, where things are much simpler and yet one has the whole complexity and richness of the situation in general. Understanding this example helps us to grasp better the general case. We then present Milnor's theorem for holomorphic functions in its two usual ways: as a fibration on a small sphere around a critical point,

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and also as a fibration of the so-called Milnor tube. As mentioned before, we only sketch here some of the main ideas and refer to the literature for complete proofs.

In section 3 we mention briefly generalisations of Milnor's theorem that have been given by various authors, such as Helmut Hamm, Lê Dũng Tráng and others. We then look, in §4, at the case of real analytic functions with a Milnor fibration. We explain briefly Milnor's theorem for real singularities and two of its main differences with the complex case, namely: i) that the theorem holds only for analytic map-germs $f : (\mathbb{R}^{n+k}, \underline{0}) \to (\mathbb{R}^k, 0)$ which are submersions on a punctured neighbourhood of $\underline{0} \in \mathbb{R}^{n+k}$, a rather stringent hypothesis that we call *the Milnor condition* for the map-germ; and ii) that even if f satisfies the Milnor condition, it is not always true that the projection map of the associated fibration can be taken to be the obvious map $f/\|f\|$, as in the complex case. When this is also satisfied we say that f satisfies *the strong Milnor condition*.

In §5 we study a specially interesting family of real analytic singularities with the strong Milnor condition, called the *twisted Pham-Brieskorn singularities* in [37]. These have a rich geometry behind, coming from the theory of holomorphic vector fields, and I believe that the understanding of their associated Milnor fibrations can provide interesting new insights on some aspects of the geometry of singularities.

The difference between satisfying the Milnor condition and the strong Milnor condition was first addressed by Jacquemard in [14], though this terminology was introduced later in [33]. The technique used in [35], and explained in §5, to prove that the twisted Pham-Brieskorn singularities satisfy the strong Milnor condition was used later in [32] by Ruas and Dos Santos to improve the results of [14] using also Bekka's $(c)-$regularity. This actually gives an interesting method for studying the geometry of analytic maps into $\mathbb{R}^2$ that we describe in §6. This method is used in §7 for holomorphic maps $(\mathbb{C}^n, \underline{0}) \xrightarrow{f} (\mathbb{C}, 0)$, where it yields a slight refinement of the classical Milnor fibration theorem for complex singularities. We obtain a fibre bundle $\mathbb{B}_\varepsilon \setminus f^{-1}(0) \to \mathbb{S}^1$, for every sufficiently small ball $\mathbb{B}_\varepsilon$ around $\underline{0}$, whose restriction to $\partial\mathbb{B}_\varepsilon$ is the classical Milnor fibration on the sphere, and its restriction to a Milnor tube $\mathbb{B}_\varepsilon \cap f^{-1}(\partial\mathbb{D}_\eta)$ is the other classical fibration (up to multiplication by the scalar η). So this has the nice feature of unifying the two classical fibrations.

A particularly interesting class of real analytic maps is provided by maps

$\mathbb{C}^n \to \mathbb{C}$ of the form $f\bar{g}$ with f, g holomorphic functions. This type of maps have already appeared in articles by N. A'Campo [1], L. Rudolph [34, 12] and A. Pichon [29], as well as in [30]. In §8 we present a fibration theorem from [31] for such germs. A key point, first observed in [12], is noting that away from the zero loci $V(fg)$ of f and g the map $f\bar{g}/|f\bar{g}|$ equals the map $(f/g)/|f/g|$. Then one can follow the proof of Milnor's fibration theorem in [24], essentially step by step, and arrive to a fibration theorem for maps $f\bar{g}$. Of course this can be also regarded as a result for meromorphic map-germs, and a similar theorem has been also proved in [4] by the same method. In [31] we ask that $f\bar{g}$ has an isolated critical value, while in [4] this condition is replaced by demanding the germ f/g to be *semi-tame*. We remark that both of these theorems yield fibrations $\mathbb{S}_\varepsilon \setminus V(fg) \to \mathbb{S}^1$. Local fibrations for meromorphic germs have been studied previously by Gusein-Zade, Luengo and Melle in a series of articles (see for instance [9, 10]); these are local fibrations of "Milnor tubes", not on the sphere. In [5] we study the relation between all these local fibrations. In particular, for semi-tame germs in two complex variables we show that the fibre of the Milnor fibration on the sphere is essentially the connected sum of the fibres of the local fibrations of [9, 10] at 0 and ∞.

Finally, in §9 we explain briefly one of the main results in [31]. So far we have discussed fibrations of the complement of the link of certain singularities, but we have not payed special attention to what happens near the link. There is actually a richer structure one can look at, the *multilink structure*. Beware that a link here has two distinct meanings. On one hand it means the intersection $V \cap \mathbb{S}_\varepsilon$ of some analytic set $V \subset \mathbb{C}^N$ with a small sphere around a singular point. But a link also means a finite, disjoint union of knots (codimension 2 connected submanifolds) in some manifold M. For simplicity we restrict to the case $M = \mathbb{S}^3$, so a link L is a disjoint union of circles embedded in the 3-sphere, $L = K_1 \cup \cdots \cup K_r$. A multilink means that we have fixed an orientation on each component K_i and we have assign a multiplicity $n_i \in \mathbb{Z}$ to each, with the convention that $-n_i\, K_i = n_i\, (-K_i)$, where $-K_i$ is K_i with the opposite orientation. A *fibration* of a multilink means a fibration of its complement whose behaviour near each K_i "takes into account" the multilink structure (see §9).

For instance, if $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ is holomorphic and $f = \prod_{i=1}^{l} f_i^{n_i}$ is

its decomposition into irreducible factors, then

$$L_f = \bigcup_{i=1}^{l} n_i K_{f_i} \,,$$

is the multilink associated with f, where $K_{f_i} = \mathbb{S}^3_\varepsilon \cap f_i^{-1}(0)$, and the Milnor fibration $f(x)/|f(x)| : \mathbb{S}^3_\varepsilon \setminus L_f \to \mathbb{S}^1$ is a fibration of the multilink L_f. In §9 we briefly explain the corresponding theorem of [31] for germs $f\bar{g}$.

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CONTENT

§1: An example: Pham-Brieskorn singularities.
§2: The classical fibration theorem of Milnor.
§3: A glance on generalisations.
§4: Real analytic germs with a Milnor fibration.
§5: Twisted Pham-Brieskorn singularities.
§6: A method for studying maps into $\mathbb{R}^2$.
§7: A refinement of Milnor's classical theorem.
§8: Singularities $f\bar{g}$ and Milnor fibrations for meromorphic germs.
§9: Fibrations of multilinks.

1 An example: Pham-Brieskorn singularities

Let us begin with an example which is the paradigm for Milnor's Fibration Theorem and was a motivation for it. Consider the Pham-Brieskorn polynomial

$$f(\underline{z}) = z_0^{a_0} + \cdots + z_n^{a_n} \quad , \ a_i > 1 \,,$$

where $\underline{z} = (z_0, ..., z_n)$. It is clear that the origin $\underline{0} \in \mathbb{C}^{n+1}$ is the only critical point of f, so the fibres $V_t = f^{-1}(t)$ are all complex n-manifolds for $t \neq 0$ and $V = f^{-1}(0)$ is a complex hypersurface with an isolated singularity at $\underline{0}$.

We want to study the topology of V and of the $V_t's$. For this let d be the least common multiple of the a_i and define an action Γ of the non-zero complex numbers $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$ by:

$$\lambda \cdot (z_0, \cdots, z_n) \mapsto (\lambda^{d/a_0} z_0, \cdots, \lambda^{d/a_n} z_n) .$$

Notice this action satisfies:

$$f(\lambda \cdot (z_0, \cdots, z_n)) \; = \; \lambda^d \cdot f(z_0, \cdots, z_n) .$$

Hence V is an invariant set of the action and one has the following properties:

Property 1.1 Restricting the action to $t \in \mathbb{R}^+$ we get a real analytic flow (or a vector field) on $\mathbb{C}^{n+1}$ whose orbits are real lines (arcs) which converge to $\underline{0}$ when t tends to 0, they escape to ∞ when $t \to \infty$ being transversal to all spheres around $\underline{0}$, and they leave V invariant (*i.e.* V is union of orbits).

Property 1.2 Restricting the action to the unit circle $\{e^{i\theta}\}$ we get an $\mathbb{S}^1$-action on $\mathbb{C}^{n+1}$ such that each sphere around $\underline{0}$ is invariant and:

$$f(e^{i\theta} \cdot (z_0, ..., z_n)) = e^{id\theta} f(z_0, \cdots, z_n) ,$$

that is, if we set $\zeta = f(z_0, \cdots, z_n)$, then multiplication by $e^{i\theta}$ in $\mathbb{C}^{n+1}$ transports the fibre $f^{-1}(\zeta)$ into the fibre over $e^{id\theta} \cdot \zeta$.

Property 1.3 The real analytic flow defined by restricting the $\mathbb{C}^*$-action to $t \in \mathbb{R}^+$ has the additional property that for points in $\mathbb{C}^{n+1} \setminus V$, the argument of the complex number $f(\underline{z})$ is constant on each orbit, *i.e.* $f(\underline{z})/|f(\underline{z})| = f(t\underline{z})/|f(t\underline{z})|$ for $t \in \mathbb{R}^+$, and the norm of $f(\underline{z})$ is an strictly increasing function of t.

