Characteristic classes of singular varieties
Hirzebruch theory and motivic theory

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Definition (Poincaré)

Let $X$ be a triangulated compact (smooth or singular) variety, the Euler-Poincaré characteristic of $X$ is defined as

$$e(X) = \sum_{i=0}^{m} (-1)^i k_i$$

where $m = \dim_{\mathbb{R}} X$ and $k_i$ is the number of $i$-dimensional simplices.
Example 1 (Lhuilier)

Let $X$ be a complex algebraic curve, i.e. a compact Riemann surface. $X$ is homeomorphic to a sphere with $g$ handles.

The Euler - Poincaré characteristic of $X$ is

$$e(X) = 2 - 2g.$$  

Example 2

Euler - Poincaré characteristic of the pinched torus is $e(X) = 1$. 

Theorem (Poincaré-Hopf)

Let $X$ be a compact manifold and let $v$ be a (continuous) vector field with (finitely many) isolated singularities $(a_j)_{j \in J}$ of index $I(v, a_j)$, then

$$e(X) = \sum_{j \in J} I(v, a_j).$$
Preamble : 2. The arithmetic genus

Let $X$ be a complex algebraic manifold, $n = \dim_{\mathbb{C}} X$. Let $g_i$ be the number of $\mathbb{C}$-linearly independent holomorphic differential $i$-forms on $X$.

- $g_0$ is the number of linearly independent holomorphic functions, i.e. the number of connected components of $X$,
- $g_n$ is called geometric genus of $X$,
- $g_1$ is called irregularity of $X$,

**Definition (Arithmetic Genus)**

The *arithmetic genus* of $X$ is defined as :

$$\chi(X) := \sum_{i=0}^{n} (-1)^i g_i$$
Example

Let $X$ be a complex algebraic curve, i.e. a compact Riemann surface. $X$ is homeomorphic to a sphere with $g$ handles. Then $g_0 = 1$ and $g_1 = g_n = g$.

The arithmetic genus of $X$ is:

$$
\chi(X) = 1 - g
$$
The **Todd genus** $T(X)$ has been defined (by Todd) in terms of Eger-Todd fundamental classes (polar varieties), using Severi results. The Eger-Todd classes are homological Chern classes of $X$.

Todd “proved” that

$$T(X) = \chi(X).$$

In fact, the Todd proof uses a Severi Lemma which has never been completely proved. The result has been proved by Hirzebruch.
Definition (Thom-Hirzebruch)

Let $M$ be a (real) compact oriented $4k$-dimensional manifold. Let $x$ and $y$ two elements of $H^{2k}(M; \mathbb{R})$, then

$$\langle x \cup y, [M] \rangle \in \mathbb{R}$$

defines a bilinear form on the vector space $H^{2k}(M; \mathbb{R})$.

The index (or signature) of $M$, denoted by $\text{sign}(M)$, is defined as the index of this form, i.e. the number of positive eigenvalues minus the number of negative eigenvalues.
What shall we do?

<table>
<thead>
<tr>
<th>$X$ manifold number</th>
<th>$X$ manifold cohomology classes</th>
<th>$X$ singular variety homology classes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e(X)$</td>
<td>Chern</td>
<td>Schwartz-MacPherson</td>
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<td></td>
<td>_</td>
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<td>$\chi(X)$</td>
<td>Todd</td>
<td>Baum-Fulton-MacPherson</td>
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<td></td>
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<tr>
<td>sign($X$)</td>
<td>Thom-Hirzebruch</td>
<td>Cappell-Shaneson</td>
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</table>

Hirzebruch Theory  Motivic Theory (BSY)
Hirzebruch Series

\[ Q_y(\alpha) := \frac{\alpha (1 + y)}{1 - e^{-\alpha (1+y)}} - \alpha y \in \mathbb{Q}[y][[\alpha]] \]

- \[ Q_{-1}(\alpha) = 1 + \alpha \quad y = -1 \]
- \[ Q_0(\alpha) = \frac{\alpha}{1 - e^{-\alpha}} \quad y = 0 \]
- \[ Q_1(\alpha) = \frac{\alpha}{\tanh \alpha} \quad y = 1 \]
Characteristic Classes of Manifolds.

