Lipschitz geometry of minimal surface singularities

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Joint work with Walter Neumann and Helge Pedersen

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Theorem. (Neumann, Pedersen, -, 2014)
Minimal surface singularities are normally embedded.
Conversely, any rational singularity which is normally embedded is minimal.

1. Lipschitz geometry, motivation 1
2. Minimal singularities, motivation 2
3. Lipschitz geometry of curves and normal surfaces
4. Ingredients of the proof
Topology of complex singularities

$X \subset \mathbb{C}^n$ complex algebraic variety, $0 \in X$.

**Question:** how looks $X$ in a neighborhood of 0?

**Topological point of view:**
Link $X : X^{(\epsilon)} = X \cap S^{2n-1}_\epsilon \epsilon << 1$.

**Theorem. (Conical structure theorem)**
For all $0 < \epsilon, \epsilon' << 1$,
$X^{(\epsilon)}$ is homeomorphic to $X^{(\epsilon')}$, and

$$(X \cap B_\epsilon, 0) \xrightarrow{\text{homeo}} (\text{Cone } (X^{(\epsilon)}), 0)$$

**Geometrical point of view:**
how changes $X^{(\epsilon)}$ metrically when $\epsilon \to 0$?
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Lipschitz geometry

\((X, 0) \subset (\mathbb{C}^n, 0)\) a complex analytic germ.

- outer distance \(d_{\text{out}}(x, y) := \|x - y\|_{\mathbb{C}^n}\)
- inner distance \(d_{\text{inn}} := \text{infimum of length of rectifiable arcs from } x \text{ to } y \text{ on } X\)

**Lipschitz category**

Objects: complex germs \((X, 0)\)
Morphisms: local bilipschitz homeomorphisms \((X, 0) \to (X', 0)\)

Analytical type \(\to\) outer lipschitz geometry \(\to\) inner lipschitz geometry

**Definition.** \((X, 0)\) is *normally embedded* if inner and outer metrics on \((X, 0)\) are bilipschitz equivalent, i.e., exists \(K \geq 1\),

\[d_{\text{inn}}(x, y) \leq K \ d_{\text{out}}(x, y)\]
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Motivation 1

Describing singularities satisfying remarkable metric properties.

**Theorem.** (Birbrair, Fernandes, Lê, Sampaio, 2014) Lipschitz regular germs of complex algebraic sets are smooth.

Normal surfaces:

**Theorem.** (Pedersen, 2011) Complete description of rational singularities which are metrically conical.

**Theorem.** (Neumann, Pedersen, -, 2014) The rational singularities which are normally embedded are the minimal singularities
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Minimal singularities

**Definition.** (Kollár, 1985) A germ of analytic space \((S, 0)\) is *minimal* if it is reduced, Cohen-Macaulay, if its tangent cone is reduced, and if Abyankar’s inequality

\[
m(S, 0) \geq edim(S, 0) - dim(S, 0) + 1
\]

is an equality.

**Proposition.** A germ of reduced curve \((C, 0) \subset (\mathbb{C}^n, 0)\) is minimal if it consists of \(n\) smooth transversal components whose tangent lines span an \(n\)-dimensional vector space.

**Proposition.** A normal surface singularity \((S, 0) \subset (\mathbb{C}^n, 0)\) is minimal if it is rational with a reduced fundamental cycle, i.e., if \(\pi: S' \to S\) is the minimal resolution, \(\pi^{-1}(0) = \bigcup E_i\), and if \(h: (S, 0) \to (\mathbb{C}, 0)\) is a general linear form then the multiplicity of \(h \circ \pi\) along each \(E_i\) equals 1.
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Resolution

**Theorem.** (Hironaka, 1964) Any algebraic variety on an algebraically closed field with characteristic zero admits a resolution.

Surface case:

**Theorem.** (Walker, with Jung’s method, 1935) Any algebraic surface admits a resolution.

**Theorem.** (Zariski, 1939) Any algebraic surface admits a resolution which is a finite composition of normalized blow-ups of points.

**Theorem.** (Spivakovsky, 1980) Any algebraic surface admits a resolution which is a finite composition of normalized Nash transforms.

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**Idea of the proof:** Consider the pair \((m(S, 0), \eta(S, 0))\) where \(m(S, 0)\) is the multiplicity of the surface and \(\eta(S, 0)\) the multiplicity of the discriminant \(\Delta\) of a generic projection \(\ell: (S, 0) \to (\mathbb{C}^2, 0)\). After a finite number of normalized blow-ups, one obtains a surface whose all singularities have pairs \((m, \eta)\) strictly less than the initial \((m(S, 0), \eta(S, 0))\).

The case of **minimal singularities** is special and has to be treated independently.
Partial Jung resolution with generic projections

If \( m(S_1, p_1) = m(S, 0) \) then \( \ell: (S_1, p_1) \to (U_1, \ell(p_1)) \) is a generic projection. Then one iterates the process.

\[
\begin{array}{c}
(S_f, p_f) \xrightarrow{e'_{f-1}} \cdots \\
\downarrow \ell_f \\
U_f \xrightarrow{e'_{f-1}} \cdots \\
\end{array}
\quad
\begin{array}{c}
(S_2, p_2) \xrightarrow{e'_1} (S_1, p_1) \xrightarrow{e'_0} (S, 0) \\
\downarrow \ell_2 \\
U_2 \xrightarrow{e_1} U_1 \xrightarrow{e_0} U_0 \\
\end{array}
\]

If \( m(S_f, p_f) = \cdots = m(S_1, p_1) = m(S, 0) \) and the strict transform of \( \Delta \) at \( \ell(p_f) \) is smooth, then either \( \eta(S_f, p_f) < \eta(S, 0) \) or \( (S, 0) \) is minimal.
Duality question (Lê)

Resolution by normalized blow-ups
- Minimal singularities
- Generic hyperplane sections

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Lipschitz geometry of curves

Let \((C', 0) \subset (\mathbb{C}^n, 0)\) be a complex curve.