Each of these three properties has important implications. The first property (1.1) implies:

Property 1.4 The variety V intersects transversally every $(2n+1)$-sphere $\mathbb{S}_r$ around the origin; hence the intersection $K_r = V \cap \mathbb{S}_r$ is a smooth manifold of real dimension $2n-1$ embedded as a codimension 2 submanifold of the sphere $\mathbb{S}_r$;

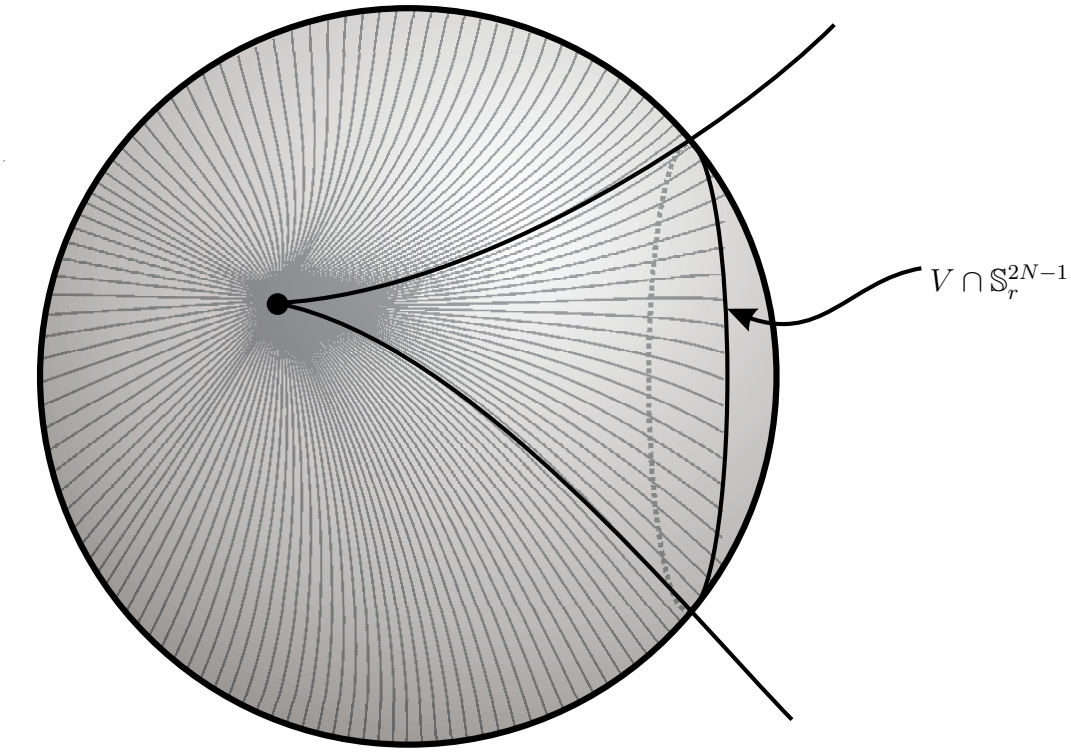

Figure 1: The conical structure

Property 1.5 The flow determines a 1-parameter group of diffeomorphisms that preserve V, thus the diffeomorphism type of K_r is independent of the choice of the sphere $\mathbb{S}_r$; and

Property 1.6 The embedded topological type of V in $\mathbb{C}^{n+1}$ is determined by the pair $(\mathbb{S}_r, K_r)$; more precisely, the pair $(\mathbb{C}^{n+1}, V)$ is homeomorphic to the (global) cone over the pair $(\mathbb{S}_r, K_r)$.

The manifold $K = K_r$ (for some r) is called the *link* of the singularity. The manifolds that arise in this way have been studied by several authors obtaining remarkable results. Notice that the link is a codimension 2 submanifold of the sphere, so it is a knot if it is connected (which is always the

case when $n > 1$ or if V is irreducible, by [24]). This type of knots $(\mathbb{S}_r, K)$ are called *algebraic knots*, and the above properties 1.3 to 1.5 imply that the embedded topological type of V in $\mathbb{C}^{n+1}$ is determined by the knot $(\mathbb{S}_r, K)$. We refer to Chapter 1 in [37] for an overview of the topic, including a large bibliography.

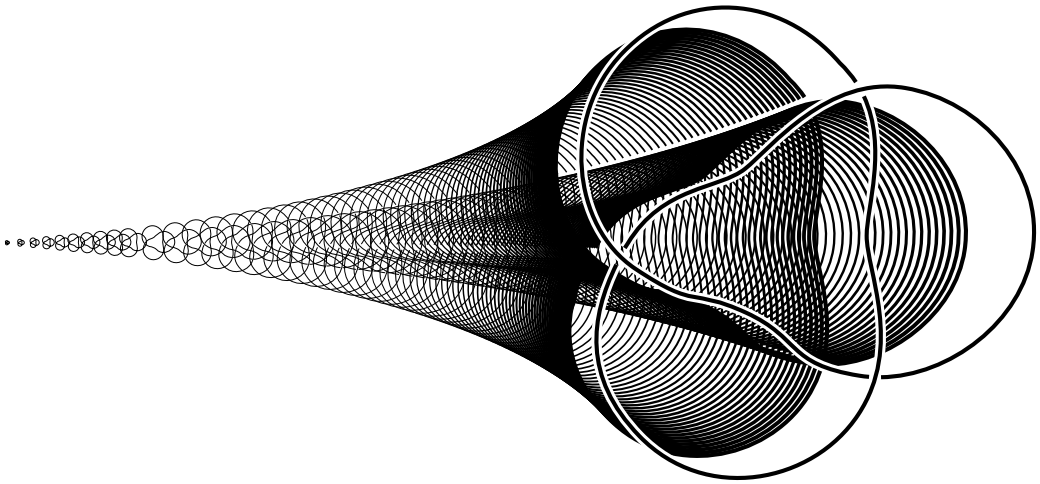

Figure 2: The trefoil knot determines the curve $\{x^2 = y^3\}$

On the other hand, for $t \neq 0$ the fibres $f^{-1}(t)$ are all complex manifolds, because $0 \in \mathbb{C}$ is the only critical value of f. Furthermore, the second property (1.2) above implies that for constant $|t| = \delta > 0$ the manifolds $f^{-1}(t)$ are all diffeomorphic and we have a flow on the "tube" $f^{-1}(C_\delta)$ given by the $\mathbb{S}^1$-action, where $C_\delta = \{t \in \mathbb{C} \,\big|\, |t| = \delta\}$. This flow is transversal to all the fibres and carries fibres into fibres. Thus we have a (locally trivial) fibre bundle over the circle C_δ:

$$f : f^{-1}(C_\delta) \longrightarrow C_\delta \,.$$

Let us focus our attention near the origin, say restricted to the unit ball $\mathbb{B} := \mathbb{B}^{2n+2} \subset \mathbb{C}^{n+1}$. Since $V = f^{-1}(0)$ meets $\mathbb{S}^{2n+1} = \partial\mathbb{B}$ transversally, it follows that for $\eta > 0$ sufficiently small, all fibres $f^{-1}(t)$ with $|t| = \eta$ meet $\mathbb{S}^{2n+1}$ transversally. Let C_δ be the circle in $\mathbb{C}$ around 0 of radius δ with $\eta > \delta > 0$, and look at the set $f^{-1}(C_\delta)$. Since the action of $\mathbb{S}^1$ on $\mathbb{C}^{n+1}$ is by isometries, it preserves the ball $\mathbb{B}$ and determines a fibre bundle:

$$f : \mathbb{B} \cap f^{-1}(C_\delta) \longrightarrow C_\delta \,. \tag{1.7}$$

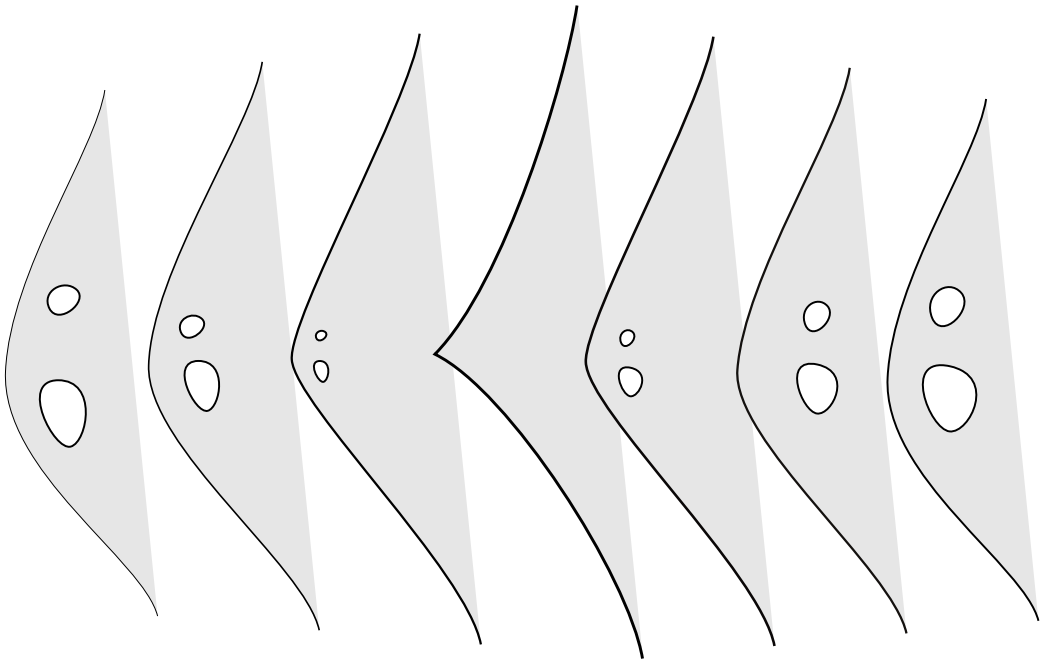

Figure 3: The Milnor fibration

Notice that one can also restrict the map f to the unit sphere $\mathbb{S}^{2n+1}$ and define

$$\phi := \frac{f}{|f|} \,:\, \mathbb{S}^{2n+1} \setminus K \longrightarrow \mathbb{S}^1 \,. \tag{1.8}$$

It is not hard to see that the map ϕ has no critical points at all, so its fibres are smooth submanifolds of the sphere, and we see from the previous discussion that the $\mathbb{S}^1$ action Γ on $\mathbb{C}^{n+1}$ preserves the unit sphere and leaves the link K invariant; thus it also carries its complement $\mathbb{S}^{2n+1} \setminus K$ into itself. Furthermore, (1.2) tells us that the orbits of this action are also transversal to the fibres of ϕ and carry fibres of ϕ into fibres of ϕ, showing that (1.8) is a locally trivial fibre bundle too.

In other words, given the polynomial map f, we have already associated to it two different fibre bundles. One of them is given by 1.7 and the other by 1.8. We claim that these two fibre bundles are actually equivalent. To prove this, notice that we loose nothing if in 1.8, instead of removing K, we

remove from $\mathbb{S}^{2n+1}$ a compact tubular neighborhood $N(K) = \mathbb{S}^{2n+1} \cap f^{-1}(\mathbb{D}_\delta)$ of the link, where $\mathbb{D}_\delta$ is the disc in $\mathbb{C}$ with boundary C_δ. Let us now define a diffeomorphism

$$h : \mathbb{B} \cap f^{-1}(C_\delta) \to \mathbb{S}^{2n+1} \setminus N(K)\,,$$

as follows: for each point $\underline{z}$ in $f^{-1}(t) \cap \overset{\circ}{\mathbb{B}}$, follow the orbit of the $\mathbb{R}^+$-action that passes through $\underline{z}$ till this orbit meets $\mathbb{S}^{2n+1}$ at some point, say $\tilde{\underline{z}}$. Then define $h(\underline{z}) = \tilde{\underline{z}}$. That this map is a well defined diffeomorphism follows from Property 1.3; and h obviously extends as the identity to the points in $\partial N(K) \subset \mathbb{S}^{2n+1}$.