Let $X$ be a complex manifold with dimension $\dim_{\mathbb{C}} X = n$, let us denote by

$$c^*(TX) = \sum_{j=0}^{n} c^j(TX), \quad c^j(TX) \in H^{2j}(X; \mathbb{Z})$$

the total Chern class of the (complex) tangent bundle $TX$.

**Definition**

The *Chern roots* $\alpha_i$ of $TX$ are defined by:

$$\sum_{j=0}^{n} c^j(TX) \, t^j = \prod_{i=1}^{n} (1 + \alpha_i t)$$

$\alpha_i \in H^2(X; \mathbb{Z})$. 

One defines the Todd-Hirzebruch class: $\widetilde{td}(y)(TX) := \prod_{i=1}^{n} Q_y(\alpha_i)$

\[
\begin{cases}
    c^*(TX) = \prod_{i=1}^{n} (1 + \alpha_i) & y = -1 \\
    td^*(TX) = \prod_{i=1}^{n} \left( \frac{\alpha_i}{1-e^{-\alpha_i}} \right) & y = 0 \\
    L^*(TX) = \prod_{i=1}^{n} \left( \frac{\alpha_i}{\tanh \alpha_i} \right) & y = 1
\end{cases}
\]

Chern class, Todd class, Thom-Hirzebruch $L$-class.
The $\chi_y$-characteristic

Let $X$ be a complex projective manifold.

**Definition**

One defines the $\chi_y$-characteristic of $X$ by

$$
\chi_y(X) := \sum_{p=0}^{\infty} \left( \sum_{i=0}^{\infty} (-1)^i \dim_\mathbb{C} H^i(X, \wedge^p T^* X) \right) \cdot y^p
$$

- $y = -1$ \quad $\chi_{-1}(X) = e(X)$, Euler - Poincaré characteristic of $X$ (Hodge)
- $y = 0$ \quad $\chi_0(X) = \chi(X)$, arithmetic genus of $X$ (definition)
- $y = 1$ \quad $\chi_1(X) = \text{sign}(X)$, signature of $X$ (Hodge)
\[ \chi_y(X) := \sum_{p=0}^{\infty} \left( \sum_{i=0}^{\infty} (-1)^i \dim_{\mathbb{C}} H^i(X, \wedge^p T^* X) \right) \cdot y^p \]

- \[ y = -1 \quad e(X) \]
- \[ y = 0 \quad \chi(X) \]
- \[ y = 1 \quad \text{sign}(X) \]

\[ \widetilde{td}_y(TX) := \prod_{i=1}^{n} Q_y(\alpha_i) \]

**Hirzebruch Riemann-Roch Theorem**

One has:

\[ \chi_y(X) = \int_X \widetilde{td}_y(TX) \cap [X] \quad \in \mathbb{Q}[y]. \]
The three particular cases

- \( e(X) = \int_X c^*(TX) \cap [X] \) 
  Euler - Poincaré characteristic of \( X \)
  \( Poincaré-Hopf \) Theorem

- \( \chi(X) = \int_X td^*(TX) \cap [X] \) 
  arithmetic genus of \( X \)
  \( Hirzebruch-Riemann-Roch \) Theorem

- \( \text{sign}(X) = \int_X L^*(TX) \cap [X] \) 
  signature of \( X \)
  \( Hirzebruch \, signature \) Theorem
Question

What can we do for singular varieties?
Three generalisations in the case of singular varieties.

**Chern Transformation (MacPherson)**

\[ \mathbb{F}(X) : \text{Group of constructible functions (ex. } 1_X) \]

\[ c_* : \mathbb{F}(X) \to H_*(X) \]

One defines \( c_*(X) := c_*(1_X) : \) Schwartz-MacPherson class of \( X \).

**Todd Transformation (Baum-Fulton-MacPherson)**

\[ G_0(X) : \text{Grothendieck Group of coherent sheaves (ex. } \mathcal{O}_X) \]

\[ td_* : G_0(X) \to H_*(X) \otimes \mathbb{Q} \]

One defines \( td_*(X) := td_*([\mathcal{O}_X]) \).

**L-Transformation (Cappell-Shaneson)**

\[ \Omega(X) : \text{Group of constructible self-dual sheaves (ex. } \mathcal{IC}_X) \]

\[ L_* : \Omega(X) \to H_{2*}(X; \mathbb{Q}) \]

One defines \( L_*(X) := L_*([\mathcal{IC}_X]) \).
Problem:

The three transformations are defined on different spaces:

\[ F(X), \quad G_0(X) \quad \text{and} \quad \Omega(X) \]
Where the “motivic” arrives...