**Theorem.** \((C', 0)\) is inner bilipschitz equivalent to the cone over its link. (*inner metrically conical*).

**Theorem.** (Teissier) Let \(p: \mathbb{C}^n \to \mathbb{C}^2\) be a generic projection for \((C', 0)\). Then \((C', 0)\) is outer Lipschitz equivalent to the plane curve \((p(C'), 0)\).

**Theorem.** (Pham, Teissier, 1969) Let \((C, 0)\) be a plane curve germ. The outer Lipschitz geometry of \((C, 0)\) determines and is determined by the embedded topology of \((C, 0)\).
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Take a generic projection on a complex line $\ell: C \to \mathbb{C}$. **Inner metric** is Lipschitz equivalent to the lifted metric from the complex line. So $(C, 0)$ ismetrically conical. **Outer geometry** determines the characteristic exponents of $(C, 0)$, so the embedded topology of $(C, 0)$.
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Take a generic projection on a complex line \( \ell : C \to \mathbb{C} \).

**Inner metric** is Lipschitz equivalent to the lifted metric from the complex line. So \((C, 0)\) is metrically conical.

**Outer geometry** determines the characteristic exponents of \((C, 0)\), so the embedded topology of \((C, 0)\).
Lemma. Outer geometry of \((C, 0)\) is determined by inner geometry + distance between vertically aligned points with respect to a generic projection \(\ell: (C, 0) \rightarrow (\mathbb{C}, 0)\).

So outer geometry is determined by embedded topology.
Proposition. An irreducible curve is normally embedded iff it is smooth.

proof. Assume \((C, 0)\) is not smooth. Let \(q\) be a characteristic exponent of \(C, 0\). Consider a radial arc \(\gamma : [0, 1) \subset \mathbb{C}\). There exist two arcs \(\gamma_1 : [0, 1) \to S\) and \(\gamma_2 : [0, 1) \to S\) such that \(\ell \circ \gamma_i = \gamma\) and \(d_{out}(\gamma_1(t), \gamma_2(t)) = O(t^q)\). Since \(d_{inn}(\gamma_1(t), \gamma_2(t)) = O(t)\), we then have

\[
\lim_{t \to 0} \frac{d_{out}(\gamma_1(t), \gamma_2(t))}{d_{inn}(\gamma_1(t), \gamma_2(t))} = 0
\]

which implies that \((C, 0)\) is not normally embedded.
Geometry of surfaces

Again,

**Lemma.** Outer geometry of \((S, 0)\) is determined by inner geometry + distance between vertically aligned points with respect to a generic projection \(\ell: (S, 0) \to (\mathbb{C}^2, 0)\).
Why is a minimal surface normally embedded?

A hint

Abyankar inequality

\[ m(S, 0) \geq edim(S, 0) - 1 \]

is an equality.

There is “enough room” in the ambient space so that the \( m(S, 0) \) sheets of a generic projection \( \ell : (S, 0) \to (\mathbb{C}^2, 0) \) are far from each other.
Measuring outer distances

**Lemma.** Outer Lipschitz geometry of \((S, 0)\) is determined by inner geometry + distance between vertically aligned points with respect to a generic projection \(\ell: (S, 0) \to (\mathbb{C}^2, 0)\).

**Inner distance:** \(\ell\) is bilipschitz outside a zone around the polar curve. So outside the polar zone, inner distance is measured by distance in \(\mathbb{C}^2\). Geometric decomposition (Birbrair, Neumann, -, Acta Math. 2014)

**Distance between vertically aligned points:**

**Definition.** A **test curve** is an irreducible plane curve which is not a branch of \(\Delta\).

Distance between vertically aligned points over a point \(p \in \mathbb{C}^2\) is measured by Lipschitz geometry of the liftings \(\ell^{-1}(\gamma)\) of test curves \(\gamma\).

Only a selection of test curves is needed: they are related with resolution of the discriminant curve \(\Delta\).

Particular test curves are the generic lines. Their liftings are **generic hyperplane sections** of the surface.
Ingredients of the proof

**Theorem.** \((S, 0)\) minimal is normally embedded.

1. A characterization of normal embedding using lifting of test curves.
2. The description of \(\Delta\) by R. Bondil as a union of \(A_n\)-curves.
3. A description of liftings of test curves through resolution of \((S, 0)\) by Jung’s method.

**First step.** Let \(\ell: (S, 0) \to (\mathbb{C}^2, 0)\) be a generic projection and let \(\gamma\) be a generic line in \(\mathbb{C}^2\). Then \(\ell^{-1}(\gamma)\) is a generic hyperplane section of \((S, 0)\).

**Theorem.** (Kollár, 1985) Any generic hyperplane section of a minimal singularity is minimal.

Consequence: The curve germ \((\ell^{-1}(\gamma), 0)\) has a minimal singularity, so it is a union of transversal tangent lines. Therefore the outer distance between two vertically aligned points on \((\ell^{-1}(\gamma), 0)\) at distance \(t\) from the origin is \(O(t)\) as well as their inner distance.
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Joyeux anniversaire, Pepe !!!