Summarizing: Property 1.1 implies one has a conical structure on V, Property 1.2 implies one has the fibrations 1.7 and 1.8, and Property 1.3 implies that these two fibrations are equivalent. Both of these are known as *the Milnor fibration* of f, and we will indicate in the following section how they generalize to other situations.

2 The classical fibration theorem of Milnor

Consider now, more generally, a holomorphic function

$$(U \subset \mathbb{C}^{n+1}, \underline{0}) \xrightarrow{f} (\mathbb{C}, 0)\,,$$

defined on an open neighbourhood U of the origin in $\mathbb{C}^{n+1}$ with a critical value at $0 \in \mathbb{C}$. Assume for simplicity that $\underline{0} \in \mathbb{C}^{n+1}$ is the only critical point of f in U. Let V be the singular variety defined by f, *i.e.* $V := \{f^{-1}(0)\}$; thus $V^* = (V \setminus \{\underline{0}\})$ is a smooth complex manifold of dimension n. We know (by work of Milnor and others) that one has in this general setting similar properties to those of the Pham-Brieskorn singularities explained in the previous section, the main difference being that in general one must restrict the discussion to a "sufficiently small" neighbourhood of the singular point. Let us explain this briefly. First, one has that V is locally a cone [24]: there exists $\varepsilon > 0$ sufficiently small so that given the ball $\mathbb{B}_\varepsilon$ in $\mathbb{C}^{n+1}$ centred at $\underline{0}$ of radius ε, one can construct a vector field (flow) similar to the one in the previous section given by the $\mathbb{R}^+$-action: its orbits are transversal to all the spheres in $\mathbb{B}_\varepsilon$ centred at $\underline{0}$, and it leaves $V \cap \mathbb{B}_\varepsilon$ invariant. Hence each sphere $\mathbb{S}_{\varepsilon'}$ in $\mathbb{C}^{n+1}$ centred at $\underline{0}$ of radius $\varepsilon' \leq \varepsilon$ meets V transversally. The intersection $K_\varepsilon = V \cap \mathbb{S}_\varepsilon$ is a smooth manifold of real dimension $2n-1$

embedded as a submanifold of the $(2n+1)$-sphere $\mathbb{S}_\varepsilon$. The diffeomorphism type of the manifold K_ε and the isotopy class of the pair $(\mathbb{S}_\varepsilon, K_\varepsilon)$ do not depend on the choice of the sphere $\mathbb{S}_\varepsilon$; the pair $(\mathbb{B}_\varepsilon, K_\varepsilon)$ is homeomorphic to the cone over the pair $(\mathbb{S}_\varepsilon, K_\varepsilon)$, where ($\mathbb{B}_\varepsilon$ is the ball bounded by $\mathbb{S}_\varepsilon$, so that the topology of V near $\underline{0}$, and its embedding in $\mathbb{C}^{n+1}$, are determined by the pair $(\mathbb{S}_\varepsilon, K_\varepsilon)$. As mentioned in Section 1, the manifold $K = K_\varepsilon$ is called *the link of the singularity* and the pair $(\mathbb{S}_\varepsilon, K_\varepsilon)$ is called *an algebraic knot*, a name introduced by Lê Dũng Tráng in 1971, [16].

One may thus consider the obvious map:

$$\phi = \frac{f}{|f|} : (\mathbb{S}_\varepsilon \setminus K_\varepsilon) \to \mathbb{S}^1 .$$

Theorem 2.1 (Milnor [24]) *This is a (locally trivial) C^∞ fibered bundle.*

Milnor gave two proofs of this theorem; we already had glimpses of both of them in the previous section; each proof brings out different insights and lends itself to different generalisations. Let us sketch the key-points in each of them.

1^{st} **Proof:** This is along the lines of the above proof of 1.8. The idea is simple: first show that the map ϕ has no critical points at all, so the fibres of ϕ are all smooth, codimension-1 submanifolds of $(\mathbb{S}_\varepsilon \setminus K_\varepsilon)$; then construct a tangent vector field on $(\mathbb{S}_\varepsilon \setminus K_\varepsilon)$ which is transversal to the fibres of ϕ and the corresponding flow moves at constant speed with respect to the argument of the complex number $\phi(\underline{z})$, so it carries fibres of ϕ into fibres of ϕ. This proves one has a product structure around each fibre of ϕ. For this, to begin, Milnor shows that the critical points of $(\mathbb{S}_\varepsilon \setminus K_\varepsilon) \xrightarrow{\phi} \mathbb{S}^1$, if there were such points, are exactly the points $\underline{z} = (z_0, \ldots, z_n)$ where the vector $\big(i\,\mathrm{grad}(\log(f(\underline{z})))\big)$ is a real multiple of $\underline{z}$. To prove this, set

$$\phi(\underline{z}) = \frac{f}{|f|}(\underline{z}) := e^{i\theta(\underline{z})} ,$$

so one has:

$$\theta(\underline{z}) = Re\,(-i\,log\,f(\underline{z}))\,.$$

An easy computation shows that given any curve $\underline{z} = p(t)$ in $\mathbb{C}^{n+1} \setminus f^{-1}(0)$, the chain rule implies:

$$d\theta(p(t))/dt \;=\; Re\,\langle \frac{dp}{dt}(t), i\,grad\,log\,f(\underline{z})\rangle\,, \tag{2.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual hermitian product in $\mathbb{C}^{n+1}$. Hence, given a vector $v(\underline{z})$ in $\mathbb{C}^{n+1}$ based at $\underline{z}$, the directional derivative of $\theta(\underline{z})$ in the direction of $v(\underline{z})$ is:

$$Re\,\langle v(\underline{z}), i\,grad\,log\,f(\underline{z})\rangle\,.$$

Since the real part of the hermitian product is the usual inner product in $\mathbb{R}^{2n}$, it follows that if $v(\underline{z})$ is tangent to the sphere $\mathbb{S}^{2n-1}_{\varepsilon}$, then the corresponding directional derivative vanishes whenever $\big(i\,\mathrm{grad}(\log(f(\underline{z})))\big)$ is orthogonal to the sphere, *i.e.* when it is a real multiple of $\underline{z}$; conversely, if this inner product vanishes for all vectors tangent to the sphere then $\underline{z}$ is a critical point of ϕ, and the claim follows.

Once we know how to characterize the critical points of ϕ and how the argument of the complex number $\phi(\underline{z})$ varies as $\underline{z}$ moves along paths in $\mathbb{S}_{\varepsilon} \setminus K$, Milnor uses his *Curve Selection Lemma* (see [24]) to conclude that ϕ has no critical points at all. This part is a little technical and we refer to Milnor's book (Chapter 4) for details. It follows that all fibres of ϕ are smooth submanifolds of the sphere $\mathbb{S}_{\varepsilon}$ of real codimension 1. In order to show that ϕ is actually the projection map of a C^{∞} fibre bundle one must prove that one has a local product structure around each fibre. This is achieved in [24] by constructing a vector field w on $\mathbb{S}_{\varepsilon} \setminus K$ satisfying:

i) the real part of the hermitian product $\langle w(\underline{z}), i\,grad\,log\,f(\underline{z})\rangle$ is identically equal to 1; recall that this is the directional derivative of the argument of ϕ in the direction of $w(\underline{z})$.

ii) the absolute value of the corresponding imaginary part is less than 1:

$$|Re\,\langle w(\underline{z}),\, grad\,log\,f(\underline{z})\rangle| \,<\, 1\,.$$

Consider now the integral curves of this vector field, *i.e.* the solutions $p(t)$ of the differential equation $d\underline{z}/dt = w(t)$. Set $e^{i\theta(\underline{z})} = \phi(\underline{z})$ as before. Since the directional derivative of $\theta(\underline{z})$ in the direction $w(t)$ is identically equal to 1 we have:

$$\theta(p(t)) \,=\, t + constant\,.$$

Therefore the path $p(t)$ projects to a path which winds around the unit circle in the positive direction with unit velocity. In other words, these paths are transversal to the fibres of ϕ and for each t they carry a point $\underline{z} \in \phi^{-1}(e^{it_o})$ into a point in $\phi^{-1}(e^{it_o+t})$. If there is a real number $t_o > 0$ so that all these paths are defined for at least a time t_o then, being solutions of the above differential equation, they will carry each fibre of ϕ diffeomorphically into all

the nearby fibres, proving that one has a local product structure and ϕ is the projection of a locally trivial fibre bundle. Milnor proves this by showing that condition (ii) above implies that all these paths are actually defined for all $t \in \mathbb{R}$, so we arrive to Theorem 2.1. $\square$

2^{nd} **Proof:** This has two big steps, the first is showing that given a map-germ $(\mathbb{C}^{n+1}, \underline{0}) \xrightarrow{f} (\mathbb{C}, 0)$ one has a fibre bundle as in 1.7:

Theorem 2.3 *Let $\mathbb{S}_\varepsilon$ be a sufficiently small sphere in $\mathbb{C}^{n+1}$ centred at $\underline{0}$ and choose $\delta > 0$ small enough with respect to ε so that all the fibres $f^{-1}(t)$ with $|t| \leq \delta$ meet $\mathbb{S}_\varepsilon$ transversally. Let $C_\delta = \partial \mathbb{D}_\delta$ be the circle in $\mathbb{C}$ of radius δ and centred at 0, and set $N(\varepsilon, \delta) = f^{-1}(C_\delta) \cap \mathbb{B}_\varepsilon$. Then:*

$$f|_{N(\varepsilon,\delta)} : N(\varepsilon, \delta) \longrightarrow C_\delta \cong \mathbb{S}^1 ,$$

is a C^∞ fibre bundle.

The manifold $N(\varepsilon, \delta)$ is usually called a *Milnor tube* for f.

This result is essentially an extension of Ehresmann's fibration lemma. The idea of the proof is that given $\varepsilon > 0$ as above, if f has an isolated critical point at $\underline{0}$, then Thom's transversality implies that all the fibres $f^{-1}(t)$ with $\varepsilon >> |t| > 0$ are transversal to $\mathbb{S}_\varepsilon$. Furthermore, since $0 \in \mathbb{C}$ is an isolated critical point, the map f in 2.3 is a submersion and we can lift a vector field on the circle C_δ to a vector field on the tube $N(\varepsilon, \delta)$, transversal to the fibres of f. We may further choose this lifting to be an integrable vector field tangent to $N(\varepsilon, \delta) \cap \mathbb{S}_\varepsilon$. Since the fibres are compact, we can assume that for each fibre F_o there is a time $t_o > 0$ so that all solutions passing by F_o are defined for at least time t_o. Then the corresponding local flow identifies F_o with all nearby fibres, which are thus diffeomorphic to F_o and one has a local product structure, hence a fibre bundle.