**Definition**

The Grothendieck relative group of algebraic varieties over $X$

$$K_0(var/X)$$

is the quotient of the free abelian group of isomorphy classes of algebraic maps $Y \rightarrow X$, modulo the “additivy relation”:

$$[Y \rightarrow X] = [Z \rightarrow Y \rightarrow X] + [Y \setminus Z \rightarrow Y \rightarrow X]$$

for closed algebraic sub-spaces $Z$ in $Y$. 
Theorem

The map $e : K_0(var/X) \longrightarrow \mathbb{F}(X)$ defined by $e([f : Y \to X]) := f_! \mathbf{1}_Y$ is the unique group morphism which commutes with direct images for proper maps and such that $e([id_X]) = 1_X$ for $X$ smooth and pure dimensional.

Theorem

There is an unique group morphism $m_C : K_0(var/X) \longrightarrow G_0(X)$ which commutes with direct images for proper maps and such that $m_C([id_X]) = [\mathcal{O}_X]$ for $X$ smooth and pure dimensional.

Theorem

The morphism $sd : K_0(var/X) \longrightarrow \Omega(X)$ defined by

$$sd([f : Y \to X]) := [Rf_* \mathcal{O}_Y \dim_\mathbb{C}(Y) + \dim_\mathbb{C}(X)]$$

is the unique group morphism which commutes with direct images for proper maps and such that $sd([id_X]) = [\mathcal{O}_X[2 \dim_\mathbb{C}(X)]] = [\mathcal{IC}_X]$ for $X$ smooth and pure dimensional.
Theorem

There is an unique group morphism

\[ T_y : K_0(\text{var}/X) \longrightarrow H_\ast(X) \otimes \mathbb{Q}[y] \]

which commutes with direct images for proper maps and such that

\[ T_y([id_X]) = \widetilde{td}_y(TX) \cap [X] \text{ for } X \text{ smooth and pure dimensional.} \]

In particular, one has: \( T_{-1}([id_X]) = c_\ast(X) \)

Remark

If a complex algebraic variety \( X \) has only rational singularities (for example if \( X \) is a toric variety), then:

\[ mC([id_X]) = [\mathcal{O}_X] \in G_0(X) \text{ and in this case } T_0([id_X]) = td_\ast(X). \]

That is not true in general!
The main result

**Theorem**

*The following diagram commutes:*

\[
\begin{array}{ccc}
\mathbb{F}(X) & \xleftarrow{e} & K_0(\text{var}/X) \\
\downarrow c_* & & \downarrow T_y \\
H_*(X) \otimes \mathbb{Q} & \xleftarrow{y = -1} & H_*(X) \otimes \mathbb{Q}[y]
\end{array}
\]
The main result

**Theorem**

The following diagram commutes:

\[ \begin{array}{ccc}
K_0(\text{var}/X) & \xrightarrow{mC} & G_0(X) \\
T_y \downarrow & & \downarrow td_* \\
H_*(X) \otimes \mathbb{Q}[y] & \xrightarrow{y=0} & H_*(X) \otimes \mathbb{Q}
\end{array} \]
The main result

Theorem

The following diagram commutes:

\[
\begin{array}{cccc}
K_0(\text{var}/X) & & & \\
& \downarrow^{sd} & & \\
T_y & \downarrow & \Omega(X) & \\
& & & \\
H_\ast(X) \otimes \mathbb{Q}[y] & & & L_\ast \downarrow \\
& & & \\
y=1 & \downarrow & \Rightarrow & \\
& & & \\
H_\ast(X) \otimes \mathbb{Q} & & & \\
\end{array}
\]
The main result

Theorem

The following tripod diagram commute:

\[ \begin{array}{ccc}
F(X) & \xleftarrow{e} & K_0(\text{var}/X) & \xrightarrow{mc} & G_0(X) \\
F(X) \otimes \mathbb{Q} & \xleftarrow{y=-1} & H_*(X) \otimes \mathbb{Q}[y] & \xrightarrow{L_*} & H_*(X) \otimes \mathbb{Q} \\
H_*(X) \otimes \mathbb{Q} & \xleftarrow{y=1} & H_*(X) \otimes \mathbb{Q} \\
\end{array} \]
Thanks for your attention
Joyeux anniversaire

Pepe