We remark that this theorem holds for all map-germs $(\mathbb{C}^{n+1}, \underline{0}) \xrightarrow{f} (\mathbb{C}, 0)$, however if f has a non-isolated critical point at $\underline{0} \in \mathbb{C}^{n+1}$ then one must use Hironaka's theorem in [13], that f has the Thom a_f-property, to conclude that the tube $N(\varepsilon, \delta)$ is transversal to $\mathbb{S}_\varepsilon$ and prove 2.3. This was done by Lê in [15] where he actually extends 2.3 to all map-germs defined on complex varieties.

The second step for proving (2.1) in this way is to show [24, Ch. 5] that there exists a vector field on $\mathbb{B}_\varepsilon \setminus f^{-1}(0)$ whose solutions move away from the origin being transversal to all the spheres around $\underline{0}$, transversal to all

the tubes $f^{-1}(C'_\delta)$ and the argument of the complex number $f(\underline{z})$ is constant along the path of each solution. This allows us to "inflate" the Milnor tube $f^{-1}(\partial\mathbb{D}_\delta) \cap \mathbb{B}_\varepsilon$ in 2.3 to become the complement of a neighbourhood $T(\varepsilon, \delta)$ of the link K in the sphere $\mathbb{S}_\varepsilon$, taking the fibres of (2.3) into the fibres of the map ϕ in Theorem 2.1. Then one must show that this fibration extends to $T(\varepsilon, \delta) \setminus K_\varepsilon$.

Notice that 2.3 implies that the fibres of Milnor's fibration are diffeomorphic to complex Stein manifolds of dimension n in $\mathbb{C}^{n+1}$, hence a theorem of Andreotti-Frankel implies they have the homotopy type of a CW-complex of middle-dimension n. This is also proved by Milnor in his book, where he proves too that the link K is always $(n-2)$-connected.

When f has an isolated critical point at $\underline{0}$, Milnor proves more: the fibres of ϕ have the homotopy type of a bouquet $\wedge\mathbb{S}^n$ of n-spheres. The number of spheres in this wedge is by definition *the Milnor number* of f, an important invariant of f. For instance the Milnor fibre of the Morse singularity $z_o^2 + z_1^2 + \cdots z_n^2$ is diffeomorphic to the total space of the tangent bundle of the unit sphere $\mathbb{S}^n$, so it has Milnor number 1.

3 A glance on generalisations

Several natural generalisations of Milnor's fibration theorem have been considered by various authors. We briefly mention here some of them.

i) As noticed by Hamm in [11], Milnor's fibration theorem extends, with essentially the same proof, to the case of holomorphic functions defined on complex analytic spaces with an isolated singularity. In this more general setting the fibre still has the homotopy type of a CW-complex of middle dimension, but it may not be homotopically equivalent to a bouquet of spheres (see for instance the articles of Hamm-Lê about rectified homological, or homotopical, depth.) Hamm's interest in [11] was mainly to study complete intersection germs

$$f : (\mathbb{C}^{n+k}, \underline{0}) \to (\mathbb{C}^k, \underline{0}) .$$

Since the critical points of f are the points where the rank of the jacobian matrix drops down, if $k > 1$ and $\underline{0} \in \mathbb{C}^k$ is a critical value, then the critical values of f are an analytic subset $\Delta \subset \mathbb{C}^k$ of dimension > 0 containing $\underline{0}$; the set Δ is called *the discriminant* of f. Hence one can not possibly have

a fibration over a small punctured disc $\mathbb{D}_\delta$ around $\underline{0} \in \mathbb{C}^k$ as one does when $k = 1$. However, if $\underline{0} \in \mathbb{C}^{n+k}$ is an isolated singularity of $V = f^{-1}(\underline{0})$, then Hamm proved that one has a fibration over $\mathbb{D}_\delta \setminus \Delta$ analogous to that in (2.3). A key for understanding this fibration (see for instance [21] for details) is to notice that in this case one can choose the functions $(f_1, ..., f_k)$ that define the germ $(V, \underline{0})$ so that

$$\widehat{V} = f_1^{-1}(0) \cap ... \cap f_{k-1}^{-1}(0)$$

is a complete intersection germ of dimension $n+1$ with an isolated singularity, and the restriction of f_k to $\widehat{V}$ has an isolated critical point at $\underline{0}$; so one has the corresponding Milnor fibration.

It is remarkable (Hamm's theorem) that for complete intersection germs the fibres have the homotopy type of a bouquet of spheres of middle dimension, just as for hypersurfaces.

We remark that this fibration theorem for complete intersection germs is false in general if the critical points of f are non-isolated in V, unlike the hypersurface case were the theorem holds for every map-germ. This fact was first noticed by Lê Dũng Tráng (see *Lê's example* in [37]), and the reason behind (noticed also by Lê) is that Hironaka's result in [13], that every holomorphic map into $\mathbb{C}$ has the Thom a_f-property, is false in general for maps into $\mathbb{C}^k$ with $k > 1$.

ii) Given $\tilde{f} : (\mathbb{C}^N, \underline{0}) \to (\mathbb{C}, 0)$ holomorphic and an analytic singular variety $X \subset \mathbb{C}^N$, one may consider the restriction f of $\tilde{f}$ to X. The concept of *critical points* of f on X makes perfect sense once we equip X with a Whitney stratification (this goes back to the work of Thom, Lazzeri, Goreski-MacPherson, Lê and others). It is proved in [15] that one has in this case a fibration theorem as in (2.3); these are called *Milnor-Lê fibrations* and they have given rise to a vast literature. We refer particularly to Lê's paper [17] and to [22, 38, 39], which contain remarkable generalisations of Milnor's theorem about the topology of the Milnor fibre. Somehow these spring out from the handle-body decomposition of the Milnor fibre given in [18].

iii) Consider polynomial maps $f, g : (\mathbb{C}^n, \underline{0}) \to (\mathbb{C}, 0)$ and the meromorphic map f/g defined away from the zero loci of f, g. In [9, 10] Gusein-Zade, Luengo and Melle proved that one has a local fibration around each point in $\mathbb{C}P^1$, and they studied carefully some aspects of the corresponding local

fibrations at 0 and ∞. These fibrations appear also in [31, 4] and we discuss them briefly in section 8 below.

iv) Consider real analytic germs $f : (U \subset \mathbb{R}^{n+k}, \underline{0}) \to (\mathbb{R}^k, \underline{0})$. This situation was first considered by Milnor himself in his book [24] and several authors have worked on this topic afterwards. This is explained in sections 4, 5 and 6 below (we refer also to [37] for more on the subject). A specially interesting case is when the real analytic map f is of the form $h\bar{g}$ with h and g holomorphic, or a sum of functions of this type. These singularities are also discussed in section 8.

4 Real analytic germs with a Milnor fibration

For real analytic germs, Milnor's fibration theorem in [23, 24] states:

Theorem 4.1 *Let* $(\mathbb{R}^{n+k}, \underline{0}) \xrightarrow{f} (\mathbb{R}^k, \underline{0})$, $n \geq 0$, *be the germ of a real analytic map with an isolated critical point at the origin. Then for every sufficiently small sphere* $\mathbb{S}_\varepsilon = \partial \mathbb{B}_\varepsilon$ *around* $\underline{0} \in \mathbb{R}^{n+k}$ *one has that the complement* $\mathbb{S}_\varepsilon \setminus K$ *of the link* $K = f^{-1}(\underline{0}) \cap \mathbb{S}_\varepsilon$ *fibres over the sphere* $\mathbb{S}^{k-1}$.

The proof of this result is by noticing first that for $\delta > 0$ sufficiently small the tube $f^{-1}(\mathbb{S}^{k-1}_\delta) \cap \mathbb{B}_\varepsilon$ fibres over the sphere $\mathbb{S}^{k-1}_\delta \subset \mathbb{C}$ of radius δ (just as in 2.3 above), and then constructing a vector field that "inflates" this tube taking it to the complement of (a regular neighbourhood of) the link in the sphere; see [24] or [31] for details.

When the map f is from $\mathbb{C}^{n+1}$ into $\mathbb{C}$ and is holomorphic, Milnor shows that one actually has a much richer structure:

i) first, one does not actually need to have an isolated critical point of f to have such a fibration: here the critical value is automatically isolated and that is enough in this case to have a fibration. For real analytic germs, isolated critical value is not enough in general and we need to ask for the Thom a_f-property (see [31]).

ii) for holomorphic germs the projection map $\phi : \mathbb{S}_\varepsilon \setminus K \to \mathbb{S}^1$ can be taken to be the obvious map $\phi = f/|f|$; as Milnor shows in his book, this statement is false in general when f is not holomorphic, even if one does have the fibration in 4.1. In [14, 32] the authors give sufficient conditions to insure that a map

f as in 4.1 defines a fibration where the projection map can be taken to be $f/|f|$. We shall return to this point in §6.

The geometry of the Milnor fibrations associated with holomorphic singularity germs has given rise to a rich literature, both within singularity theory itself, as well as in nearby areas like fibered (or Neuwirth-Stallings) knots and links, open-book decompositions, the results of Lawson and others about codimension 1 foliations, etc. That is not the case for real analytic germs. There are various reasons for this, in particular because it is difficult to find examples of real analytic singularities with an isolated critical point, and it is even harder to study their underlying geometry and topology.

Several natural -related- problems arise, as for instance: i) find interesting families of real analytic germs with a rich geometry and a Milnor fibration; ii) relax the conditions in Milnor's fibration theorems in order to include larger families.

These questions have been addressed by several authors in various ways, and in the previous section we mentioned some of these. Let us discuss briefly some general facts about real analytic germs with a Milnor fibration as above. For simplicity we restrict the discussion to real analytic functions $f : (\mathbb{R}^n, \underline{0}) \to (\mathbb{R}^2, 0)$, $n > 2$.

Definition 4.2 We say that the map f *satisfies the Milnor condition at* $\underline{0}$ if the derivative $Df(x)$ has rank 2 at every point $x \in U - \underline{0}$, where U is an open neighbourhood of $\underline{0} \in \mathbb{R}^n$, *i.e.* if f is a local submersion at every point in a punctured neighbourhood of $\underline{0} \in \mathbb{R}^n$.

One has the above theorem (4.1) of Milnor for functions satisfying (4.2). For instance, every complex valued holomorphic function with an isolated critical point in its domain satisfies these conditions, and so does if we compose such functions with a real analytic local diffeomorphism in the source $\mathbb{C}^{n+1}$, or in the target $\mathbb{C}$, but these are somehow "fake" examples of real analytic functions. The interesting point is to find examples which are honestly real.

As Milnor pointed out in his book, the hypothesis of Df having maximal rank everywhere near $\underline{0}$ is too stringent; the generic case is to have real curves in $\mathbb{R}^2$ converging to $(0, 0)$ whose inverse image contains points where the jacobian matrix has rank less than 2. And even if we have that $(0, 0) \in \mathbb{R}^2$ is an isolated critical value, for (4.2) we need more: $\underline{0} \in \mathbb{R}^n$ must be an isolated critical point.

Milnor actually asked whether there exist "non-trivial" examples satisfying the condition of (4.2). This question was answered positively by Looijenga [20] for n even and by Church and Lamotke [6] for n odd, using Looijenga's technique. The key point in [20] is to show that if $(\mathbb{S}^{2m-1}, K)$ is a fibred knot with non-empty K, which is odd (in the sense that the antipodal map of $\mathbb{S}^{2m-1}$ preserves K and the fibration of $\mathbb{S}^{2m-1} \setminus K$ over $\mathbb{S}^1$ is equivariant with respect to the antipodal maps of both spheres), then one has that $(\mathbb{S}^{2m-1}, K)$ arises from the isolated singularity of some real polynomial map. Furthermore, given such a fibred knot $(\mathbb{S}^{2m-1}, K)$, the connected sum $(\mathbb{S}^{2m-1}, K)\#(\mathbb{S}^{2m-1}, K)$ is again fibred and odd. Hence there exist many non-complex examples of isolated singularities satisfying the Milnor condition. However, essentially no explicit examples of such singularities were given.

An explicit non-trivial example of a real analytic singularity satisfying the Milnor condition at $\underline{0}$, other than those of Milnor, was given by A'Campo in [1]. This is the map $\mathbb{C}^{m+2} \to \mathbb{C}$ defined by

$$(u, v, z_1, ..., z_m) \longmapsto uv(\bar{u} + \bar{v}) + z_1^2 + ... + z_m^2 \,, \tag{4.3}$$

which is not holomorphic due to the presence of complex conjugation.

Later Perron in [26] proved that the figure-eight knot, which was known to be fibered, can also be realised by real algebraic equations. Lee Rudolph, in his review of Perron's article, gives a very nice set of equations that define this knot: if $z = x + iy$ and $w = u + iv$ are complex coordinates for R^4, then the figure-eight knot is the link of the 2-dimensional singularity defined by the real polynomials:

$$g(z, w) = w^3 - 3(x^2 + y^2)(1 + iy)w - 2x, \quad f(z, w) = g(z^2, w) \,.$$

In [35, 33] there are given infinite families of singularities satisfying Milnor's condition, which are somehow in the same vein as (4.3). Before explaining these examples, let us look at a more subtle question for which we introduce the following notation from [33]:

Definition 4.4 Let $f = (f_1, f_2) : (\mathbb{R}^n, \underline{0}) \to (\mathbb{R}^2, \underline{0})$, $n \geq 2$, be analytic and satisfy the Milnor condition at $\underline{0}$. Let $K = f_1^{-1}(0) \cap f_2^{-1}(0) \cap \mathbb{S}_\varepsilon$ be its link; f satisfies *the strong Milnor condition at* $\underline{0}$ if for every sufficiently small sphere $\mathbb{S}_\varepsilon$ around $\underline{0}$ the map

$$\frac{f}{|f|} : \mathbb{S}_\varepsilon - K \;\to\; \mathbb{S}^1 \,,$$

is the projection of a fibre bundle.

As shown by Milnor himself in [23, p. 99], there exist maps that satisfy condition (4.2) but not the stronger condition (4.4).

So the question is: given a real analytic map-germ f satisfying the Milnor condition (4.2), when does it satisfy the strong Milnor condition? This question was first studied by Jacquemard in [14] and we return to this point in section 6, but before doing so we describe below, as an example, a family of real singularities with a rich geometry and satisfying (4.4).

5 Twisted Pham-Brieskorn singularities

In the previous section we mentioned Looijenga's construction of real singularities having the Milnor condition. We also gave the corresponding examples of A'Campo and Perron. The first explicit family of non-trivial examples of real singularities satisfying the (strong) Milnor condition (4.4) were given in [35] (see also [33, 37]). These are defined in $\mathbb{R}^{2n} \cong \mathbb{C}^n$ by:

$$f(\underline{z}) \;=\; z_1^{a_1}\,\bar{z}_{\sigma_1} + \cdots + z_n^{a_n}\,\bar{z}_{\sigma_n} \quad , \quad a_i > 1\,, \tag{5.1}$$

where $\sigma = (\sigma_1, \cdots, \sigma_n)$ is any permutation of $(1, \cdots, n)$. Notice these are reminiscent of the Pham-Brieskorn singularities $f(\underline{z}) = z_1^{a_1} + \cdots + z_n^{a_n}$ and they are called in [37] *twisted Pham-Brieskorn singularities*; the permutation σ being *the twisting*. A reason for this name is that when there is no twisting, *i.e.* when σ is the identity, these singularities are topologically equivalent to Pham-Brieskorn singularities, by [33].

The variety defined by such a polynomial map consists of the points in $\mathbb{C}^n$ where the holomorphic vector field:

$$\xi(\underline{z}) = \sum_{i=1}^{n} z_i^{a_i}\,\frac{\partial}{\partial z_{\sigma_i}}\,, \tag{5.2}$$

is tangent to the spheres around the origin $\underline{0} \in \mathbb{C}^n$. In general, given a holomorphic vector field $\xi(\underline{z}) = \sum_{i=1}^{n} \alpha_i(\underline{z})\,\frac{\partial}{\partial z_i}$ with an isolated singularity at $\underline{0}$, one can look at the holomorphic foliation $\mathcal{F}$ by complex curves that it defines, and consider the points of contact of this foliation with the spheres around the origin, *i.e.* the points were $\mathcal{F}$ is tangent to some sphere centred at $\underline{0}$. This variety is defined by the equation:

$$V_\xi \;=\; \{\underline{z} \in \mathbb{C}^n \mid \langle \xi(\underline{z}), \underline{z} \rangle = 0\}\,,$$

where $\langle \xi(\underline{z}), \underline{z} \rangle = \sum_{i=1}^{n} \alpha_i(\underline{z}) \bar{z}_i$ is the usual hermitian product. Notice that V_ξ is therefore defined by two real analytic equations, which are the real and imaginary parts of the analytic map $\underline{z} \mapsto \langle \xi(\underline{z}), \underline{z} \rangle \in \mathbb{C}$.

One can prove in general that V_ξ is smooth of codimension 2 at a point $\underline{z}$ iff the contact of $\mathcal{F}$ and the sphere through $\underline{z}$ and centre $\underline{0}$ is generic, and this is equivalent to saying that $\underline{z}$ is either a local minimal point or a saddle in its corresponding leaf in $\mathcal{F}$ (see [37, Chap. VI]). Besides this, it is rather hard (and interesting) to say anything about the varieties that one gets in this way (see [37]). For instance consider the linear vector field in $\mathbb{C}^3$,

$$\xi(\underline{z}) = \sum_{i=1}^{3} \lambda_i \, z_i \, \frac{\partial}{\partial z_i} \, ,$$

where the eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ are either $(1, i, 1+i)$, $(1, i, -1-i)$ or $(1, i, -i)$. In the first case we have a vector field in the so-called *Poincaré domain*, which means that $0 \in \mathbb{C}$ is not contained in the convex hull of the eigenvalues. It is an easy exercise to show that in this case the only solution to the equation $\langle \xi(\underline{z}), \underline{z} \rangle = 0$ is $\underline{z} = \underline{0}$, hence $V_\xi = \{\underline{0}\}$. In the second case we have a vector field in the *Siegel domain* with generic eigenvalues. The variety V_ξ is a cone with vertex at $\underline{0}$ and base its intersection with the unit sphere, which is the link K_ξ. Notice that the natural diagonal action of the 3-torus $\mathbb{T}^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ on $\mathbb{C}^3$ preserves V_ξ and is by isometries, so it preserves the link K_ξ; it is easy to show that this action of $\mathbb{T}^3$ on K_ξ is free and transitive, hence K_ξ is $\mathbb{T}^3$.

In the third case the variety V_ξ is defined by the equations $|z_1| = 0$ and $|z_2| = |z_3|$, so it is a real 3-dimensional quadric in the plane $\{z_2, z_3\}$.

Surprisingly, for the vector fields in (5.2) one has an amazing regularity. The following more general theorem is proved in [35, 33] (see also [37]):

Theorem 5.3 *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial map of the form:*

$$f(z_1, \ldots, z_n) \; = \; \lambda_1 \, z_1^{a_1} \, \bar{z}_{\sigma_1} \; + \cdots + \; \lambda_n \, z_n^{a_n} \, \bar{z}_{\sigma_n} \, ,$$

where the λ_i are arbitrary non-zero complex numbers, all a_i are ≥ 2 and $\sigma = (\sigma_1, \cdots, \sigma_n)$ is a permutation of $(1, \cdots, n)$. Then:

i. The variety $V = f^{-1}(0)$ is a (codimension 2) real algebraic complete intersection in $\mathbb{C}^n$ with an isolated singularity at $\underline{0}$.

ii. f satisfies the strong Milnor condition at $\underline{0}$.

We notice that this result is extended in [33] by allowing some exponents to be 1 in certain cases, depending on the permutation σ.

The proof of the first statement in the theorem is by a straightforward computation, showing that $\underline{0}$ is actually the only point where the rank of the jacobian matrix is less than 2. This implies (i) and also implies that f satisfies the Milnor condition at 0.

The proof that these singularities satisfy the strong Milnor condition uses a construction that proves actually more. Given f we consider a 1-parameter family of real analytic maps h_θ, $\theta \in [0, \pi)$, where each h_θ is the composition of f followed by the projection into the line in $\mathbb{C}$ that has angle θ with the positive real axis. This family fills out the entire $\mathbb{C}^n$ by real analytic hypersurfaces X_θ, which meet at $V = f^{-1}(0)$ and they have each a unique singularity at $\underline{0} \in \mathbb{C}^n$. One can define a diagonal action of $\mathbb{R}^+$ on $\mathbb{C}^n$, whose orbits are transversal to all the spheres around $\underline{0}$ and leave each X_θ invariant. Thus one has a global conical structure: each X_θ meets transversally all the spheres around $\underline{0}$ and its link is a smooth codimension 1 submanifold of the unit sphere. Furthermore, one has also an $\mathbb{S}^1$ action on $\mathbb{C}^n$ that permutes the varieties X_θ, leaving V invariant, and endows the whole space $\mathbb{C}^n \setminus V$ with the structure of a fibre bundle over $\mathbb{S}^1$; each fibre is transversal to all the spheres around $\underline{0}$ and each pair of antipodal fibres is naturally glued together along V forming the analytic space X_θ.

The specific $\mathbb{R}^+$ and $\mathbb{S}^1$ actions can be described explicitly as follows: look at the permutation σ of the indices $\{1, ..., n\}$ in the definition of f and split it into cycles. Then split $\mathbb{C}^n$ into linear subspaces according to the cycles of σ. A 1-cycle corresponds to a monomial of the form $z_i^{a_i}\bar{z}_i$; a two-cycle corresponds to a monomial $z_i^{a_i}\bar{z}_j + z_j^{a_j}\bar{z}_i$, and so on. Given an r-cycle, $1 \leq r \leq n$, re-label the components so that this cycle is:

$$z_1^{a_1}\bar{z}_2 + z_2^{a_2}\bar{z}_3 + \cdots + z_r^{a_r}\bar{z}_1\,,$$

then the $\mathbb{R}^+$-action is of the form

$$t \cdot (z_1, \cdots, z_r) \;=\; (t^{m_1} z_1, \cdots, t^{m_r} z_r)\,,$$

where the m_i are rational numbers determined by the equation $A \cdot (m_1, \cdots, m_r) =$

$(1, \cdots, 1)$, with A being the non-singular matrix:

$$A = \begin{pmatrix} a_2 & 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & a_3 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_r & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & a_1 \end{pmatrix}$$

Of course we can multiply them by the least common multiple of the denominators in order to get integral weights. Similarly, the $\mathbb{S}^1$-action is defined by:

$$e^{i\theta} \cdot (z_1, \cdots, z_r) = (e^{i\theta s_1} z_1, \cdots, e^{i\theta s_r} z_r),$$

where the s_i are the unique solution to $B \cdot (s_1, \cdots, s_r) = (1, \cdots, 1)$ where B is the matrix:

$$B = \begin{pmatrix} -1 & a_1 & 0 & \cdots & & \cdots & 0 \\ 0 & -1 & a_2 & 0 & & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & 0 & & -1 & a_{r-1} \\ a_r & 0 & 0 & \cdots & \cdots & 0 & -1 \end{pmatrix}$$

The obvious problem now is to study the topology of these singularities. Notice that the monodromy map of the corresponding fibre bundle over $\mathbb{S}^1$ is just the first return map of this $\mathbb{S}^1$-action, so it is periodic and its period can be computed explicitly from the weights of the action.

The simplest of these fibrations is when the permutation $(\sigma_1, \cdots, \sigma_n)$ is the identity, so the corresponding singularities are of the form:

$$f(\underline{z}) = z_1^{a_1} \bar{z}_1 + \cdots + z_n^{a_n} \bar{z}_n \quad , \; a_i > 1 \,.$$

As mentioned before, this case was studied in [33] where it is proved that these singularities are topologically (not analytically) equivalent to the Pham-Brieskorn singularities:

$$z_1^{a_1 - 1} + \cdots + z_n^{a_n - 1} \,,$$

whose topology is well understood. This implies, for instance, that the famous Poincaré's homology sphere $\Sigma(2, 3, 5)$ can be regarded as the set of points in the unit sphere $\mathbb{S}^5$ in $\mathbb{C}^3$ where the holomorphic vector field (z_1^3, z_2^4, z_3^6) is tangent to the sphere.

For $n = 2$ the remaining case is when f is of the form:

$$f(z_1, z_2) \;=\; z_1^p \bar{z}_2 \;+\; z_2^q \bar{z}_1 \quad , \; p, q > 1 \, .$$

This case was studied thoroughly in [30]. The results in that article, together with [28], imply that the link $K = f^{-1}(0) \cap \mathbb{S}^3$ of this singularity is isotopic to the link of the complex singularity defined by

$$\widehat{h}(z_1, z_2) \;=\; z_1 \, z_2 \, (z_1^{p+1} \;+\; z_2^{q+1}) \, , \tag{5.4}$$

but their corresponding Milnor fibrations are not equivalent. In fact one has that the real analytic singularity

$$h(z_1, z_2) \;=\; \bar{z}_1 \, \bar{z}_2 \, (z_1^{p+1} \;+\; z_2^{q+1}) \, , \tag{5.5}$$

also satisfies condition (4.4) and its Milnor fibration is equivalent to that of f, by [28, 30]. The components of K corresponding to the axes $z_1 z_2 = 0$ get different orientations in 5.4 and 5.5, in a sense that can be made precise, and this implies (by [28]) that the corresponding Milnor fibrations:

$$\mathbb{S}^3 \setminus K \xrightarrow{\frac{h}{|h|}} \mathbb{S}^1 \qquad \text{and} \qquad \mathbb{S}^3 \setminus K \xrightarrow{\frac{\widehat{h}}{|\widehat{h}|}} \mathbb{S}^1 \, ,$$

are not equivalent: the fibres have different Euler characteristic and the monodromy maps have different period (see [30]).

6 A method for studying maps into $\mathbb{R}^2$

As mentioned before, the problem of studying real singularities satisfying the strong Milnor condition (4.4) was first studied in [14] and turns out to be quite subtle. Such singularities define open-book decompositions on the spheres with the link as binding.

Given an analytic map-germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^2, 0)$ that satisfies the Milnor condition (4.2), Jacquemard gave two conditions that were sufficient to guarantee that f actually satisfies the strong Milnor condition. He proved:

Theorem 6.1 *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^2, 0)$ be an analytic map-germ. If the components $(f_1, f_2) = f$ satisfy the following two conditions, then f satisfies the strong Milnor condition:*

A) *there exists a neighbourhood U of the origin in $\mathbb{R}^n$ and a real number $0 < \rho < 1$ such that for all $x \in U - 0$ one has:*

$$\frac{|\langle grad\, f_1(x), grad\, f_2(x)\rangle|}{||grad\, f_1(x)|| \cdot ||grad\, f_2(x)||} \leq 1 - \rho\,,$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^n$; and,

B) *if ε_n denotes the local ring of analytic map-germs at the origin in $\mathbb{R}^n$, then the integral closures in ε_n of the ideals generated by the partial derivatives*

$$\Big(\frac{\partial f_1}{\partial x_1}\,,\, \frac{\partial f_1}{\partial x_2}\,, \cdots\,,\, \frac{\partial f_1}{\partial x_n}\Big) \quad \textit{and} \quad \Big(\frac{\partial f_2}{\partial x_1}\,,\, \frac{\partial f_2}{\partial x_2}\,, \cdots\,,\, \frac{\partial f_2}{\partial x_n}\Big)$$

coincide.

It was noted in [33] that the above condition (B) can be relaxed and (6.1) still holds. It is enough to demand only condition ($B_{\mathbb{R}}$): that the real integral closures of the jacobian ideals of f_1 and f_2 coincide.

This slight improvement of (6.1) was used in [33] to prove a stability theorem for real singularities with the strong Milnor condition. This was used also by Ruas and Dos Santos in [32] where they refine (6.1) using "regularity conditions" instead of Jacquemard's conditions. Their construction is inspired in the proof of (5.3) that we sketched in the previous section (see also [36]), but beautifully taken into a general setting using Bekka's (c)-regularity.

This provides a new method for studying the geometry of maps into $\mathbb{R}^2$ and motivates the refinement of Milnor's theorem that we give in the following section. This same method can be adapted to study, more generally, the geometry of maps $\mathbb{R}^{n+k} \to \mathbb{R}^k$, though we do not know whether or not one can actually get something interesting in this way for $k \neq 2$.

Let us explain briefly the main ideas in [32]. Consider a map-germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^2, 0)$ satisfying the Milnor condition at 0. As before, let $\pi_\theta : \mathbb{C} \to \mathcal{L}_\theta$ be the projection into the line $\mathcal{L}_\theta$ forming an angle θ with the horizontal axis in $\mathbb{C} = \mathbb{R}^2$, and consider the family of maps $h_\theta = \pi_\theta \circ f$. The projections π_θ are all submersions, hence the $\{h_\theta\}$ are all real valued analytic maps with an isolated critical point at 0. Setting $V = f^{-1}(0)$ and $X_\theta = h_\theta^{-1}(0)$, is easy to see that one has (see for instance [37, VI.4] for details):

$$\mathbb{B} = \cup X_\theta \quad \text{and} \quad V = \cap X_\theta = X_{\theta_1} \cap X_{\theta_2}\ , \ \text{for each pair } \theta_1 \neq \theta_2\,.$$

The main result of [32] says:

Theorem 6.2 *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^2, 0)$ be a real analytic map-germ with an isolated critical point at the origin that satisfies the Milnor condition at 0. If the family $h_\theta : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ satisfies the $(c)-$regularity condition of Bekka with respect to the function $\rho(x, \theta) = \sum_i x_i^2$, then f satisfies the strong Milnor condition.*

They also prove that if a function f satisfies Jacquemard's conditions A and B (or even $B_{\mathbb{R}}$) as in 6.1, then the corresponding family $\{h_\theta\}$ is $(c)-$regular at 0, but not conversely. Hence 6.2 improves 6.1. As a corollary they obtain that for quasi-homogeneous germs (as for instance the twisted Pham-Brieskorn singularities) the Milnor condition (4.2) and the strong Milnor condition (4.4) are actually equivalent.

The $(c)-$regularity is reminiscent of Thom's a_f-condition and can be explained as follows (cf. [2, 3]). Define $F : (\mathbb{R}^n \times [0, \pi)) \to \mathbb{R}$ by $F(x, \theta) = h_\theta(x)$ and set $Y_f = 0 \times [0, \pi)$ and $X_f = F^{-1}(0) \setminus Y_f$. Notice that if we set $X_\theta^* = X_\theta - \{0\} \subset \mathbb{R}^n$ for each $\theta \in [0, \pi)$, then one has $X_f = \bigcup X_\theta^*$ for all $\theta \in [0, \pi)$. Now consider the *control function* on $\mathbb{R}^n \times [0, \pi)$ defined by $\rho(x, \theta) = \sum_i x_i^2$. The pair (X_f, Y_f) is $(c)-$regular at a point $y_0 \in Y_f$ with respect to ρ if for each sequence of points $\{x_i\} \to y_0$, $x_i \in X_f$, such that the sequence of planes $\{ker\, D\rho(x_i) \cap T_{x_i} X_f\}$ converges to a plane T in the grassmannian of $(dim\, X_f - 1)$-planes in $\mathbb{R}^n$, one has $T_{y_0} Y_f \subset T$. The family $\{h_\theta\}$ is $(c)-$regular at 0 with respect to the control function ρ if the pair (X_f, Y_f) is $(c)-$regular at every point $y_0 \in Y_f$.

We remark that in the proof of 5.3 described above we used an action of $\mathbb{R}^+$ that leaves all X_θ invariant in order to prove that one has a *uniform conical structure* for all of them; in particular they are all transverse to every small sphere around 0. Then we used an $\mathbb{S}^1$-action that leaves invariant all spheres around 0 and permutes the X_θ to exhibit the fibre bundle structure. Two key points in [32] are to notice first that the $(c)-$regularity condition implies that the manifolds X_θ^* are all transversal to every sufficiently small sphere centred at 0, and then use [2, 40] to get a family of germs of homeomorphisms $h_\theta : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, that leave invariant each small sphere around the origin and permute the X_θ. Hence the hypersurfaces X_θ are homeomorphic and one has a local product structure. This yields to 6.2.

7 A refinement of Milnor's classical theorem

Let us now adapt the previous method for studying the classical Milnor's fibration theorem for complex singularitites. We only sketch the main points here, the proofs are given in [7].

Consider a holomorphic function $f : (\mathbb{B}, \underline{0}) \to (\mathbb{C}, 0)$, defined on an open ball $\mathbb{B}$ in $\mathbb{C}^m$, $m \geq 2$, centred at $\underline{0}$, with an isolated critical value at $0 \in \mathbb{C}$. We are going to study the geometry of f by looking at the 1-parameter family of real valued functions associated to it as above.

Recall there are two equivalent ways of defining the Milnor fibration, either as:

$$\phi = \frac{f}{|f|} : \mathbb{S}_\varepsilon \setminus K \longrightarrow \mathbb{S}^1 \,, \tag{7.1}$$

where $K = f^{-1}(0) \cap \mathbb{S}_\varepsilon$ is the link, or equivalently by:

$$f : N(\varepsilon, \eta) \;\longrightarrow \partial\mathbb{D}_\eta \,, \tag{7.2}$$

where $\varepsilon >> \eta > 0$ are sufficiently small, $\mathbb{D}_\eta \subset \mathbb{C}$ is the disc of radius η around $0 \in \mathbb{C}$, $\mathbb{B}_\varepsilon$ is the ball of radius ε around $\underline{0} \in \mathbb{C}^n$ and $N(\varepsilon, \eta)$ is the *Milnor tube* $\mathbb{B}_\varepsilon \cap f^{-1}(\partial\mathbb{D}_\eta)$.

For each $\theta \in [0, \pi)$ consider the real line $\mathcal{L}_\theta \subset \mathbb{C}$ passing through the origin with an angle θ with respect to the real axis, measured in the usual way. Let $\pi_\theta \colon \mathbb{C} \to \mathcal{L}_\theta$ be the orthogonal projection and set $h_\theta = \pi_\theta \circ f$, so that h_0 and $h_{\frac{\pi}{2}}$ are, respectively, the real and imaginary parts of f. Since each π_θ is a submersion everywhere, one has that the critical points of f are exactly the critical points of each h_θ. Hence $\{h_t\}$ is a 1-parameter family of real analytic functions with a critical value at 0, and if we set $X_\theta = h_\theta^{-1}(0)$ and $V = f^{-1}(0)$, then each X_θ is a real hypersurface with $Sing\, X_\theta = Sing\, V$.

As before (see for instance [37, VI.4] for details), it is an exercise to show that one has a decomposition:

$$\mathbb{B} = \cup X_\theta \quad \text{and} \quad V = \cap X_\theta = X_{\theta_1} \cap X_{\theta_2} \text{ , for each pair } \theta_1 \neq \theta_2 \,, \tag{7.3}$$

that we denote by $\mathcal{X}_\mathbb{B}$. That is, $\mathcal{X}_\mathbb{B}$ denotes the ball $\mathbb{B}$ regarded as union of the hypersurfaces X_θ, with $\theta \in [0, \pi)$. Then, using [24] one can easily prove:

Lemma 7.4 *For every sufficiently small ball $\mathbb{B}_\varepsilon \subset \mathbb{B}$ around $\underline{0}$, one has that each manifold $X_t \backslash V$ meets the boundary sphere $\mathbb{S}_\varepsilon = \partial\mathbb{B}_\varepsilon$ transversally. Thus one has a decomposition $(\mathcal{X}_{\mathbb{S}_\varepsilon \setminus K})$, where $K = V \cap \mathbb{S}_\varepsilon$ is the link, induced from*

the above decomposition $\mathcal{X}_{\mathbb{B}_\varepsilon}$ restricted to $\mathbb{B}_\varepsilon \setminus V$, and $\mathcal{X}_{\mathbb{B}_\varepsilon \setminus V}$ is diffeomorphic to the cylinder $\mathcal{X}_{\mathbb{S}_\varepsilon \setminus K}) \times [0,1)$.

Now observe that each X_t is naturally the union of three sets: the points $z \in \mathbb{B}$ that map to 0 under the projection $\pi_\theta \colon \mathbb{C} \to \mathcal{L}_\theta$, and the points that map under this projection into the two half lines of $\mathcal{L}_\theta \setminus \{0\}$. Write this as:

$$X_\theta = E_\theta \cup V \cup E_{\theta+\pi} \, .$$

Similarly one has:

$$(X_\theta \cap \mathbb{S}_\varepsilon) = (E_\theta \cap \mathbb{S}_\varepsilon) \cup (V \cap \mathbb{S}_\varepsilon) \cup (E_{\theta+\pi} \cap \mathbb{S}_\varepsilon) \, .$$

The previous lemma, together with Milnor's proof of his Theorem 4.8 in [24] imply that there exists a differentiable flow $\{\phi_t\}$ on $\mathbb{B}_\varepsilon \setminus V$, defined for all $t \in \mathbb{R}$, which leaves invariant each sphere in $\mathbb{B}_\varepsilon$ around $\underline{0}$, whose orbits are transversal to the manifolds $X_\theta \setminus V$ and permutes these manifolds: for each fixed time t, the flow carries each $X_\theta \setminus V$ into $X_{\theta+t} \setminus V$, where the angle $\theta + t$ must be taken modulo π. In particular, for $t = \pi$ the flow interchanges the two halves of $X_\theta \setminus V$.

Thus we arrive to the following slight refinement of Milnor's fibration theorem:

Theorem 7.5 *Let $f : (\mathbb{B}_\varepsilon, \underline{0}) \to (\mathbb{C}, 0)$ be holomorphic with a critical value at $0 \in \mathbb{C}$. For each $\theta \in [0, \pi)$, let h_θ be the composition of f followed by the projection into the real line in $\mathbb{C}$ with angle θ. Then:*

i) *The h_θ are all real analytic maps, whose critical points are those of f. Thus each $X_\theta = h_\theta^{-1}(0)$ is a real analytic hypersurface in $\mathbb{B}$ that contains $V = f^{-1}(0)$ and its singular set is $Sing(V)$.*

ii) *The union of all X_θ is the whole ball B_ε and the intersection of any two of them is V.*

iii) *The manifolds $X_\theta \setminus V$ are the fibres of the fibre bundle:*

$$\psi(z) = \frac{f(z)}{|f(z)|} : \mathbb{B}_\varepsilon \setminus V \longrightarrow \mathbb{S}^1 \, ,$$

whose restriction to the sphere gives Milnor's fibration (7.1), and whose restriction to $f^{-1}(\partial \mathbb{D}_\eta) \cap \mathbb{B}_\varepsilon$, $\varepsilon >> \eta > 0$, gives the equivalent Milnor-Lê fibration (7.2), up to multiplication by η..

In [7] we actually prove this theorem (as in [23]) for holomorphic map-germs defined on analytic spaces with arbitrary singularities.

8 Singularities $f\bar{g}$ and Milnor fibrations for meromorphic germs

We notice that the singularities in (5.5) as well as that in A'Campo's example (4.3) for $m = 0$, are all of the form $f\bar{g}$ with f, g being holomorphic functions in two complex variables. This type of singularities also appear in Lee Rudolph's work [34] and in [12]. This motivated the study in [29, 31] of singularities in $\mathbb{C}^n$ of the form $f\bar{g}$. Notice that the zero-set of $f\bar{g}$ is $\{f = 0\} \cup \{g = 0\}$, so for $n > 2$ the link $K_{fg} = \{f\bar{g} = 0\} \cap \mathbb{S}^{2n-1}_\varepsilon$ is necessarily a singular variety and $f\bar{g}$ must have non-isolated critical points. Also notice that, since the link of $f\bar{g}$ is also the link of fg which is a holomorphic function, Milnor's theorem tells us that its complement $\mathbb{S}^{2n-1}_\varepsilon \setminus K_{fg}$ fibres over the circle $\mathbb{S}^1$. But the same manifold can fibre over the circle in very many different ways, and Milnor's fibration (2.1) is one of them. The following theorem is proved in [31].

Theorem 8.1 *Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be holomorphic map-germs such that the function $f\bar{g}$ has an isolated critical value at 0. Then,*

$$\phi = \frac{f\bar{g}}{|f\bar{g}|} : \mathbb{S}^{2n-1}_\varepsilon \setminus K_{fg} \longrightarrow S^1 \ ,$$

is a C^∞ fibre bundle. (In general not equivalent to the corresponding fibration for the holomorphic germ fg.)

Notice that in Milnor's theorem (2.1) the origin $0 \in \mathbb{C}$ is automatically an isolated critical value (unless it is a regular value) because the function in question is holomorphic. However maps of the form $f\bar{g}$ can have non-isolated critical values.

The starting point for the proof in [31] of this result is to notice that, as pointed out in [12], on $\mathbb{S}^{2n-1}_\varepsilon \setminus K_{fg}$ the map ϕ coincides with $\frac{f/g}{|f/g|}$, so Theorem 8.1 can be regarded as a statement for meromorphic maps. Then the proof follows step by step the method used by Milnor in [24] to prove his fibration theorem for complex singularities: first show that ϕ has no critical points, so that all the fibres $\phi^{-1}(e^{i\theta})$ are smooth manifolds of codimension 1 in $\mathbb{S}^{2n-1}_\varepsilon$; then construct a smooth, complete vector field on the complement $\mathbb{S}^{2n-1}_\varepsilon \setminus K_{fg}$ whose integral lines are transversal to the fibres of ϕ and the corresponding flow moves the points at constant speed with respect to the

argument of $\phi(\underline{z})$, so it carries each fibre diffeomorphically into the nearby fibres, giving a local product structure.

We notice that a similar theorem is proved in [4] replacing the condition that $f\bar{g}$ has an isolated critical value by the condition that the meromorphic germ f/g is semi-tame at the origin (cf. [25]).

It is of course interesting to compare this fibration with the local fibrations for meromorphic germs studied in [9, 10]; this is done in [5]. More precisely, given f, g as above, one has a well defined map $f/g : \mathbb{C}^n \setminus V(fg) \to \mathbb{C} \setminus \{0\}$, where $V(fg) = f^{-1}(0) \cup g^{-1}(0)$. One can consider local "Milnor tubes" for this map around 0, ∞, and around each critical point of f, as in theorem (2.3); it is proved in [9, 10] that each of these tubes has a natural fibre bundle structure. This is achieved in those articles by "going up" to a resolution of the singularity and looking at the induced map from there into $\mathbb{C}$.

I am grateful to David Massey for explaining me how to obtain these fibrations from the Milnor-Lê fibration theorem for holomorphic germs. The idea is simple and beautiful, so we sketch it now. For simplicity we do it for the local fibration at 0, but this can be easily adapted to all critical points of f/g. Consider the hypersurface X in $\mathbb{C}^{n+1}$ defined by

$$X = \{(z,t) \in \mathbb{C}^n \times \mathbb{C} \,|\, f(\underline{z}) = t\, g(\underline{z})\} \,,$$

and let $\pi : X \to \mathbb{C}$ be the obvious map $(z,t) \mapsto t$. By Lê's theorem in [15], for every $\varepsilon, \eta > 0$ sufficiently small, $\varepsilon >> \eta$, one has a fibre bundle

$$\mathbb{B}_\varepsilon^{2n+2} \cap X \cap \pi^{-1}(\partial \mathbb{D}_\eta) \xrightarrow{\pi} \partial \mathbb{D}_\eta \,,$$

where $\mathbb{D}_\eta$ is a small disc around 0 in $\mathbb{C}$. The fibres of this bundle are $\mathbb{B}_\varepsilon^{2n+2} \cap \big(\{f = g = 0\} \cup \{f/g = t \text{ with } g \neq 0\}\big)$. Furthermore, by the 1^{st} Thom-Mather Isotopy Theorem (see [19]) one can lift the usual vector field on $\partial \mathbb{D}_\eta$ to a stratified, integrable vector field on $\mathbb{B}_\varepsilon^{2n+2} \cap X \cap \pi^{-1}(\partial \mathbb{D}_\eta)$, which is transversal to all the fibres of π and preserves the strata $\{f = g = 0\}$ in the fibres. This vector field can be also chosen to be tangent to the link K of X. Thus one gets the usual Milnor-Lê fibration. Now remove from each fibre the stratum $\{f = g = 0\}$ to get a fibre bundle over $\partial \mathbb{D}_\eta$ with fibre $f/g = t \cap (\mathbb{B}_\varepsilon^{2n+2} \cap X)$. This is the local fibration for f/g at 0. Of course we can do the same for the germ g/f and get the local fibration for f/g at ∞.

In [5] we show that, with appropriate conditions, these are both sub-fibrations of the fibration given in 8.1.

9 Fibrations of multilinks

The material in this section is taken from [31] and we refer to that article and to [8] for more on the subject. See also the seminal article [28] of Anne Pichon, as well as [4, 29, 30].

So far we have envisaged fibre bundle decompositions of the complement of the link of some (real or complex) singularity, but we have ignored significant additional information that one has: what is the behaviour of the fibration near the link? Let us illustrate this with a trivial example. Take your favourite holomorphic map $f : \mathbb{C}^2 \to \mathbb{C}$, for instance $(z_1, z_2) \mapsto z_1^2 + z_2^3$, whose link K_f is the trefoil knot in $\mathbb{S}^3$. Milnor's theorem tells us that

$$\frac{f}{|f|} : \mathbb{S}^3 \setminus K_f \to \mathbb{S}^1 ,$$

is a fibre bundle. Now take the same map f to some power $k > 1$. The vanishing set is that of f, so the link is the same K_f, and Milnor's theorem says that

$$\frac{f^k}{|f^k|} : \mathbb{S}^3 \setminus K_f \to \mathbb{S}^1 ,$$

is also a fibre bundle. In the first case one actually has an *open-book decomposition* of the 3-sphere, *i.e.* we can find a tubular neighbourhood N of K_f in $\mathbb{S}^3$ diffeomorphic to $\mathbb{S}^1 \times \mathbb{D}^2$ where the projection map $\frac{f}{|f|}$ takes the form $(t, z) \mapsto \frac{z}{|z|}$. In the second case the corresponding map is $(t, z) \mapsto \frac{z^k}{|z^k|}$ and this can be considered as a "generalized open-book" (or rather a multilink fibration, as we explain below).

Similarly, consider the singularities given by equations (5.4) and (5.5). In both cases the link is the union of *the Hopf link*

$$L_0 = \mathbb{S}^3 \cap \big(\{z_1 = 0\} \cup \{z_2 = 0\}\big)$$

with the link L_1 of $\{z_1^{p+1} + z_2^{q+1} = 0\}$, which is a torus link of type $(p+1, q+1)$. Choosing appropriate orientations one has that in a neighbourhood of each component of L_0 the projection map corresponding to the holomorphic map in (5.4) is $(t, z) \mapsto \frac{z}{|z|}$, while for the map in (5.5) it is $(t, z) \mapsto \frac{\bar{z}}{|z|}$. Around the components in L_1 both projection maps coincide.

What we see in these examples is the *multilink structure*. Let us make this concept precise. For simplicity we restrict to the case of links in $\mathbb{S}^3$. A

link in $\mathbb{S}^3$ is a finite union of disjoint circles $\mathbb{S}^1$ embedded in the 3-sphere; its connected components are called *knots*.

We let $L = K_1 \cup \ldots \cup K_l$ be the connected components of a link L in $\mathbb{S}^3$. We say that L is *oriented* if an orientation is fixed on each K_i. Notice that the normal bundle of each K_i in $\mathbb{S}^3$ is trivial. Hence, if L is oriented then one has orientation preserving diffeomorphisms $N(K_i) \cong D^2 \times \mathbb{S}^1$ defined on a neighbourhood of each K_i, that carry the oriented circles K_i into $\{0\} \times \mathbb{S}^1$ with its usual orientation.

A *multilink* is the data of an oriented link $L = K_1 \cup \ldots \cup K_l$ together with a *multiplicity* $n_i \in \mathbb{Z}$ associated with each component K_i. We denote such a multilink by

$$L = n_1 K_1 \cup \ldots \cup n_l K_l \,,$$

and we fix the convention that $n_i K_i = (-n_i)(-K_i)$, where $-K_i$ means K_i with the opposite orientation.

Definition 9.1 A multilink $L = n_1 K_1 \cup \ldots \cup n_l K_l$ in $\mathbb{S}^3$ is *fiberable* if there exists a map $\Phi : \mathbb{S}^3 \setminus L \longrightarrow \mathbb{S}^1$ which satisfies:

i) the map Φ is a C^∞ locally trivial fibration; and

ii) for each $i = 1, \ldots, l$, there exists a tubular neighbourhood $N(K_i)$ of K_i in $\mathbb{S}^3 \setminus (L \setminus K_i)$, an orientation-preserving diffeomorphism $\tau : \mathbb{S}^1 \times \mathbb{D}^2 \to N(K_i)$ such that $\tau(\mathbb{S}^1 \times \{0\}) = K_i$, and an integer $k_i \in \mathbb{Z}$ such that for all $(t, z) \in \mathbb{S}^1 \times (\mathbb{D}^2 \setminus \{0\})$ one has:

$$(\Phi \circ \tau)(t, z) = \left(\frac{z}{|z|}\right)^{n_i} t^{k_i}$$

In this case we say that Φ is a *fibration of the multilink* L.

For instance, if $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ is holomorphic and $f = \prod_{i=1}^{l} f_i^{n_i}$ is its decomposition into irreducible factors, then the multilink associated with f is

$$L_f = \bigcup_{i=1}^{l} n_i K_{f_i} \,,$$

where $K_{f_i} = \mathbb{S}^3_\varepsilon \cap f_i^{-1}(0)$, and the Milnor fibration $f(x)/|f(x)| : \mathbb{S}^3_\varepsilon \setminus L_f \to \mathbb{S}^1$ is a fibration of the multilink L_f.

A *plumbing multilink* in $\mathbb{S}^3$ is a multilink $L = n_1 K_1 \cup \ldots \cup n_l K_l$ such that there is a plumbing decomposition of $\mathbb{S}^3$ for which each K_i is an $\mathbb{S}^1$-leaf. These

links are classified in [8], where there is also an easy combinatorial algorithm (11.2) to determine from the plumbing graph of the corresponding plumbing decomposition, whether or not a given plumbing multilink is fiberable. This is purely topological. Notice that all links defined by a holomorphic equation are automatically plumbing links.

One has the following theorem of [31], which improves an earlier version given in [29] (see also [4]).

Theorem 9.2 *Let $f, g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be two holomorphic germs with no common branches. Then the real analytic germ $f\bar{g} : (\mathbb{S}^3, p) \to (\mathbb{R}^2, 0)$ has 0 as an isolated critical value if and only if the multilink $L_f - L_g$ is fibred. Moreover, if these conditions hold, then the Milnor fibration (of 8.1):*

$$\frac{f\bar{g}}{|f\bar{g}|} : \mathbb{S}^3 \setminus (L_f \cup L_g) \longrightarrow \mathbb{S}^1$$

is a fibration of the multilink $L_f - L_g$.

We refer to [31] for the proof and details about this result, which is actually proved in the more general setting of plumbing multilinks in any 3-manifold which is the link of an isolated complex surface singularity (X, p). There is also given in [31] one more equivalence in the theorem: the multilink $L_f - L_g$ is fibred iff f and g have different multiplicities at each rupture vertex of the graph of a resolution of the holomorphic germ $fg : (X, p) \to (\mathbb{C}, 0)$.

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